

Optimal output feedback control of Timed Event Graphs including disturbances in a resource sharing environment

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Abstract: Timed Event Graphs (TEGs) constitute a subclass of timed Petri nets that model synchronization and delay phenomena, but not conflict or choice. In a suitable mathematical framework (idempotent semirings such as the min-plus algebra), the temporal evolution of TEGs can be described by a set of linear equations. Recently, a method has been proposed for the optimal control of TEGs that share one or more resources based on a prespecified priority policy. In this paper, we aim at finding a solution on how to deal with disturbances in TEGs with shared resources, which is not possible under the current feedforward-based approaches.

Keywords: Timed event graphs, idempotent semirings, min-plus-algebra, just-in-time control.

1. INTRODUCTION

Timed event graphs (TEGs) are a subclass of timed Petri nets which are characterized by the fact that each place has exactly one upstream and one downstream transition and all arcs have weight one. In a suitable mathematical framework like an idempotent semiring, the temporal evolution of a TEG can be described linearly, for instance in the min-plus algebra (Bacelli et al., 1992). Resource sharing is an important phenomenon in many applications but cannot be treated in the standard TEG framework. An example is a railway system where multiple trains share the same track so that trains occupy the track with delay between each other. Consequently, a choice must be made as to which train can use the track at a given time. Modeling conflict or choice requires that multiple users have access to the same resource.

The phenomenon of resource sharing has already been discussed in various literature sources. In van den Boom and De Schutter (2004), the authors implement switching max-plus linear systems to model systems that occupy resources simultaneously. The optimal switching sequence between modes is determined by using model predictive control. In Corrêa et al. (2009), additional inequalities are introduced to model resource sharing. Addad et al. (2012) proves that network conflicting timed event graphs can be represented by linear time-varying equations in the max-plus algebra while the policy of resource allocation is arbitrary.

Works regarding control of TEGs are less abundant. Hardouin et al. (2018) summarizes general approaches towards different control strategies. It considers feedforward control with a given output-reference and an additional prefilter, output-feedback control and state-feedback con-

trol, each with a prespecified transfer matrix restriction. In Menguy et al. (2000), the authors apply a technique to optimally (just-in-time) update the control input in case of a change in the output-reference. Moradi et al. (2017) provides a control technique for an output-reference-based approach with two subsystems sharing the same resources. The approach is based on a prespecified priority policy among the users competing for the resource. Schafaschek et al. (2020) builds on Menguy et al. (2000) and Moradi et al. (2017) by extending their results to an arbitrary number of subsystems and shared resources while also incorporating an output-reference update for resource-sharing TEGs. However, no work could be found that offers a control strategy for a resource sharing system that is affected by disturbances.

In practice, it is common that disturbances occur in control systems. The purpose of this paper is to develop a method that responds to disturbances in such a way that the optimal inputs are updated so as to continue to be *just-in-time*, taking into account the disturbances, preserving the past up to them and also considering the dependencies of each participant of the resource-sharing system. The control technique proposed here is a output feedback control strategy based on the output-reference, which is updated every time a disturbance is detected.

Section 2 summarizes basic results on semirings and elaborates on resource sharing. In Section 3, a method for optimal control for a single TEG with disturbances is proposed. Section 4 offers a control strategy for resource sharing systems that are affected by disturbances. Section 2 is strongly oriented to Schafaschek et al. (2020), and is included for the sake of making the paper largely self-contained. Sections 3 and 4 present the main contributions of this paper. Section 5 gives the conclusion.

2. ALGEBRAIC PRELIMINARIES

This section summarizes basic results on timed event graphs and idempotent semirings, residuation theory, resource sharing and control of TEGs. For an extended discussion the interested reader is invited to pursue Bacelli et al. (1992), Blyth and Janowitz (1972), De Schutter et al. (2020) and Hardouin et al. (2018).

2.1 Semiring theory

A set \mathcal{D} is an *idempotent semiring* (or dioid) if the following properties hold: \mathcal{D} is equipped with two binary operators, denoted \oplus (sum) and \otimes (product); \oplus is associative, commutative and idempotent, i.e., $\forall a \in \mathcal{D}, a \oplus a = a$, and has a neutral element (also called zero element) denoted by ε ; \otimes is associative, distributes over \oplus , and has a neutral element (also called unit element), denoted by e ; the element ε is absorbing for \otimes , i.e., $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon$. As in classical algebra, the multiplication sign \otimes is often omitted. An order relation can be defined over \mathcal{D} by

$$(\forall a, b \in \mathcal{D}) a \preceq b \Leftrightarrow a \oplus b = b.$$

Note that ε is the bottom element, i.e., $(\forall a \in \mathcal{D}) \varepsilon \preceq a$.

An idempotent semiring is *complete* if it is closed for infinite sums and if \otimes distributes over infinite sums. For a complete idempotent semiring, the top element is defined as $\top = \bigoplus_{x \in \mathcal{D}} x$, and the greatest lower bound operation, denoted \wedge , by

$$(\forall a, b \in \mathcal{D}) a \wedge b = \bigoplus_{x \preceq a, x \preceq b} x.$$

\wedge is associative, commutative, and idempotent, and we have $a \oplus b = b \Leftrightarrow a \preceq b \Leftrightarrow a \wedge b = a$.

Example 1. The set $\overline{\mathbb{Z}} \stackrel{\text{def}}{=} \mathbb{Z} \cup \{+\infty, -\infty\}$, with the minimum operation as \oplus and conventional addition as \otimes , forms a complete idempotent semiring called *min-plus algebra*, denoted $\overline{\mathbb{Z}}_{\min}$, in which $\varepsilon = +\infty$, $e = 0$, and $\top = -\infty$. Note that in $\overline{\mathbb{Z}}_{\min}$ we have $3 \oplus 7 = 3$, so $7 \preceq 3$; the order is reversed with respect to the conventional order over \mathbb{Z} . \diamond

A mapping $\Pi : \mathcal{D} \rightarrow \mathcal{C}$, with \mathcal{D} and \mathcal{C} two idempotent semirings, is isotone if it is order-preserving, i.e., $(\forall a, b \in \mathcal{D}) a \preceq b \Rightarrow \Pi(a) \preceq \Pi(b)$.

Remark 2. The composition of two isotone mappings is isotone. \diamond

Remark 3. (Hardouin et al., 2018). Let $\Pi : \mathcal{D} \rightarrow \mathcal{D}$ be an isotone mapping with \mathcal{D} a complete idempotent semiring and let $\mathcal{Y} = \{x \in \mathcal{D} \mid \Pi(x) = x\}$ be the set of fixed points of Π . $\bigwedge_{y \in \mathcal{Y}} y$ is the least fixed point of Π and $\bigoplus_{y \in \mathcal{Y}} y$ is the greatest fixed point of Π . \diamond

Algorithms exist (e.g. Hardouin et al. (2018)) which allow to compute, in a finite number of steps, the least and greatest fixed points of isotone mappings over complete idempotent semirings, provided such fixed points are finite.

In a complete idempotent semiring \mathcal{D} , the Kleene star operator on $a \in \mathcal{D}$ is defined as $a^* = \bigoplus_{i \geq 0} a^i$, with $a^0 = e$.

Remark 4. The implicit equation $x = ax \oplus b$ over a complete idempotent semiring \mathcal{D} admits $x = a^*b$ as least solution (see Bacelli et al. (1992)). \diamond

2.2 Semirings of formal power series

Let $s = \{s(t)\}_{t \in \overline{\mathbb{Z}}}$ be a sequence over $\overline{\mathbb{Z}}_{\min}$. The δ -transform of s is a formal power series in δ with coefficients in $\overline{\mathbb{Z}}_{\min}$ and exponents in $\overline{\mathbb{Z}}$, defined by

$$s = \bigoplus_{t \in \overline{\mathbb{Z}}} s(t) \delta^t.$$

We denote both the sequence and its δ -transform by the same symbol, as no ambiguity will occur. Since

$$s \otimes \delta = \bigoplus_{t \in \overline{\mathbb{Z}}} s(t) \otimes \delta^{t+1} = \bigoplus_{t \in \overline{\mathbb{Z}}} s(t-1) \otimes \delta^t,$$

multiplication by δ can be seen as a backward shift operator.

Definition 5. The set of formal power series in δ with exponents in $\overline{\mathbb{Z}}$ and coefficients in $\overline{\mathbb{Z}}_{\min}$, with addition and multiplication defined by

$$\begin{aligned} s \oplus s' &= \bigoplus_{t \in \overline{\mathbb{Z}}} (s(t) \oplus s'(t)) \delta^t, \\ s \otimes s' &= \bigoplus_{t \in \overline{\mathbb{Z}}} \left(\bigoplus_{\tau \in \overline{\mathbb{Z}}} (s(\tau) \otimes s'(t-\tau)) \right) \delta^t, \end{aligned}$$

is a complete idempotent semiring, denoted $\overline{\mathbb{Z}}_{\min}[[\delta]]$. Note that the order in $\overline{\mathbb{Z}}_{\min}[[\delta]]$ is induced by the order in $\overline{\mathbb{Z}}_{\min}$, i.e., $s \preceq s' \Leftrightarrow (\forall t \in \overline{\mathbb{Z}}) s(t) \preceq s'(t)$. \diamond

In this paper we will use sequences to represent the number of firings of transitions in TEGs, so that, e.g., $s(t)$ represents the accumulated number of firings of a transition up to time t . Such series are clearly nonincreasing (in the order of $\overline{\mathbb{Z}}_{\min}$), meaning their δ -transforms obey $s(t-1) \succeq s(t)$ for all t . We will henceforth refer to such series as *counters*.

Definition 6. The set of counters (i.e., nonincreasing power series) in $\overline{\mathbb{Z}}_{\min}[[\delta]]$ is a complete idempotent semiring, named $\overline{\mathbb{Z}}_{\min, \delta}[[\delta]]$, with zero element s_ε given by $s_\varepsilon(t) = \varepsilon$ for all t , unit element s_e given by $s_e(t) = e$ for $t \leq 0$ and $s_e(t) = \varepsilon$ for $t > 0$, and top element s_\top given by $s_\top(t) = \top$ for all t . We will denote this semiring by Σ , for brevity. \diamond

Counters can be represented compactly by omitting terms $s(t)\delta^t$ whenever $s(t) = s(t+1)$. For example, a counter s with $s(t) = e$ for $t \leq 3$, $s(t) = 1$ for $4 \leq t \leq 7$, $s(t) = 3$ for $8 \leq t \leq 12$, and $s(t) = 6$ for $t \geq 13$ can be written $s = e\delta^3 \oplus 1\delta^7 \oplus 3\delta^{12} \oplus 6\delta^{+\infty}$.

2.3 Residuation theory

Residuation theory provides, under certain conditions, greatest (resp. least) solutions to inequalities such as $f(x) \preceq b$ (resp. $f(x) \succeq b$).

Definition 7. An isotone mapping $f : \mathcal{D} \rightarrow \mathcal{C}$, with \mathcal{D} and \mathcal{C} complete idempotent semirings, is said to be residuated if for all $y \in \mathcal{C}$ there exists a greatest solution to the inequality $f(x) \preceq y$. This greatest solution is denoted $f^\sharp(y)$, and the mapping $f^\sharp : \mathcal{C} \rightarrow \mathcal{D}, y \mapsto \bigoplus \{x \in \mathcal{D} \mid f(x) \preceq y\}$, is called the *residual* of f .

Mapping f is said to be dually residuated if for all $y \in \mathcal{C}$ there exists a least solution to the inequality $f(x) \succeq y$. This least solution is denoted $f^\flat(y)$, and the mapping $f^\flat : \mathcal{C} \rightarrow \mathcal{D}, y \mapsto \bigwedge \{x \in \mathcal{D} \mid f(x) \succeq y\}$, is called the *dual residual* of f . \diamond

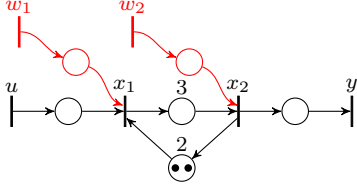


Fig. 1. A SISO timed event graph with uncontrollable input transitions w_1 and w_2 .

Note that, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ and $f^\flat(y)$ yield its greatest and least solutions, respectively.

Theorem 8. (Blyth and Janowitz, 1972). Mapping f as in Def. 7 is residuated if and only if there exists a unique isotone mapping $f^\sharp : \mathcal{C} \rightarrow \mathcal{D}$ such that $f \circ f^\sharp \preceq \text{Id}_{\mathcal{C}}$ and $f^\sharp \circ f \succeq \text{Id}_{\mathcal{D}}$, where $\text{Id}_{\mathcal{C}}$ and $\text{Id}_{\mathcal{D}}$ are the identity mappings on \mathcal{C} and \mathcal{D} , respectively. \diamond

Remark 9. For $a \in \mathcal{D}$, mapping $L_a : \mathcal{D} \rightarrow \mathcal{D}$, $x \mapsto a \otimes x$, is residuated; its residual is denoted by $L_a^\sharp(y) = a \wp y$. \diamond

Theorem 10. (Blyth and Janowitz, 1972). Mapping f as in Def. 7 is dually residuated if and only if $f(\top) = \top$ and $(\forall \mathcal{A} \subseteq \mathcal{D}) f(\bigwedge_{x \in \mathcal{A}} x) = \bigwedge_{x \in \mathcal{A}} f(x)$. \diamond

2.4 TEG models in idempotent semirings

Timed event graphs (TEGs) are timed Petri nets in which each place has exactly one upstream and one downstream transition and all arcs have weight 1. With each place p is associated a *holding time* (can be equal to 0), representing the minimum time every token needs to spend in p before it can contribute to the firing of its downstream transitions. In a TEG, we can distinguish input transitions (those that are not affected by the firing of other transitions), output transitions (those that do not affect the firing of other transitions), and internal transitions (those that are neither input nor output transitions). Input transitions can be controllable or uncontrollable. In this paper, we will limit our discussion to SISO TEGs, i.e., TEGs with only one controllable input and one output transition, which we denote respectively by u and y ; internal transitions are denoted by x_i . Disturbances are modelled as uncontrollable input transition w_i , one for each internal transition. A disturbance is manifested when w_i fails to fire. An example of a SISO TEG is shown in Fig 1. We interpret the place with holding time 3 between x_1 and x_2 , initially empty, as the operation of the system, and the bottom place with holding time 2 between x_2 and x_1 , with two initial tokens, as a double-capacity resource. Under this interpretation, the firings of transitions x_1 and x_2 represent resource-allocation and resource-release events, respectively.

A TEG is said to be operating under the *earliest firing rule* if every transition fires as soon as it is enabled.

With each transition x_i , we associate a sequence $\{x_i(t)\}_{t \in \mathbb{Z}}$, for simplicity denoted by the same symbol, where $x_i(t)$ represents the accumulated number of firings of x_i up to and including time t . Similarly, we associate sequences $\{u(t)\}_{t \in \mathbb{Z}}$, $\{w_i(t)\}_{t \in \mathbb{Z}}$ and $\{y(t)\}_{t \in \mathbb{Z}}$ with transitions u , w and y , respectively. In \mathbb{Z}_{\min} , the number of firings of transition x_1 of the TEG from Fig. 1 follows, under the earliest firing rule,

$$(\forall t \in \mathbb{Z}) x_1(t) = u(t) \oplus 2x_2(t-2) \oplus w_1(t),$$

which, through the δ -transform, can be expressed in Σ as

$$x_1 = u \oplus 2\delta^2 x_2 \oplus w_1.$$

We can obtain similar relations for x_2 and y and, defining vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, write

$$\begin{aligned} x &= \begin{bmatrix} s_\varepsilon & 2\delta^2 \\ e\delta^3 & s_\varepsilon \end{bmatrix} x \oplus \begin{bmatrix} e\delta^0 \\ s_\varepsilon \end{bmatrix} u \oplus \begin{bmatrix} e\delta^0 & s_\varepsilon \\ s_\varepsilon & e\delta^0 \end{bmatrix} w, \\ y &= [s_\varepsilon \ e\delta^0] x. \end{aligned}$$

In general, a TEG can be described by implicit equations over Σ of the form

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw, \\ y &= Cx, \end{aligned} \quad (1)$$

with R being the identity matrix, meaning there is a disturbance for each internal transition. From Remark 4, the least solution of (1) is given by

$$y = CA^*Bu \oplus CA^*Rw, \quad (2)$$

where $G = CA^*B$ is referred to as the *transfer function*. For the system in Fig. 1, the (scalar) transfer function is $G = e\delta^3(2\delta^5)^*$ and $CA^*R = [e\delta^3(2\delta^5)^* \ (2\delta^5)^*]$.

Remark 11. Assuming that for every time instant t at which no disturbance occurs on transition x_i we have $w(t) = A^*Bu$ and hence $x(t) = A^*Bu$. It follows, for a disturbance-free system, $y = CA^*Bu = CA^*Rw$. A disturbance leads to $w(t) \succeq A^*Bu$, hence $x(t) = A^*Bu \oplus A^*Rw$ and thus to $y = CA^*Bu \oplus CA^*Rw \succeq CA^*Bu$. Clearly the disturbance effect will be observable only if $y \succ CA^*Bu$. \diamond

2.5 Optimal control of disturbance-free TEGs

Assume that a TEG to be controlled is modeled by equations (1), without the disturbance term Rw , and that an output-reference $z \in \Sigma$ is given. Under the just-in-time paradigm, we aim at firing the input transition u the least possible number of times while guaranteeing that the output transition y fires, by each time instant, at least as many times as specified by z . In other words, we seek the greatest u (in the order of Σ) such that $y = G \otimes u \preceq z$. Based on (2), without the disturbance term CA^*Rw , and Remark 9, the solution is directly obtained by

$$u_{\text{opt}} = G \wp z. \quad (3)$$

2.6 Modeling and optimal control of TEGs with resource sharing

Consider the resource-sharing system with disturbance-free TEGs S^1, \dots, S^K in Fig. 2. H^k represents the internal dynamics of S^k . β may, in general, be a TEG or just a single place describing the capacity of the resource. Let us assume that input transitions (u^k) are connected to resource-allocation transitions (x_A^k) via a single place with zero delay and no initial tokens, the same being true for the connection between resource-release transitions (x_R^k) and output transitions (y^k). Such system cannot be modelled by linear equations such as (1) (without Rw). Therefore, the Hadamard product of series is introduced (Hardouin et al., 2008).

Definition 12. The Hadamard product of $s_1, s_2 \in \Sigma$, written $s_1 \odot s_2$, is the counter defined as follows:

$$(\forall t \in \overline{\mathbb{Z}}) (s_1 \odot s_2)(t) = s_1(t) \otimes s_2(t).$$

This operation is commutative, distributes over \oplus and \wedge , has neutral element $e_\odot = e\delta^{+\infty}$, and $s_\varepsilon = +\infty\delta^{-\infty}$ is absorbing for it (i.e., $(\forall s \in \Sigma) s \odot s_\varepsilon = s_\varepsilon$). \diamond

The relationship among counters x_A^k and x_R^k , $k \in 1, \dots, K$ for the system in Fig. 2 can be expressed by

$$\beta \otimes \left(\bigodot_{k=1}^K x_R^k \right) \preceq \bigodot_{k=1}^K x_A^k. \quad (4)$$

which states that the totally combined firings of the allocation transitions at any given time cannot exceed those of the release transitions together with the capacity at the same time.

For a system like the one from Fig. 2, assuming output-references z^k to be given for each S^k , it is in general impossible for each subsystem to follow a just-in-time schedule with respect to their output-references since the resource is limited. We adopt a priority policy such that S^k has higher priority than S^{k+1} , for all $k \in \{1, \dots, K-1\}$. It is based on a simple rule: for each $k \in \{2, \dots, K\}$ and for all $j \in \{1, \dots, k-1\}$, S^k cannot interfere with the performance of S^j . For highest-priority subsystem S^1 we simply compute its optimal input by $u_{\text{opt}}^1 = x_{A_{\text{opt}}}^1 = G^1 \mathbb{q} z^1$ (cf. Section 2.5). For S^2 , in order to find the optimal input we must take into account the optimal schedule $x_{A_{\text{opt}}}^1$. This is achieved, based on (6), by finding the greatest fixed point of the mapping $\Phi^2 : \Sigma \rightarrow \Sigma$,

$$\Phi^2(x_A^2) = G^2 \mathbb{q} \left[\left(\beta \mathbb{q} (x_{A_{\text{opt}}}^1 \odot x_A^2) \right) \odot^\# x_{R_{\text{opt}}}^1 \right] \wedge G^2 \mathbb{q} z^2 \wedge x_A^2. \quad (5)$$

More generally, for S^k the optimal $x_{A_{\text{opt}}}^k$ is given by the greatest fixed point of $\Phi^k : \Sigma \rightarrow \Sigma$,

$$\Phi^k(x_A^k) = G^k \mathbb{q} \left[\left(\beta \mathbb{q} \left(\bigodot_{i=1}^{k-1} x_{A_{\text{opt}}}^i \odot x_A^k \right) \right) \odot^\# \left(\bigodot_{i=1}^{k-1} x_{R_{\text{opt}}}^i \right) \right] \wedge G^k \mathbb{q} z^k \wedge x_A^k.$$

Note that the Hadamard product is (dually) residuated.

Proposition 13. (Hardouin et al., 2008). For any $a \in \Sigma$, the mapping $\Pi_a : \Sigma \rightarrow \Sigma$, $x \mapsto a \odot x$, is residuated. For any $b \in \Sigma$, $\Pi_a^\#(b)$, denoted $b \odot^\# a$, is the greatest $x \in \Sigma$ such that $a \odot x \preceq b$. \diamond

Proposition 14. (Schafaschek et al., 2020). Let $\tilde{\Sigma} = \{s \in \Sigma \mid (\forall t \in \overline{\mathbb{Z}}) s(t) \notin \{\varepsilon, \top\}\}$. For any $a \in \tilde{\Sigma}$, the mapping $\Pi_a : \Sigma \rightarrow \Sigma$, $x \mapsto a \odot x$, is dually residuated. For any $b \in \Sigma$, $\Pi_a^b(b)$, denoted $b \odot^b a$, is the least $x \in \Sigma$ such that $a \odot x \succeq b$. \diamond

Example 15. Consider the system from Fig. 3, where subsystems S^1 and S^2 share a resource with capacity 2. The transfer function for S^1 is $G^1 = e\delta^3(2\delta^5)^*$ and, for S^2 , we obtain $G^2 = e\delta^5(2\delta^7)^*$. In this example, $\beta = 2\delta^2$. The references $z^1 = e\delta^{42} \oplus 1\delta^{46} \oplus 3\delta^{55} \oplus 6\delta^{+\infty}$ and $z^2 = e\delta^{39} \oplus 1\delta^{50} \oplus 2\delta^{54} \oplus 3\delta^{+\infty}$ are given. As S^1 has the highest priority, we can simply compute $u_{\text{opt}}^1 = x_{A_{\text{opt}}}^1 = G^1 \mathbb{q} z^1 = e\delta^{38} \oplus 1\delta^{42} \oplus 2\delta^{43} \oplus 3\delta^{47} \oplus 4\delta^{52} \oplus 6\delta^{+\infty}$ and $y_{\text{opt}}^1 = G^1 \otimes u_{\text{opt}}^1 = e\delta^{41} \oplus 1\delta^{45} \oplus 2\delta^{46} \oplus 3\delta^{50} \oplus 4\delta^{55} \oplus 6\delta^{+\infty}$ by applying (3). The

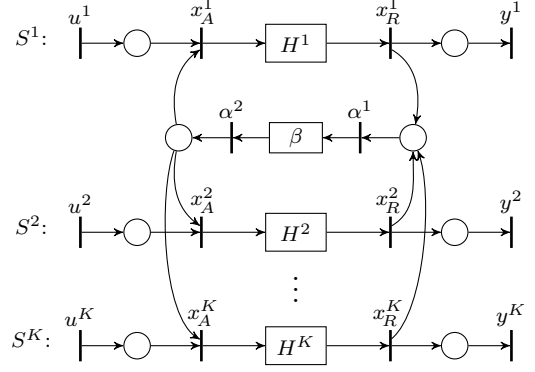


Fig. 2. A number of TEGs with a single shared resource.

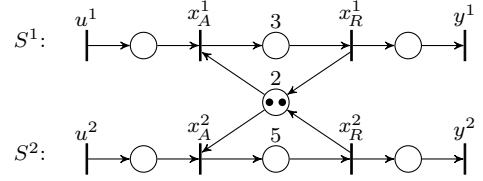


Fig. 3. Two TEGs sharing a resource with capacity 2.

resulting schedule is shown in Fig. 4, presented solely by the grey bars and its corresponding delays by the resource shown as dashed bars after the first grey bars onwards. Next, we determine $x_{A_{\text{opt}}}^2$. The greatest fixed point of (5) yields $x_{A_{\text{opt}}}^2 = e\delta^{28} \oplus 1\delta^{31} \oplus 2\delta^{35} \oplus 3\delta^{+\infty} (= u_{\text{opt}}^2)$ and $x_{R_{\text{opt}}}^2 = e\delta^{33} \oplus 1\delta^{36} \oplus 2\delta^{40} \oplus 3\delta^{+\infty} (= y_{\text{opt}}^2)$. These optimal schedules are shown in Fig. 4. Because the availability of the resource for S^2 is subject to the operation of S^1 , the firings of y^2 have to be considerably earlier than required by z^2 ; this is, however, the latest they can be so as to respect z^2 without interfering with S^1 . \diamond

3. OPTIMAL CONTROL OF TEGS WITH DISTURBANCES

In this section, we consider a scenario in which disturbances may occur during the operation of a TEG. We aim at updating the input so that its just-in-time property is maintained and also guaranteeing that the output is as close as possible to the reference. The disturbance is assumed to be detected as soon as the actual output deviates from the expected optimal output, i.e., $y \neq y_{\text{opt}}$.

Consider a TEG with transfer function G . In Section 2.5, the optimal input schedule based on an output-reference z is computed. The operation $G \mathbb{q} z = u_{\text{opt}}$ leads to the input transition firing the latest times possible so as to respect z . Therefore, $x_{\text{opt}} = A^* B u_{\text{opt}}$ is the greatest state vector and $y_{\text{opt}} = C A^* B u_{\text{opt}}$ is the greatest output which ensures $z \succeq y_{\text{opt}}$. According to Remark 11 a disturbance leads to $y = C A^* B u_{\text{opt}} \oplus C A^* R w \succeq C A^* B u_{\text{opt}} = y_{\text{opt}}$. Hence, a disturbance will be detected at the output as soon as $y \succ y_{\text{opt}}$. Furthermore, the requirement $z \succeq y$ is no more respected in this case, i.e., $y = C A^* B u_{\text{opt}} \oplus C A^* R w \not\preceq z$, since y_{opt} was the greatest series achieving this constraint. If a disturbance is detected at time T , our objective is to compute a new input schedule u'_{opt} , for which it must be guaranteed that the past up to and including time T is preserved. Since, as argued above, reference z can no

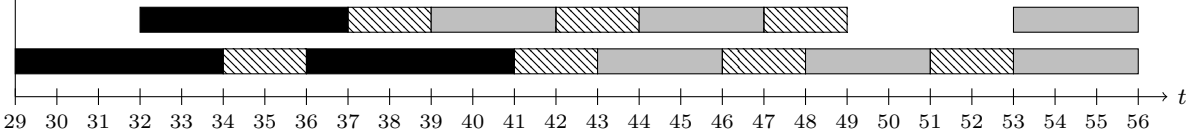


Fig. 4. Optimal schedules obtained in Example 15; the gray and black, bars represent the operation of S^1 and S^2 , whereas the dashed bars are the delays imposed by the resource.

longer be met, in order to compute this updated input a new reference z' is required that is greater than the previous reference but as close to it as possible, i.e., we look for the least reference $z' \succeq z$ that can be tracked after the disturbance is detected.

In output-feedback we have no information about the time or the affected transition of a disturbance. However, what is most relevant to us is the effect this disturbance has in the future behavior of the system, as this is what will allow us to determine the new reference (and hence the new optimal input) from that point on. In other words, we do not have access to w but we need it to estimate y and thus compute z' . Looking at the system in Fig. 1, we know that the counter x_2 is equal to the counter y . We also know that counters w_2 and x_2 are equal for the disturbance-free case (cf. Remark 11). Let a disturbance be detected at time T , i.e., $y(T) \succ y_{\text{opt}}(T)$ and $y(t) = y_{\text{opt}}(t)$ for $t < T$. Such a disturbance can be emulated in the resource-release transition as long as it is directly connected to the output transition and the place between both does not have any holding time or initial tokens. To incorporate the disturbance, we assume no new disturbances after time T and set $w_2 = r_T(y) \oplus y_{\text{opt}}$, meaning the emulated disturbance on the resource-release transition will be equal to the known delayed output up to time T , and remain equal to the optimal output from then on. As we cannot know where the disturbance occurred, for the sake of computing the future effect of the detected disturbance, we neglect w_1 and emulate the disturbance in w_2 . Therefore, we set $w_1 = x_1 = u_{\text{opt}}$. At this point, we artificially assembled w where all disturbances detected so far are represented by w_2 . The dynamic equation of the system, including the term CA^*Rw as in (2), will then manifest the effect of such disturbances in the future.

Define, inspired by Menguy et al. (2000), the mapping $r_T: \Sigma \rightarrow \Sigma$,

$$[r_T(u)](t) = \begin{cases} u(t), & \text{if } t \leq T; \\ \varepsilon, & \text{if } t > T. \end{cases} \quad (6)$$

The objective can then be restated as follows: find the greatest element u'_{opt} of the set

$$\mathcal{F}_{z'} = \{u \in \Sigma \mid CA^*Bu \oplus CA^*Rw \preceq z' \text{ and } r_T(u) = r_T(u_{\text{opt}})\},$$

where z' is the new reference still to be determined.

As a starting point, we want to determine under what conditions set $\mathcal{F}_{z'}$ is nonempty. In order to do so, let us consider the set

$$\tilde{\mathcal{F}} = \{u \in \Sigma \mid r_T(u) = r_T(u_{\text{opt}})\},$$

i.e., the set of counters that up to and including time T are identical to u_{opt} . Consider now

$$\underline{u} \stackrel{\text{def}}{=} \bigwedge_{u \in \tilde{\mathcal{F}}} u = r_T(u_{\text{opt}}).$$

Since $r_T \circ r_T = r_T$ and therefore $r_T(\underline{u}) = r_T(r_T(u_{\text{opt}})) = r_T(u_{\text{opt}})$, $\underline{u} \in \tilde{\mathcal{F}}$. Isotony of L_G , together with the fact that \oplus is order-preserving, thus implies

$$\mathcal{F}_{z'} \neq \emptyset \Leftrightarrow G\underline{u} \oplus CA^*Rw \preceq z'.$$

Proposition 16. The least $z' \succeq z$ such that $\mathcal{F}_{z'} \neq \emptyset$ is $z' = z \oplus G\underline{u} \oplus CA^*Rw$.

Proof. First, note that $z' = z \oplus G\underline{u} \oplus CA^*Rw$ clearly implies $\underline{u} \in \mathcal{F}_{z'}$, so $\mathcal{F}_{z'} \neq \emptyset$. Now, take an arbitrary $\tilde{z}' \succeq z$ such that $\mathcal{F}_{\tilde{z}'} \neq \emptyset$, and take $\tilde{u} \in \mathcal{F}_{\tilde{z}'}$. Then, $\tilde{u} \in \tilde{\mathcal{F}}$ and hence $\underline{u} \preceq \tilde{u}$, implying $z' = z \oplus G\underline{u} \oplus CA^*Rw \preceq z \oplus G\tilde{u} \oplus CA^*Rw \preceq z \oplus \tilde{z}' = \tilde{z}'$. \square

Now, based on the updated reference and as $CA^*Rw \preceq z'$ by our choice of z' , instead of considering set $\mathcal{F}_{z'}$ we can equivalently seek the greatest element of set

$$\mathcal{F} = \{u \in \Sigma \mid G \otimes u \preceq z' \text{ and } r_T(u) = r_T(u_{\text{opt}})\},$$

which is clearly nonempty since $\mathcal{F}_{z'} \subseteq \mathcal{F}$. The following theorem provides, given that certain conditions are met, a way to compute this greatest element.

Theorem 17. (Menguy et al., 2000). Let \mathcal{D} and \mathcal{C} be complete idempotent semirings, $f_1, f_2: \mathcal{D} \rightarrow \mathcal{C}$ residuated mappings, and $c_1, c_2 \in \mathcal{C}$. If the set

$$\mathcal{S} = \{x \in \mathcal{D} \mid f_1(x) \preceq c_1 \text{ and } f_2(x) = c_2\}$$

is nonempty, we have $\bigoplus_{x \in \mathcal{S}} x = f_1^\sharp(c_1) \wedge f_2^\sharp(c_2)$. \diamond

A correspondence between \mathcal{F} and \mathcal{S} can be established by taking \mathcal{D} and \mathcal{C} both as Σ , f_1 as L_G (which is well known to be residuated — see Remark 9), c_1 as z' , f_2 as r_T , and c_2 as $r_T(u_{\text{opt}})$.

Remark 18. Mapping r_T as defined in (6) is residuated, with

$$[r_T^\sharp(u)](t) = \begin{cases} u(t), & \text{if } t \leq T; \\ u(T), & \text{if } t > T. \end{cases}$$

In fact, r_T^\sharp is clearly isotone and we have $r_T \circ r_T^\sharp = r_T \preceq \text{Id}_\Sigma$ and $r_T^\sharp \circ r_T = r_T^\sharp \succeq \text{Id}_\Sigma$, so the conditions from Theorem 8 are fulfilled. \diamond

Hence, as long as set \mathcal{F} is nonempty, Theorem 17 provides the desired solution

$$u'_{\text{opt}} = G \diamond z' \wedge r_T^\sharp(u_{\text{opt}}). \quad (7)$$

Example 19. Consider the graph from Fig. 1. Suppose reference $z = e\delta^{42} \oplus 1\delta^{46} \oplus 3\delta^{54} \oplus 6\delta^{+\infty}$ is given. From (3) we get $u_{\text{opt}} = e\delta^{38} \oplus 1\delta^{41} \oplus 2\delta^{43} \oplus 3\delta^{46} \oplus 4\delta^{51} \oplus 6\delta^{+\infty}$ and $y_{\text{opt}} = e\delta^{41} \oplus 1\delta^{44} \oplus 2\delta^{46} \oplus 3\delta^{49} \oplus 4\delta^{54} \oplus 6\delta^{+\infty}$. Assume now that there is a disturbance at time $T = 39$ in transition x_1 , i.e., $w_1(39) = e$ prevents transition x_1 from firing. The disturbance is detected at $T = 42$. Without disturbances, we assume that the sequences w_1 and w_2 correspond to the

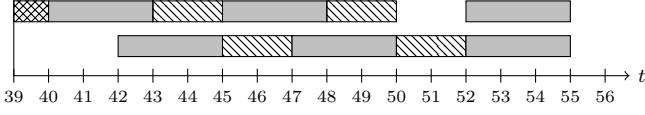


Fig. 5. Optimal schedule obtained in Example 19. To be interpreted as Fig. 4 with an additional cross-hatched bar representing the disturbance.

sequences x_1 and x_2 , respectively. With the disturbance detected at $T = 42$, however, we set $w_2 = r_T(y) \oplus y_{\text{opt}}$ as explained above. We obtain $r_T(y) = e\delta^{42} \oplus \varepsilon\delta^{+\infty}$ and assume $w_1 = x_1 = u_{\text{opt}}$. With $CA^*Rw = e\delta^{42} \oplus 1\delta^{44} \oplus 2\delta^{47} \oplus 3\delta^{49} \oplus 4\delta^{54} \oplus 6\delta^{+\infty}$, it can be easily checked that reference z is violated since $2\delta^{47} \succ 3\delta^{47}$. By applying Proposition 13, reference z is updated to $z' = e\delta^{42} \oplus 1\delta^{46} \oplus 2\delta^{47} \oplus 3\delta^{54} \oplus 6\delta^{+\infty}$. We finally obtain from (7)

$$u'_{\text{opt}} = e\delta^{38} \oplus 1\delta^{41} \oplus 2\delta^{44} \oplus 3\delta^{46} \oplus 4\delta^{51} \oplus 6\delta^{+\infty}.$$

The updated schedule is shown in Fig. 5. \diamond

4. OPTIMAL CONTROL OF TEGS WITH RESOURCE SHARING AND DISTURBANCES

In this section, as the main contribution of this paper, we aim at finding a solution on how to deal with disturbances in TEGs with resource sharing.

4.1 Problem formulation

Consider the system from Fig. 2 and assume every subsystem S^k is operating optimally with respect to its own output-reference z^k and the priority strategy (cf. Sec. 2.6). Furthermore, assume that each internal transition is additionally affected by an uncontrollable input transition. A disturbance in an arbitrary subsystem S^k will lead to a delay in its optimal schedule. It may follow that reference z^k is not achievable anymore but that is not necessarily the case, except for highest-priority subsystem S^1 which definitely operates optimally with respect to its reference (cf. Example 15). It is possible that a disturbance in a lower-priority subsystem does not interfere with its reference but leads to a delay that affects the schedule of another subsystem so that it cannot achieve its own reference anymore. Therefore, whenever a disturbance is detected in an arbitrary subsystem S^k , we have to check for all subsystems whether their references are still achievable and update their inputs if necessary. The same method from Sec. 3 of emulating the disturbance in the resource-release transition is employed. If a disturbance is detected in subsystem S^k at a certain time T , we determine the least trackable reference $z^{k'} \succeq z^k$ and, based thereon, we seek for system S^k the input $u^{k'}_{\text{opt}}$ which leads the corresponding output to optimally track the reference $z^{k'}$ while including the disturbance and preserving the input $u^{k'}_{\text{opt}}$ up to time T . The crucial difference compared to the case for a single system is that now the priority scheme must be observed and, furthermore, the past resource allocations by subsystems with lower priority must also be respected. Such allocations are relevant – despite having occurred before time T – because the respective resource releases may take place after T , thus influencing the availability of the resource in the meantime.

For the purpose of the discussion to follow, let us fix an arbitrary $k \in \{1, \dots, K\}$. When updating the input of S^k , we require minimal interference from lower-priority subsystems (i.e. all S^j with $j \in \{k+1, \dots, K\}$). This means that we have to respect past resource allocations in these subsystems but may ignore future ones. Recall that $u^j_{\text{opt}}(t)$ is the accumulated number of firings originally scheduled for u^j up to time t . Respecting the past means that the firings which have already occurred by time T (when disturbances are detected and consequently the new reference is computed) cannot be revoked. On the other hand, the prospective firings that have not taken place by time T can still be postponed and hence, from the point of view of S^k , ignored. In other words, for the sake of determining $u^{k'}_{\text{opt}}$ with minimal interference from S^j , we preserve the terms $u^j(t) = u^j_{\text{opt}}(t)$ for $t \leq T$ and neglect all new firings by making $u^j(t) = u^j_{\text{opt}}(T)$ for $t > T$. Recalling Remark 18, this is precisely captured by the counter $r^{\#}_T(u^j_{\text{opt}})$.

In sum, whenever a disturbance is detected in any of the subsystems, (i) we must compute $u^{k'}_{\text{opt}}$ in decreasing order of priority, i.e., start from $k = 1$ and proceed up to $k = K$; (ii) when calculating $u^{k'}_{\text{opt}}$ for $k > 1$, we must consider $u^{i'}_{\text{opt}}$ for every $i \in \{1, \dots, k-1\}$; (iii) when calculating $u^{k'}_{\text{opt}}$ for $k < K$, we must consider $r^{\#}_T(u^j_{\text{opt}})$ for every $j \in \{k+1, \dots, K\}$.

It will be convenient to define the following terms:

$$\begin{aligned} \mathcal{H}_A^k &= \bigodot_{i=1}^{k-1} (u^{i'}_{\text{opt}}), \\ \mathcal{H}_{R_w}^k &= \bigodot_{i=1}^{k-1} (G^i u^{i'}_{\text{opt}} \oplus (CA^*R)^i w^i), \\ \mathcal{L}_A^k &= \bigodot_{j=k+1}^K (r^{\#}_T(u^j_{\text{opt}})), \\ \mathcal{L}_{R_w}^k &= \bigodot_{j=k+1}^K (G^j r^{\#}_T(u^j_{\text{opt}}) \oplus (CA^*R)^j w^j). \end{aligned}$$

\mathcal{H}_A^k combines the counters $u^{i'}_{\text{opt}} = x^{i'}$ of all subsystems S^i with priority higher than that of S^k , referring to the already updated optimal schedules of input transitions u^i with respect to the corresponding updated references $z^{i'}$; accordingly, $\mathcal{H}_{R_w}^k$ combines the counters $y^{i'} = G^i u^{i'}_{\text{opt}} \oplus (CA^*R)^i w^i (= x^{i'}_R)$ representing the respective output events synchronized with disturbances caused by all uncontrollable input transitions w^i . Note that the terms involving w_i will only be relevant if a disturbance has been detected in S^i (and hence, according to the method introduced in Sec. 3, is emulated in w^i_2). In a similar way, \mathcal{L}_A^k combines the counters $r^{\#}_T(u^j_{\text{opt}}) (= r^{\#}_T(x^j_A))$ of all subsystems S^j with priority lower than that of S^k , representing the past firings (up to time T) of the inputs and neglecting their firings after time T , whereas $\mathcal{L}_{R_w}^k$ gathers the respective output (resource-release) events by combining the counters $G^j r^{\#}_T(u^j_{\text{opt}}) \oplus (CA^*R)^j w^j$. It should be emphasized that, even though we only consider the resource allocations by S^j up to time T , the respective

resource-release events may take place after T , so in general one may have $G^j r_T^\#(u_{\text{opt}}^j) \oplus (CA^*R)^j w^j \neq r_T^\#(x_{R_{\text{opt}}}^j) \oplus (CA^*R)^j w^j$.

Thus, based on (4) and on the foregoing discussion, in order to update u^k while ensuring minimal interference of lower-priority subsystems and also taking into account all past resource allocations as well as disturbances, we must respect

$$\beta \otimes \left(\mathcal{H}_{R_w}^k \odot (G^k u^k \oplus (CA^*R)^k w^k) \odot \mathcal{L}_{R_w}^k \right) \preceq \mathcal{H}_A^k \odot u^k \odot \mathcal{L}_A^k, \quad (8)$$

where it is understood that for $k=1$ (resp. $k=K$), the degenerate terms \mathcal{H}_A^1 and $\mathcal{H}_{R_w}^1$ (resp. \mathcal{L}_A^K and $\mathcal{L}_{R_w}^K$) are to be neglected.

The problem of determining the new optimal input $u_{\text{opt}}^{k'}$ upon the detection of a disturbance at time T can be formulated in two steps as follows:

(I) find the least $z^{k'}$ such that $z^{k'} \succeq z^k$ and that the set

$$\mathcal{F}_{z^{k'}}^k = \{u^k \in \Sigma \mid (CA^*B)^k u^k \oplus (CA^*R)^k w^k \preceq z^{k'} \text{ and (8) and } r_T(u^k) = r_T(u_{\text{opt}}^k)\}$$

is nonempty;

(II) find the greatest element of the set

$$\mathcal{F}^k = \{u^k \in \Sigma \mid G^k \otimes u^k \preceq z^{k'} \text{ and (8) and } r_T(u^k) = r_T(u_{\text{opt}}^k)\}.$$

Recall that, as argued in Sec. 3, since from (I) it follows that $(CA^*R)^k w^k \preceq z^{k'}$, in (II) we can obtain the optimal input schedule by computing the greatest element of set \mathcal{F}^k , which is the same as that of $\mathcal{F}_{z^{k'}}^k$. Note also that, as mentioned in Sec. 3, in output-feedback we do not have direct access to w^k . Therefore, we incorporate all disturbances in w_2^k since x_2^k is directly connected to the output y^k , i.e., we set $w_2^k = r_T(y^k) \oplus y_{\text{opt}}^k$. Moreover, it is assumed that $w_1^k = u_{\text{opt}}^k$. A disturbance detected in S^k at time T may cause delays in higher-priority subsystems which cannot be immediately compensated through the method we present. These delays will eventually be manifested in the output of the corresponding higher-priority subsystems and will then be dealt with as though they were disturbances in those subsystems.

4.2 Optimal control of TEGs with resource sharing and disturbances

As in Sec. 3, we start by investigating when $\mathcal{F}_{z^{k'}}^k$ is nonempty. To that end, we consider the set

$$\tilde{\mathcal{F}}^k = \{u \in \Sigma \mid (8) \text{ and } r_T(u^k) = r_T(u_{\text{opt}}^k)\}.$$

Proposition 20. (Schafaschek et al., 2020).¹ The least element of $\tilde{\mathcal{F}}^k$ is the least fixed point of mapping $\Upsilon^k : \Sigma \rightarrow \Sigma$,

$$\Upsilon^k(u) = [\beta \otimes (\mathcal{H}_{R_w}^k \odot (G^k u \oplus (CA^*R)^k w^k) \odot \mathcal{L}_{R_w}^k) \odot^\flat (\mathcal{H}_{A_w}^k \odot \mathcal{L}_{A_w}^k)] \oplus r_T(u_{\text{opt}}^k) \oplus u.$$

¹ Although the result in the referred work (Section 5.2) does not involve disturbance-related terms, these terms do not interfere with the argument, and the proof directly carries over to the case presented here.

In Schafaschek et al. (2020) it is shown that there exists a (unique) least counter in the version of set $\tilde{\mathcal{F}}^k$ for the disturbance-free case; these results can be readily applied to the present case, thus showing that there exists a least counter satisfying (8) and $r_T(u^k) = r_T(u_{\text{opt}}^k)$, denoted by

$$\underline{u}^k \stackrel{\text{def}}{=} \bigwedge_{u \in \tilde{\mathcal{F}}^k} u,$$

which can be computed as the least fixed point of mapping Υ^k . Note that, the mapping $\Pi_{(\mathcal{H}_{A_w}^k \odot \mathcal{L}_{A_w}^k)}$ is dually residuated, so Υ^k is well defined. \diamond

Isotony of L_{G^k} , together with the fact that \oplus is order-preserving, implies

$$\mathcal{F}_{z^{k'}}^k \neq \emptyset \Leftrightarrow G^k \underline{u}^k \oplus (CA^*R)^k w^k \preceq z^{k'}.$$

In order to find a solution for (I), we use the following proposition.

Proposition 21. The least counter $z^{k'} \succeq z^k$ such that $\mathcal{F}_{z^{k'}}^k \neq \emptyset$ is $z^{k'} = z^k \oplus (G^k \underline{u}^k \oplus (CA^*R)^k w^k)$.

Proof. Taking $z^{k'} = z^k \oplus (G^k \underline{u}^k \oplus (CA^*R)^k w^k)$, it can be readily checked that $\underline{u}^k \in \mathcal{F}_{z^{k'}}^k$, therefore, $\mathcal{F}_{z^{k'}}^k \neq \emptyset$; the proof then proceeds by direct analogy with that of Proposition 16. \square

We now proceed to solve (II), i.e., we look for the greatest element of set \mathcal{F}^k . The following proposition is a generalized version of Theorem 17.

Proposition 22. (Schafaschek et al. (2020)) Let \mathcal{D} and \mathcal{C} be complete idempotent semirings, $f_1, f_2 : \mathcal{D} \rightarrow \mathcal{C}$ residuated mappings, $\psi : \mathcal{D} \rightarrow \mathcal{C}$, and $c \in \mathcal{C}$. Consider the set

$$\mathcal{S}_\psi \stackrel{\text{def}}{=} \{x \in \mathcal{D} \mid f_1(x) \preceq \psi(x) \text{ and } f_2(x) = c\}$$

and the isotone mapping $\Omega : \mathcal{D} \rightarrow \mathcal{D}$,

$$\Omega(x) = x \wedge f_1^\#(\psi(x)) \wedge f_2^\#(c).$$

If $\mathcal{S}_\psi \neq \emptyset$, we have $\bigoplus_{x \in \mathcal{S}_\psi} x = \bigoplus\{x \in \mathcal{D} \mid \Omega(x) = x\}$. \diamond

We fix an arbitrary $k \in \{1, \dots, K\}$ and assume $u_{\text{opt}}^{i'}$ has been determined for each (if any) $i \in \{1, \dots, k-1\}$. Seeing that (8) is equivalent to

$$(G^k u^k \oplus (CA^*R)^k w^k) \preceq [\beta \mathfrak{q}(\mathcal{H}_{A_w}^k \odot (u^k \oplus w_1^k) \odot \mathcal{L}_{A_w}^k)] \odot^\# (\mathcal{H}_{R_w}^k \odot \mathcal{L}_{R_w}^k), \quad (9)$$

and as we know that $\underline{u}^k \in \mathcal{F}_{z^{k'}}^k$, it follows that $(CA^*R)^k w^k \preceq [\beta \mathfrak{q}(\mathcal{H}_{A_w}^k \odot (\underline{u}^k \oplus w_1^k) \odot \mathcal{L}_{A_w}^k)] \odot^\# (\mathcal{H}_{R_w}^k \odot \mathcal{L}_{R_w}^k)$. Moreover, any candidate \tilde{u}^k for a solution for (II) is an element of $\tilde{\mathcal{F}}^k$ and hence is greater than or equal to \underline{u}^k , so it also holds that $(CA^*R)^k w^k \preceq [\beta \mathfrak{q}(\mathcal{H}_{A_w}^k \odot (\tilde{u}^k \oplus w_1^k) \odot \mathcal{L}_{A_w}^k)] \odot^\# (\mathcal{H}_{R_w}^k \odot \mathcal{L}_{R_w}^k)$. Therefore, instead of using (9), we can equivalently look for the greatest u^k satisfying $G^k u^k \preceq [\beta \mathfrak{q}(\mathcal{H}_{A_w}^k \odot (u^k \oplus w_1^k) \odot \mathcal{L}_{A_w}^k)] \odot^\# (\mathcal{H}_{R_w}^k \odot \mathcal{L}_{R_w}^k)$.

By defining the mapping $\Psi^k : \Sigma \rightarrow \Sigma$,

$$\Psi^k(u) = z^{k'} \wedge [(\beta \mathfrak{q}(\mathcal{H}_{A_w}^k \odot (u^k \oplus w_1^k) \odot \mathcal{L}_{A_w}^k)) \odot^\# (\mathcal{H}_{R_w}^k \odot \mathcal{L}_{R_w}^k)]$$

we can then restate set \mathcal{F}^k as

$$\mathcal{F}^k = \{u^k \in \Sigma \mid G^k u^k \preceq \Psi^k(u) \text{ and } r_T(u^k) = r_T(u_{\text{opt}}^k)\}.$$

This reveals a correspondence between set \mathcal{F}^k and set \mathcal{S}_ψ from Proposition 22: take \mathcal{D} and \mathcal{C} both as Σ , f_1 as L_{G^k} , ψ

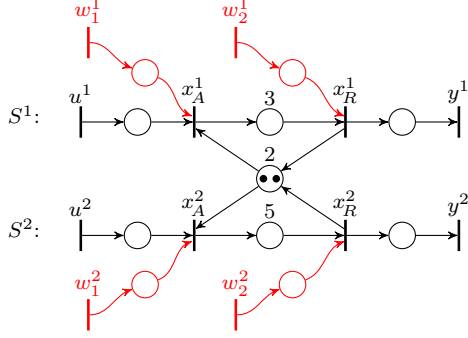


Fig. 6. Two subsystems with disturbances sharing a double capacity resource.

as Ψ^k , f_2 as r_T , and c as $r_T(u_{\text{opt}}^k)$. As long as $\mathcal{F}^k \neq \emptyset$, the conditions from the proposition hold and, recalling that $r_T^\# \circ r_T = r_T^\#$, the optimal update of u^k is the greatest fixed point of the (isotone) mapping $\Gamma^k : \Sigma \rightarrow \Sigma$,

$$\Gamma^k(u) = u \wedge G^k \circ \Psi^k(u) \wedge r_T^\#(u_{\text{opt}}). \quad (10)$$

In summary, we apply Proposition 20 to find the least element in $\tilde{\mathcal{F}}^k$, then proceed to compute the least counter $z^{k'} \succeq z^k$ such that $\mathcal{F}_{z^{k'}} \neq \emptyset$ as in Proposition 21 and, finally, obtain the greatest element in \mathcal{F}^k by calculating the greatest fixed point of mapping Γ^k as in (10).

Example 23. Consider the system from Exp. 15 with output references $z^1 = e\delta^{42} \oplus 1\delta^{46} \oplus 3\delta^{55} \oplus 6\delta^{+\infty}$ and $z^2 = e\delta^{39} \oplus 1\delta^{50} \oplus 2\delta^{54} \oplus 3\delta^{+\infty}$, a shared resource with capacity 2 and, additionally, uncontrollable input transitions for each internal transition as depicted in Fig. 6. Now, there is a disturbance in subsystem S^2 at transition x_A^2 at time 36 but it is detected at $T = 41$. With $r_T(y^2) = e\delta^{33} \oplus 1\delta^{36} \oplus 2\delta^{41}\varepsilon\delta^{+\infty}$ and $w_1^2 = x_A^2 = u_{\text{opt}}^2$ we have $(CA^*R)^2w^2 = e\delta^{33} \oplus 1\delta^{36} \oplus 2\delta^{41} \oplus 3\delta^{+\infty}$. According to the procedure stated in Sec. 4.1, we start with updating u_{opt}^1 first. For that end, the new reference $z^{1'}$ must be determined. Note that, even though the disturbance occurred in S^2 , its effect on S^1 is manifested through the terms \mathcal{L}_A^1 and \mathcal{L}_R^1 in (8). The least fixed point of mapping Υ^1 is $\underline{u}^1 = e\delta^{38} \oplus 1\delta^{43} \oplus 3\delta^{48} \oplus 4\delta^{52} \oplus 5\delta^{53} \oplus 6\delta^{+\infty}$ which, together with Proposition 21, leads to an updated output-reference $z^{1'} = e\delta^{42} \oplus 1\delta^{46} \oplus 2\delta^{47} \oplus 3\delta^{55} \oplus 5\delta^{56} \oplus 6\delta^{+\infty}$. The greatest fixed point of Γ^1 results in

$$u_{\text{opt}}^{1'} = e\delta^{38} \oplus 1\delta^{43} \oplus 3\delta^{48} \oplus 4\delta^{52} \oplus 5\delta^{53} \oplus 6\delta^{+\infty}$$

which is also the greatest element of set \mathcal{F}^1 and thus the desired solution for highest-priority subsystem S^1 . Next, the steps will be repeated for subsystem S^2 while respecting the priority policy, i.e., the updated schedule for S^1 . Since $G^2 \underline{u}^2 \oplus (CA^*R)^2w^2 \preceq z^2$, set \mathcal{F}^2 is not empty and, therefore, $z^{2'} = z^2$ and $\underline{u}^2 = u_{\text{opt}}^2$. The greatest fixed point of mapping Γ^2 results in

$$u_{\text{opt}}^{2'} = e\delta^{28} \oplus 1\delta^{31} \oplus 2\delta^{35} \oplus 3\delta^{+\infty}$$

which is also the greatest element of set \mathcal{F}^2 and thus the desired result. The updated schedule is shown in Fig 7. \diamond

5. CONCLUSION

This paper provides a method for dealing with disturbances in timed event graphs that is inspired by optimal

control of TEGs with output-reference update. It starts with a control strategy for a single TEG and is further extended to the case where multiple TEGs share the same resource. During the process, disturbances lead to a delay in the optimal schedule. A strategy is proposed that updates the reference signal and, based on it, the optimal input, so that the disturbances are included in the schedule, the past is preserved, and the output is just-in-time for the new reference; for a single TEG and a resource-sharing system. For the results, the output feedback case is assumed. Examples for a single TEG and for resource sharing are provided to show that the concept works. The class of applications could be further extended by, e.g., applying the methods to systems where multiple resources are shared.

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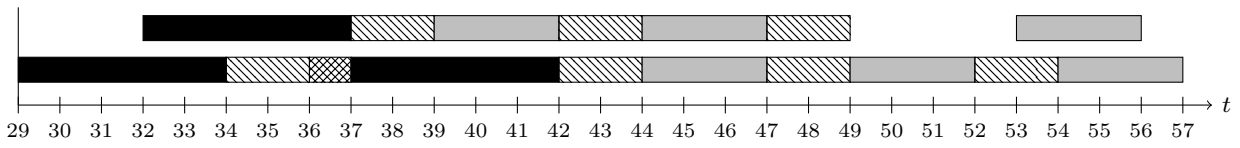


Fig. 7. Optimal schedules obtained in Example 23; to be interpreted as in Fig. 4 with an additional cross-hatched bar representing the disturbance.