Set-membership State Estimation of Max-Plus Linear Systems by using Tropical Polyhedra

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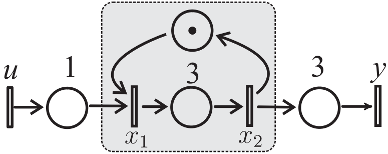
Problem Statement

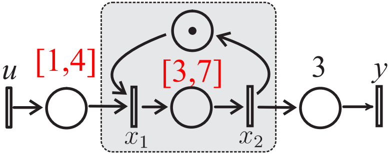


(max,+) Linear Systems and Timed Event Graphs

Timed Event Graphs behavior is perfectly described by (max,+) linear systems.
The signals considered are the firing dates of each transition. Internal transitions are denoted x_i (inputs transitions. u_i, outputs transitions y_i).

See : http//perso-laris.univ-angers.fr/~hardouin/GET_BO.html

Problem Statement



Uncertain (max,+) Linear Systems

Delays are assumed to be random values belonging to intervals.

Question ?

- Is it possible to compute a state estimation $\hat{x}$ by considering inputs u and available output measurements y ?

See :
http://perso-laris.univ-angers.fr/~hardouin/GET_BOInterval.html

Outline

- Idempotent semi-ring and (max,+) algebra
- Geometry and (max,+) algebra
- Model of Timed Event Graphs
- State Estimation : Observer synthesis
- State Estimation : Set-membership approach
- Complexity analysis
- Conclusion

Idempotent semi-ring and (max,+) algebra

Idempotent Semiring $\mathcal{T}$

- Sum $\oplus$, associative,commutative, zero element denoted ε ,
- Product $\otimes$, associative, identity element denoted e,
- Product $\otimes$ distributes with respect of sum,
  $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$,
- Zero element ε is absorbing, $a \otimes \varepsilon = \varepsilon$
- The sum is idempotent, $a \oplus a = a$.
- $a \oplus b = a \vee b = a \Leftrightarrow b \preceq a \Leftrightarrow a \wedge b = b$
  hence an idempotent semiring has a complete lattice structure, with (ε) as bottom element and ($T = \bigoplus_{x \in \mathcal{S}} x$) as top element.

Example :(max,+) algebra, $\overline{\mathbb{Z}}_{\max}$

More

Sum $\oplus$ is the operator *max*, product $\otimes$ is classical sum $+$, $\varepsilon = -\infty$ and $e = 0$, then :

$$1 \oplus 1 = 1 = max(1,1),$$
$$2 \otimes 1 = 3 = 2 + 1.$$

Idempotent semi-ring and (max,+) algebra

Fixed point equations

For order preserving (isotone) mapping, ($x \preceq y \Leftrightarrow f(x) \preceq f(y)$), it is possible to compute fixed points $f(x) = x$.

Application : $x = ax \oplus b$

Theorem : Over a complete idempotent semiring $\mathcal{T}$, the least solution to $x = ax \oplus b$ is $x = a^*b$ with $a^* = \bigoplus_{i \in \mathbb{N}_0} a^i = e \oplus a \oplus a^2 \oplus \ldots$

$*$ is called Kleene star operator.

Idempotent semi-ring and (max,+) algebra

Residuation Theory (Croisot 56, Blyth 05)

A pseudo inverse exists for order preserving mapping defined over ordered sets.

Inequality $a \otimes x \preceq b$

Over a complete idempotent semiring $\mathcal{T}$, inequality $a \otimes x \preceq b$ admits a greatest solution , denoted, $x = a \diamond\!\!\!\!\setminus b$,
($i.e.$ $a(a \diamond\!\!\!\!\setminus b) \preceq b$ and equality is achieved, if possible).

Example : (max,+) algebra $\overline{\mathbb{Z}}_{\max}$

Inequality $3 \otimes x \preceq 5$ admits a greatest solution $x = 3 \diamond\!\!\!\!\setminus 5 = 5 - 3 = 2$. It achieves equality in the scalar case.

Idempotent semi-ring and (max,+) algebra

Matrix

Let A, B, C three matrices in $\mathcal{T}^{n\times n}$

- $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$
- $(A \otimes B)_{ik} = \bigoplus_{j=1\ldots n} (A_{ij} \otimes B_{jk})$
- $(A \backslash\!\!\!\circ B)_{ik} = \bigwedge_{j=1\ldots n} (A_{ji} \backslash\!\!\!\circ B_{jk})$, where $A \backslash\!\!\!\circ B$ is the greatest matrix s.t. $AX \preceq B$
- $(B /\!\!\!\circ A)_{ik} = \bigwedge_{j=1\ldots n} (A_{ij} /\!\!\!\circ B_{kj})$, where $A /\!\!\!\circ B$ is the greatest such $XA \preceq B$
- $(X)_{ij} = A^*_{ij}$ is the greatest matrix s.t. $X \preceq A^*$

▸ More

Geometry in (max,+) algebra

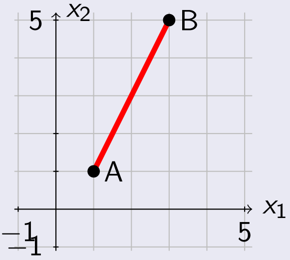
Segment in $\mathbb{R}^n$ vs $\overline{\mathbb{R}}^n_{max}$

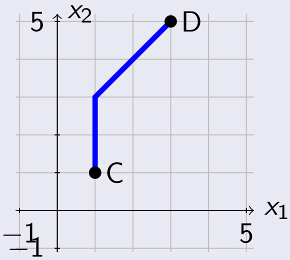
A segment $[\mathrm{A}, \mathrm{B}]$ in $\mathbb{R}^n$, is

$$\begin{aligned}[\mathrm{A},\mathrm{B}] &= \{\, t\,\mathrm{A} + (1-t)\,\mathrm{B} \mid t \in [0,1] \,\} \\ &= \{\, \lambda\,\mathrm{A} + \beta\,\mathrm{B} \mid \lambda + \beta = 1,\ \beta \geq 0, \lambda \geq 0 \,\}\end{aligned}$$

A segment $[\mathrm{C}, \mathrm{D}]$ in $\overline{\mathbb{R}}^n_{max}$ is

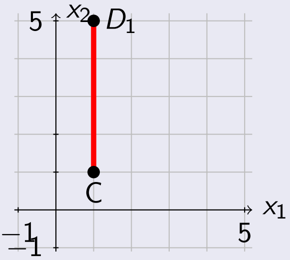
$$[\mathrm{C},\mathrm{D}] = \{\, \lambda\,\mathrm{C} \oplus \beta\,\mathrm{D} \mid \lambda \oplus \beta = e,\ \beta \geq \varepsilon, \lambda \geq \varepsilon \}$$

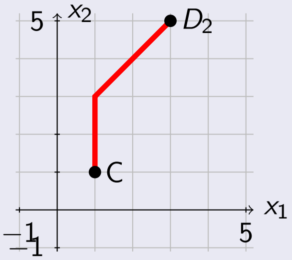




Segment in $\overline{\mathbb{R}}^2_{\max}$

A segment $[\mathrm{C}, \mathrm{D}_i]$ in $\overline{\mathbb{R}}^2_{\max}$ is $[\mathrm{C}, \mathrm{D}_i] = \{ \lambda\, \mathrm{C} \oplus \beta\ \mathrm{D}_i \mid \lambda \oplus \beta = e \}$, actually six types $i \in [1, 6]$





Geometry in (max,+) algebra

Segment in $\overline{\mathbb{R}}^2_{\max}$

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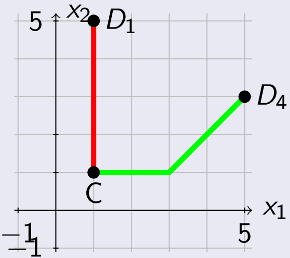

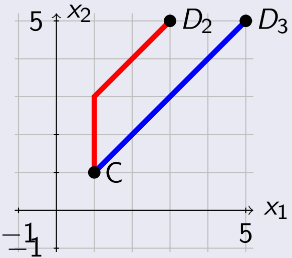

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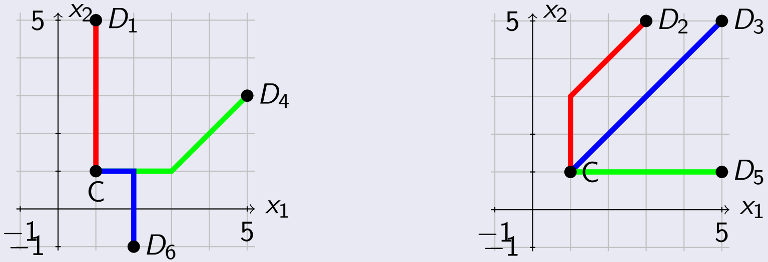
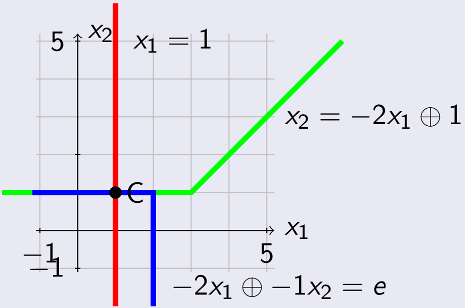
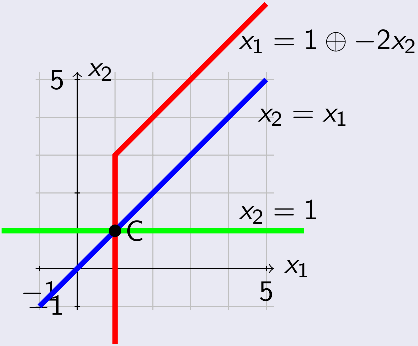


Line in $\overline{\mathbb{R}}^n_{max}$

A classical line is $a_1 x_1 + a_2 x_2 + b = c_1 x_1 + c_2 x_2 + d$. A (max,+) line is defined as : $a_1 x_1 \oplus a_2 x_2 \oplus b = c_1 x_1 \oplus c_2 x_2 \oplus d$. In $\overline{\mathbb{R}}^2_{max}$ this leads to 6 types of line.

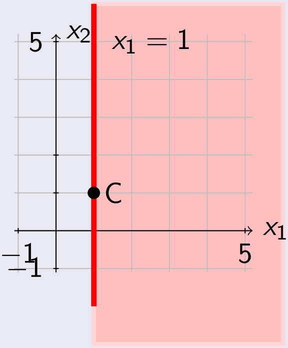


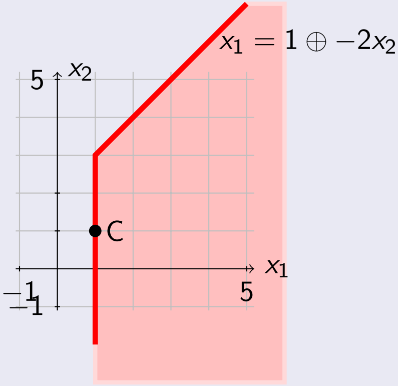


Geometry in (max,+) algebra

Half space $\overline{\mathbb{R}}^n_{\max}$

Each line separates the space in half-space.

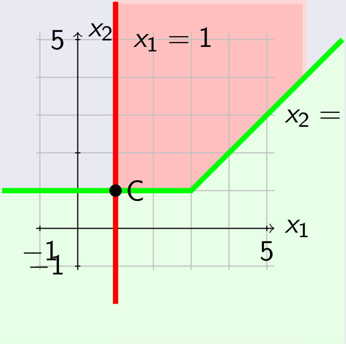


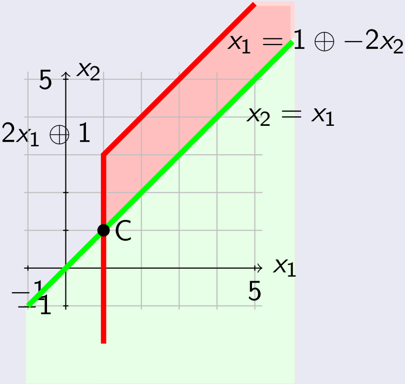


Geometry in (max,+) algebra

Half space in $\overline{\mathbb{R}}^n_{max}$

Each line separates the space in half-space.

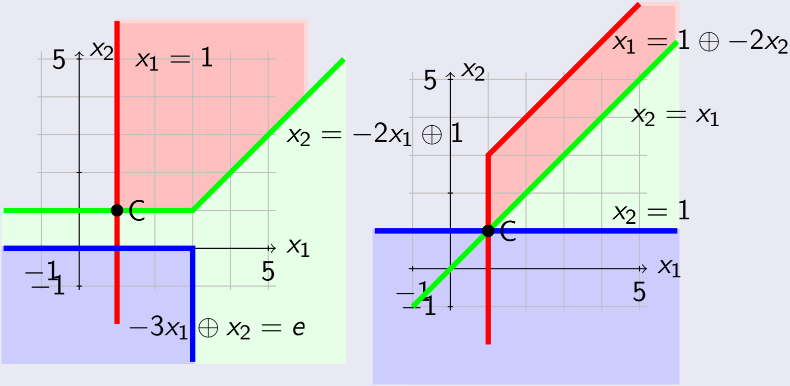




Geometry in (max,+) algebra

Half space in $\overline{\mathbb{R}}^n_{\max}$

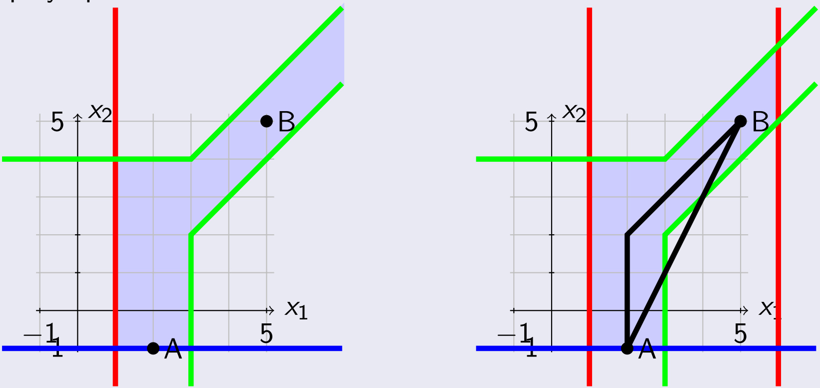
Each line separates the space in half-space.



Geometry in (max,+) algebra

Intersection of half space in $\overline{\mathbb{R}}^n_{\max}$

Intersection of half-spaces yields a max-plus polytope also called tropical polytope.



Geometry in (max,+) algebra

Tropical polytope in $\overline{\mathbb{R}}_{\max}$, external representation or $\mathcal{H}$-form

It admits an external representation representing all the constraints as the set $\mathcal{P} = \{x \in \overline{\mathbb{R}}^n_{\max} | Ax \oplus b \preceq Cx \oplus d\}$, with $A, C \in \overline{\mathbb{R}}^{q \times n}_{\max}$ and $b, d \in \overline{\mathbb{R}}^q_{\max}$.
Homogeneous representation $\mathcal{P} = \{z \in \overline{\mathbb{R}}^{n+1}_{\max} | Ez \preceq Fz\}$, where $E = (A\ b)$, $F = (C\ d)$, $z = (x^t, \alpha)^t$.

Tropical polytope in $\overline{\mathbb{R}}_{\max}$, internal representation or $\mathcal{V}$-form

A max-plus polytope admits also an internal representation
$\mathcal{P} = \{x | x = \bigoplus_{i=1}^{p} \lambda_i v^i\}$ where $V = \{v^1, ..., v^p\} \subset \overline{\mathbb{Z}}^n_{\max}$ is a set of p vertices, and $\lambda_i \in \overline{\mathbb{R}}_{\max}$.
Remark A minimal representation exists (i.e. p is minimal).

Geometry in (max,+) algebra

Transformation : $\mathcal{H}$-form to $\mathcal{V}$-form

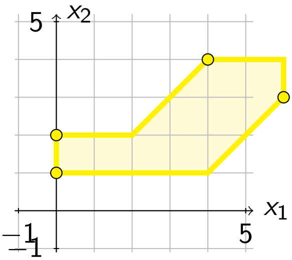
Algorithms exist (Butkovic 84, Gaubert & Allamigeon 08,13) to change from a $\mathcal{H}$-form to a $\mathcal{V}$-form, and vice-versa.
Complexity is exponential according to the size n of the state. Practically it is lower ...
Library TPLib is available :
http://www.cmap.polytechnique.fr/~allamigeon/software/

Compact tropical polytope in $\overline{\mathbb{R}}_{\max}$

By considering the polytope $\mathcal{P} = \{x | x = \bigoplus_{i=1}^{p} \lambda_i v^i\}$ with the constraint $\bigoplus_{i=1}^{p} \lambda_i = e$, the polytope is compact and then is called a compact tropical polytope denoted

$$co(V) = \{x | x = \bigoplus_{i=1}^{p} \lambda_i v^i, \ and \ \bigoplus_{i=1}^{p} \lambda_i = e\}$$

Geometry in (max,+) algebra



$\mathcal{H}$-form $\Leftrightarrow Ax \oplus b \preceq Cx \oplus d$

$$x_2 \preceq x1 \oplus 2, \quad -3x_1 \oplus 1 \preceq x_2,$$
$$e \preceq x_1, \quad -6x_1 \oplus -4x_2 \preceq e.$$

$$\Leftrightarrow$$

$$\begin{pmatrix} \varepsilon & 1 \\ -3 & \varepsilon \\ \varepsilon & \varepsilon \\ -6 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} \varepsilon \\ 1 \\ e \\ \varepsilon \end{pmatrix} \preceq \begin{pmatrix} e & \varepsilon \\ \varepsilon & e \\ \varepsilon & \varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ \varepsilon \\ \varepsilon \\ e \end{pmatrix}$$

$\mathcal{V}$-form

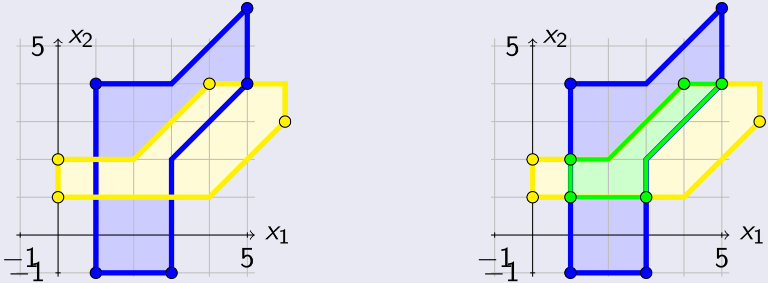
$\mathcal{P} = \{x | x = \bigoplus_{i=1}^{p} \lambda_i v^i\}$
with $\bigoplus_{i=1}^{p} \lambda_i = e$

$$V = \begin{pmatrix} e & e & 4 & 6 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

Geometry in (max,+) algebra

Intersection of tropical polytope

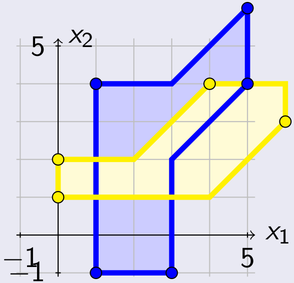
From the $\mathcal{H}$-form, it is directly obtained, the $\mathcal{V}$-form is still with an exponential complexity.

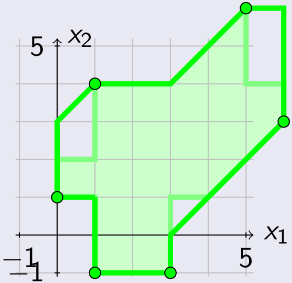


Geometry in (max,+) algebra

Union of tropical polytope is not a tropical polytope

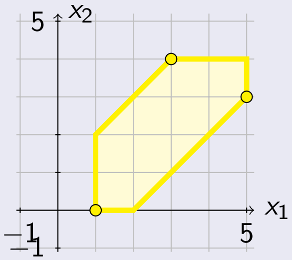
A smallest over approximation as a tropical polytope can be obtained with a polynomial complexity.





Geometry in (max,+) algebra

Difference Bound Matrix (DBM) is a tropical polytope



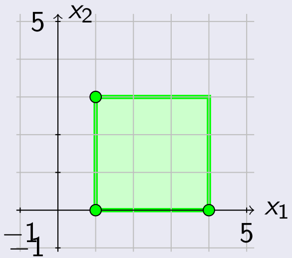
DBM (or zone) in canonical form

$$1 \leq x1 \leq 5$$
$$0 \leq x2 \leq 4$$
$$x_2 \leq x_1 + 1$$
$$x_2 \geq x1 - 2$$

$$\Leftrightarrow \begin{pmatrix} 0 & -1 & 0 \\ 5 & 0 & 2 \\ 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 5 & 0 & 2 \\ 4 & 1 & 0 \end{pmatrix}^*$$

A DBM in $\mathbb{R}^n_{max}$ corresponds to a tropical polytope with $p = n + 1$ generators ($p = 3$ yellow vertices).

Box is a DBM hence a tropical polytope



Box

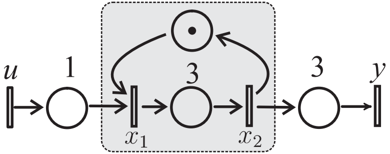
$$1 \leq x1 \leq 4$$
$$0 \leq x2 \leq 3$$

The corresponding polytope is with 3 generators (3 green vertices).

$$V = \begin{pmatrix} 1 & 1 & 4 \\ 3 & e & e \end{pmatrix}$$

A box in $\overline{\mathbb{R}}^n_{max}$ corresponds to a tropical polytope with $p = n + 1$ generators.

TEG Model in $\overline{\mathbb{Z}}_{\max}$



Firing Date [Cohen et al., IEEE TAC 85]

$x_i(k)$: date of the firing numbered k for the transition labelled x_i.

For each transition :

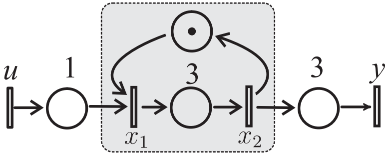
$$
\begin{aligned}
x_1(k) &= \max(1 + u(k), x_2(k-1)) \\
x_2(k) &= 3 + x_1(k) \\
y(k) &= 3 + x_2(k)
\end{aligned}
$$

In $\overline{\mathbb{Z}}_{\max}$

▸ $\overline{\mathbb{Z}}_{\max}$:

$$
\begin{aligned}
x_1(k) &= 1 \otimes u(k) \oplus x_2(k-1) \\
x_2(k) &= 3 \otimes x_1(k) \\
y(k) &= 3 \otimes x_2(k)
\end{aligned}
$$

TEG Model in $\overline{\mathbb{Z}}_{\max}$



Firing Date [Cohen et al., IEEE TAC 85]

$x_i(k)$: date of the firing numbered k for the transition labelled i.

Dynamic Model

$$\begin{aligned}
\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} &= \begin{pmatrix} \varepsilon & \varepsilon \\ 3 & \varepsilon \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & e \\ \varepsilon & \varepsilon \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} \oplus \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} u(k) \\
y(k) &= \begin{pmatrix} \varepsilon & 3 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
x(k) &= A_0 x(k) \oplus A_1 x(k-1) \oplus Bu(k) \\
y(k) &= Cx(k)
\end{aligned}$$

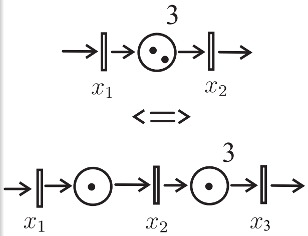
TEG Model in $\overline{\mathbb{Z}}_{max}$, Markovian Representation

State extension [Cohen et al., IEEE TAC 85]

$$x(k) = A_0 x(k) \oplus A_1 x(k-1) \oplus A_2 x(k-2) \oplus Bu(k)$$
$$y(k) = Cx(k)$$

By extending the state vector $\tilde{x} = [x^T \; x^T]^T$:

$$\begin{array}{rcl} \tilde{x}(k) & = & \tilde{A}_0 \tilde{x}(k) \oplus \tilde{A}_1 \tilde{x}(k-1) \oplus \tilde{B} u(k) \\ y(k) & = & \tilde{C} \tilde{x}(k) \end{array} .$$



TEG Model in $\overline{\mathbb{Z}}_{\max}$, Markovian Representation

Implicit to Markovian Representation

Theorem : Over a complete idempotent semiring $\mathcal{T}$, the least solution to $x = ax \oplus b$ is $x = a^*b$ with $a^* = \bigoplus_{i \in \mathbb{N}_0} a^i = e \oplus a \oplus a^2 \oplus \ldots$

$*$ is called Kleene star operator.

Application

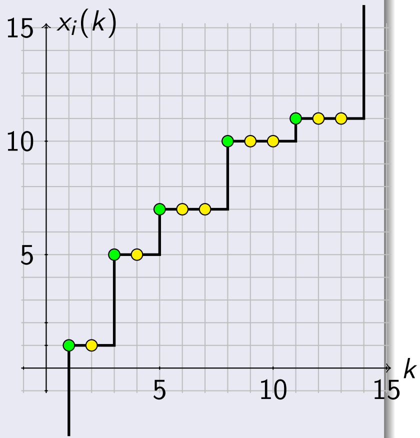
$$\begin{aligned} \tilde{x}(k) &= \tilde{A}_0\tilde{x}(k) \oplus \tilde{A}_1\tilde{x}(k-1) \oplus \tilde{B}u(k) \\ y(k) &= \tilde{C}\tilde{x}(k) \end{aligned}$$

by considering $A = \tilde{A}_0^*\tilde{A}_1$ and $B = \tilde{A}_0^*\tilde{B}$, a Markovian standard form is obtained :

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

Signal in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

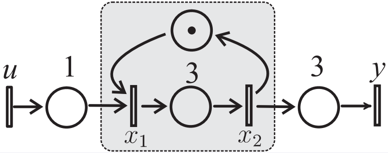
$x_i(\gamma) = 1\gamma \oplus 5\gamma^3 \oplus 7\gamma^5 \oplus 10\gamma^8 \oplus 11\gamma^{11} \oplus \top\gamma^{14}$



γ transform [Cohen, Quadrat et al. IEEE TAC 89] ▸ More

γ transform of $x(k)$ is a formal series $x(\gamma) = \bigoplus_{k\in\mathbb{N}} \gamma^k x(k)$. The set of series is a semiring denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. A series with a finite support is called a polynomial, and a monomial if there is only one element.

TEG Model in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$



The previous system in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

$$
\begin{aligned}
x(\gamma) &= Ax(\gamma) \oplus Bu(\gamma) &= \begin{pmatrix} \varepsilon & \gamma \\ 3 & \varepsilon \end{pmatrix} x(\gamma) \oplus \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} u(\gamma) \\
y(\gamma) &= Cx(\gamma) &= \begin{pmatrix} \varepsilon & 3 \end{pmatrix} x(\gamma)
\end{aligned}
$$

Transfer relations in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

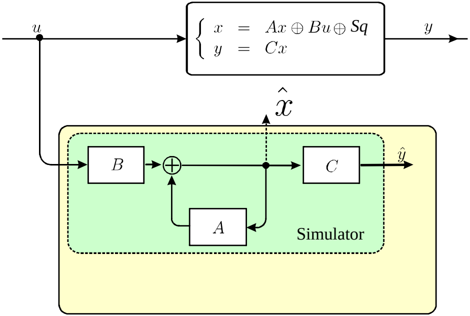
▸ More

$$
\begin{aligned}
x(\gamma) &= A^*Bu(\gamma) &= \begin{pmatrix} (3\gamma)^* & \gamma(3\gamma)^* \\ 3(3\gamma)^* & (3\gamma)^* \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} u(\gamma) \\
y(\gamma) &= CA^*Bu(\gamma) &= \begin{pmatrix} 7(3\gamma)^* \end{pmatrix} u(\gamma)
\end{aligned}
$$

Software library MinmaxGD available :

http://perso-laris.univ-angers.fr/~hardouin/outils.html

State Estimation : Observer Synthesis



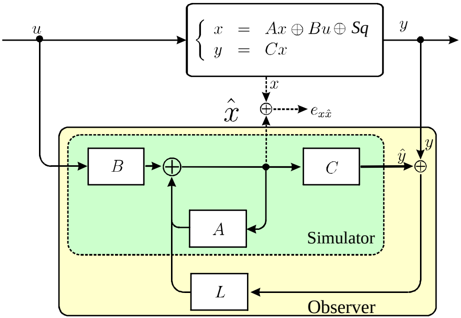
Prediction computation :

$$\hat{x}(\gamma) = Ax(\gamma) \oplus Bu(\gamma).$$

or

$$\hat{x}(k) = Ax(k-1) \oplus Bu(k).$$

State Estimation : Observer Synthesis

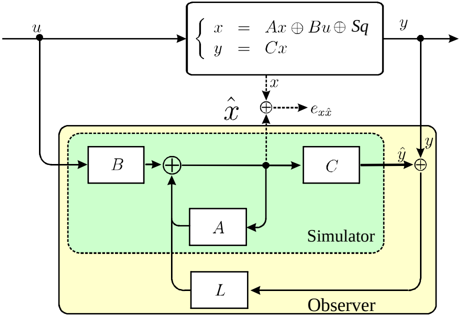


Objective :

Compute the greatest observer matrix L such that

$$\hat{x} \preceq x.$$

State Estimation : Observer Synthesis



System Equations :

Matrix S

$$x = Ax \oplus Bu \oplus Sq = A^* Bu \oplus A^* Sq$$
$$y = Cx = CA^* Bu \oplus CA^* Sq.$$

Estimated State Equations :

$$\hat{x} = A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y)$$
$$\hat{y} = C\hat{x}.$$

State Estimation : Observer Synthesis

Constraints Satisfaction :

Compute the greatest observer matrix L such that

$$\begin{aligned} (A \oplus LC)^* Bu &\preceq A^* Bu & \forall u \\ (A \oplus LC)^* LCA^* Sq &\preceq A^* Sq & \forall q, \end{aligned}$$

Constraints Satisfaction :

Compute the greatest matrix L such that

$$\begin{aligned} (A \oplus LC)^* B &\preceq A^* B \Leftrightarrow L \preceq (A^* B) /\!\!\!\circ (CA^* B) \\ (A \oplus LC)^* LCA^* S &\preceq A^* S \Leftrightarrow L \preceq (A^* S) /\!\!\!\circ (CA^* S). \end{aligned}$$

State Estimation : Observer Synthesis

Optimal Matrix : (Hardouin et al. IEEE TAC 2010, Hardouin et al. 2019)

$$L_{opt} = ((A^*B) \phi (CA^*B)) \wedge ((A^*S) \phi (CA^*S))$$

is the greatest such that

$$\hat{x} \preceq x.$$

State Estimation : Observer Synthesis : Performance Analysis

Equality of the asymptotic slope (Hardouin et al. IEEE TAC 2010)

If matrix C linking state vector to the output is connected to all connected components of the graph then

$$\sigma_\infty(\hat{x}_i) = \sigma_\infty(x_i) \ \forall i$$

Corollary :

If state x_i belongs to a connected component whose at least one transition is measured then the error $\hat{x}_i - x_i$ is bounded.

State Estimation : Set-membership approach

Uncertain system
$A(k) \in [\underline{A}, \overline{A}] = [A], B(k) \in [\underline{B}, \overline{B}] = [B], C(k) \in [\underline{C}, \overline{C}] = [C]$

Each matrices entries is supposed bounded and $A(k), B(k), C(k)$ is a realization at step k

$$\begin{aligned} x(k) &= A(k)x(k-1) \oplus B(k)u(k) \\ y(k) &= C(k)x(k) \end{aligned}$$

In the sequel, to enlighten the notation, we assume (without lost of generality) autonomous systems, i.e., $x(k) = A(k)x(k-1)$.

Indeed we can consider $\tilde{x} = (x^t u^t)^t$ and $\tilde{A} = \begin{pmatrix} A & \varepsilon \\ \varepsilon & B \end{pmatrix}$

State Estimation : Set-membership approach

Uncertain system $A(k) \in [\underline{A}, \overline{A}] = [A]$, $C(k) \in [\underline{C}, \overline{C}] = [C]$

$$\begin{array}{rcl} x(k) & = & A(k)x(k-1) \\ y(k) & = & C(k)x(k) \end{array}$$

Q1 : Assuming $x(k-1) \in \mathcal{X}_{k-1|k-1}$ a known set, is it possible to compute the set $\mathcal{X}_{k|k-1} = \left\{ Ax \mid A \in [\underline{A}, \overline{A}], x \in \mathcal{X}_{k-1|k-1} \right\}$? (prediction)
Q2 : Assuming $y(k)$ available, is it possible to compute the inverse image set $[C]^{-1}(y(k)) = \left\{ x \mid y(k) = Cx, C \in [\underline{C}, \overline{C}] \right\}$? (likelihood)
Q3 : Is it possible to compute the intersection of the two previous sets to obtain the set $\mathcal{X}_{k|k} = \mathcal{X}_{k|k-1} \cap [C]^{-1}(y(k))$? (estimation)

State Estimation : Set-membership approach

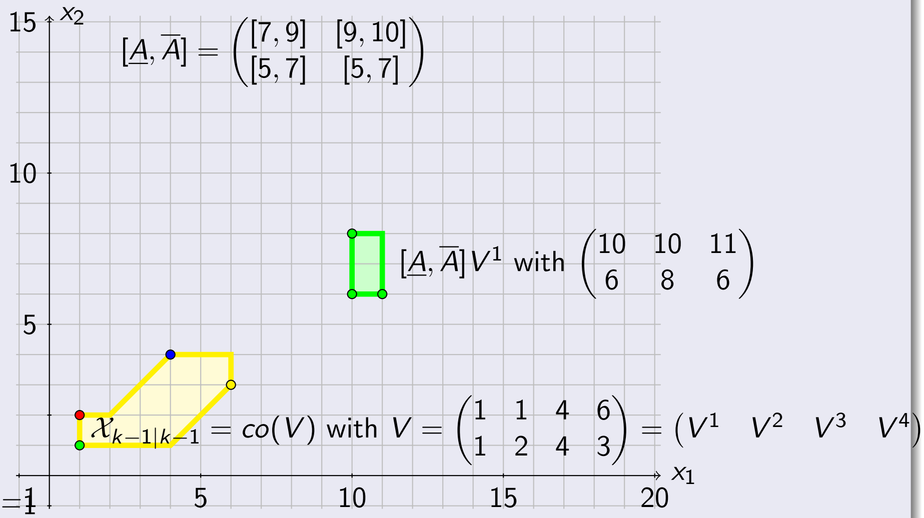
Uncertain system $A \in [\underline{A}, \overline{A}]$, $C \in [\underline{C}, \overline{C}]$

$$\begin{array}{rcl} x(k) & = & A(k)x(k-1) \\ y(k) & = & C(k)x(k) \end{array}$$

Q1 : Assuming $x(k-1) \in \mathcal{X}_{k-1|k-1}$ a known set, is it possible to compute the set $\mathcal{X}_{k|k-1} = \left\{ Ax \mid A \in [\underline{A}, \overline{A}], x \in \mathcal{X}_{k-1|k-1} \right\}$?
Assumption : $\mathcal{X}_{k-1|k-1}$ is depicted as a tropical polytope.
(Lemma 2.1 PhD Guilherme Winck (University of Angers)).

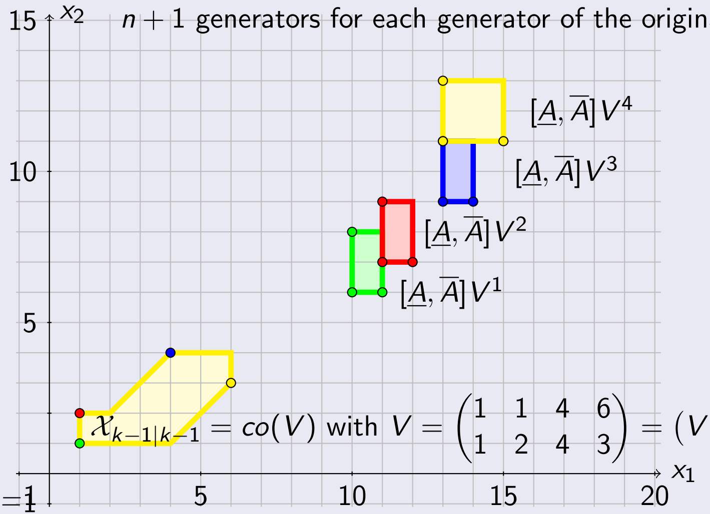
State Estimation : Set-membership approach

Q1 : Assuming $x(k-1) \in \mathcal{X}_{k-1|k-1}$ a known set, is it possible to compute the set $\mathcal{X}_{k|k-1} = \{Ax \mid A \in [\underline{A}, \overline{A}], x \in \mathcal{X}_{k-1|k-1}\}$?



State Estimation : Set-membership approach

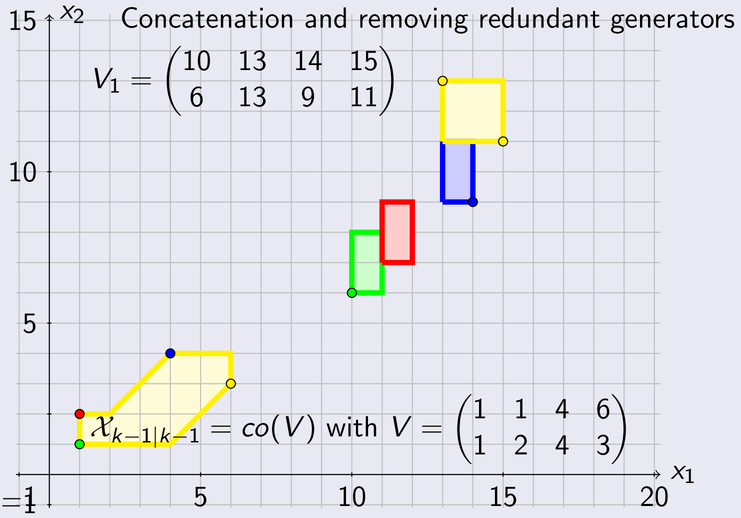
Q1 : Assuming $x(k-1) \in \mathcal{X}_{k-1|k-1}$ a known set, is it possible to compute the set $\mathcal{X}_{k|k-1} = \left\{ Ax \mid A \in [\underline{A}, \overline{A}], x \in \mathcal{X}_{k-1|k-1} \right\}$?



$\mathcal{X}_{k-1|k-1} = co(V)$ with $V = \begin{pmatrix} 1 & 1 & 4 & 6 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} V^1 & V^2 & V^3 & V^4 \end{pmatrix}$

State Estimation : Set-membership approach

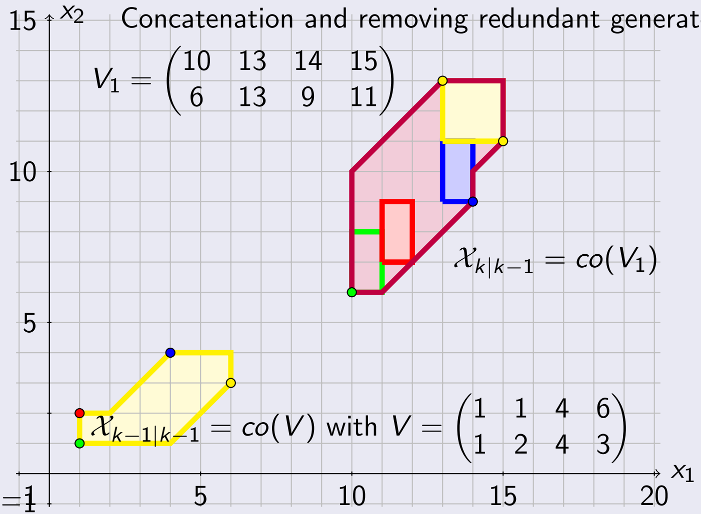
Q1 : Assuming $x(k-1) \in \mathcal{X}_{k-1|k-1}$ a known set, is it possible to compute the set $\mathcal{X}_{k|k-1} = \left\{ Ax \mid A \in [\underline{A}, \overline{A}], x \in \mathcal{X}_{k-1|k-1} \right\}$?



State Estimation : Set-membership approach

Q1 : Assuming $x(k-1) \in \mathcal{X}_{k-1|k-1}$ a known set, is it possible to compute the set $\mathcal{X}_{k|k-1} = \left\{ Ax \mid A \in [\underline{A}, \overline{A}], x \in \mathcal{X}_{k-1|k-1} \right\}$?

Concatenation and removing redundant generators



State Estimation : Set-membership approach

Uncertain system $A \in [\underline{A}, \overline{A}]$, $C \in [\underline{C}, \overline{C}]$

$$\begin{array}{rcl} x(k) & = & A(k)x(k-1) \\ y(k) & = & C(k)x(k) \end{array}$$

Q2 : Assuming $y(k)$ available, is it possible to compute the inverse image set $[C]^{-1}(y(k)) = \{x \mid y(k) = Cx, C \in [\underline{C}, \overline{C}]\}$?
The set can be written as $[C]^{-1}(y(k)) = \{x | \underline{C}x \preceq y(k) \preceq \overline{C}x\}$, which can be decomposed in two sets :

$$\mathcal{X} = \overline{\mathcal{X}} \cap \underline{\mathcal{X}}$$

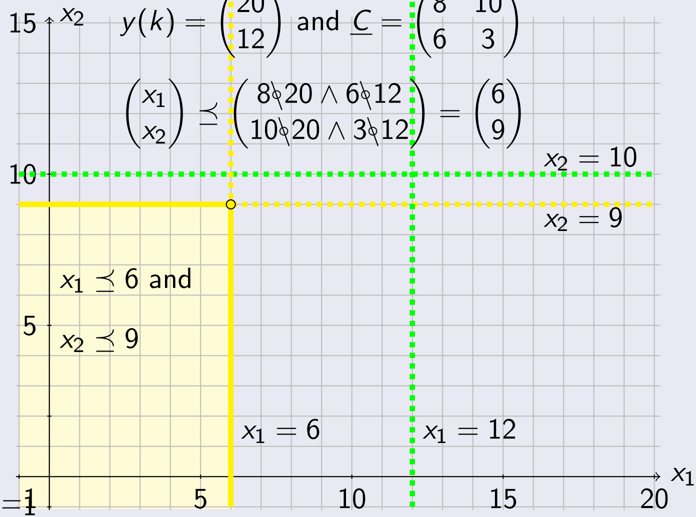
where $\overline{\mathcal{X}} = \{x | \underline{C}x \preceq y(k)\}$ and $\underline{\mathcal{X}} = \{x | y(k) \preceq \overline{C}x\}$
Renato Cândido et al., "An Algorithm to Compute the Inverse Image of a Point with Respect to a Nondeterministic Max Plus Linear System", in IEEE TAC, 2021.

State Estimation : Set-membership approach

Computation $\overline{\mathcal{X}} = \{x | \underline{C}x \preceq y(k)\} \Leftrightarrow \overline{\mathcal{X}} = \{x | x \preceq \underline{C} \circ\backslash y(k)\}$

$$y(k) = \begin{pmatrix} 20 \\ 12 \end{pmatrix} \text{ and } \underline{C} = \begin{pmatrix} 8 & 10 \\ 6 & 3 \end{pmatrix}$$

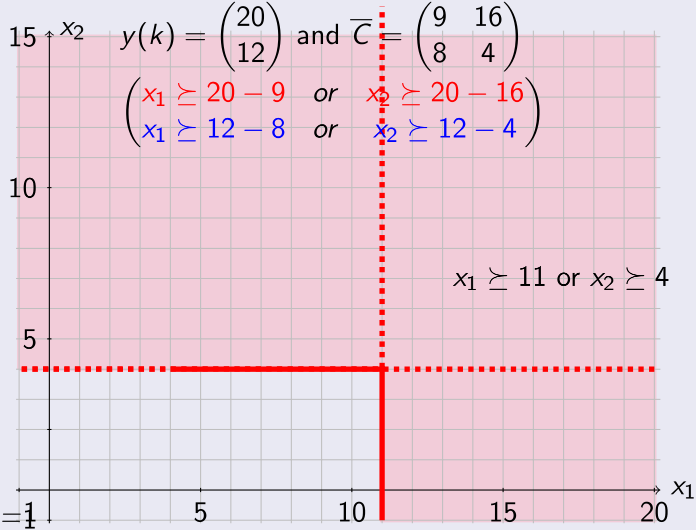
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \preceq \begin{pmatrix} 8 \circ\backslash 20 \wedge 6 \circ\backslash 12 \\ 10 \circ\backslash 20 \wedge 3 \circ\backslash 12 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$



Computation $\underline{\mathcal{X}} = \{x | y(k) \preceq \overline{C}x\}$

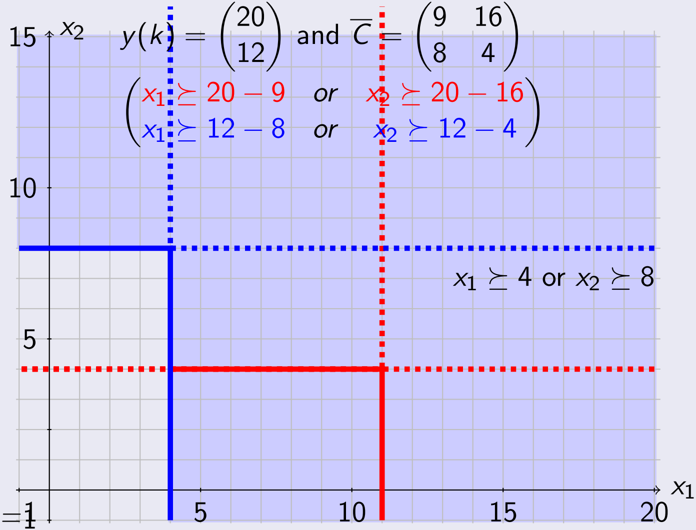
$$y(k) = \begin{pmatrix} 20 \\ 12 \end{pmatrix} \text{ and } \overline{C} = \begin{pmatrix} 9 & 16 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \succeq 20 - 9 & or & x_2 \succeq 20 - 16 \\ x_1 \succeq 12 - 8 & or & x_2 \succeq 12 - 4 \end{pmatrix}$$



State Estimation : Set-membership approach

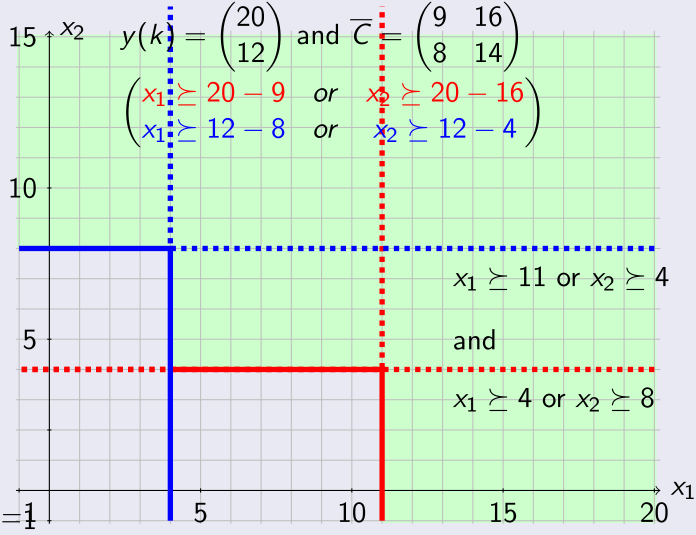
Computation $\underline{\mathcal{X}} = \{x | y(k) \preceq \overline{C} x\}$



Computation $\underline{\mathcal{X}} = \{x | y(k) \preceq \overline{C}x\}$

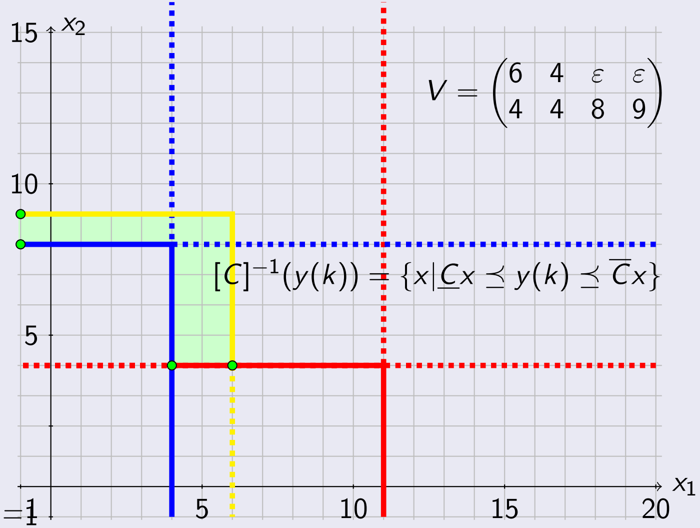
$$y(k) = \begin{pmatrix} 20 \\ 12 \end{pmatrix} \text{ and } \overline{C} = \begin{pmatrix} 9 & 16 \\ 8 & 14 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \succeq 20 - 9 & or & x_2 \succeq 20 - 16 \\ x_1 \succeq 12 - 8 & or & x_2 \succeq 12 - 4 \end{pmatrix}$$



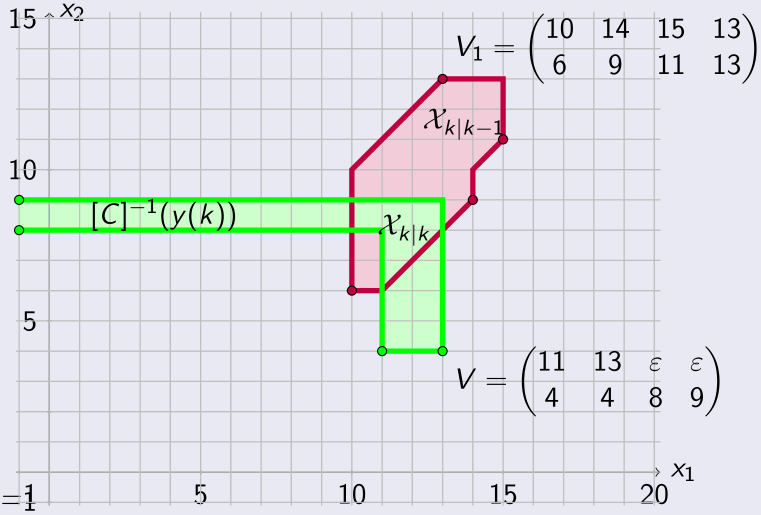
State Estimation : Set-membership approach

Q2 : Computation $[C]^{-1}(y(k)) = \overline{\mathcal{X}} \cap \underline{\mathcal{X}}$



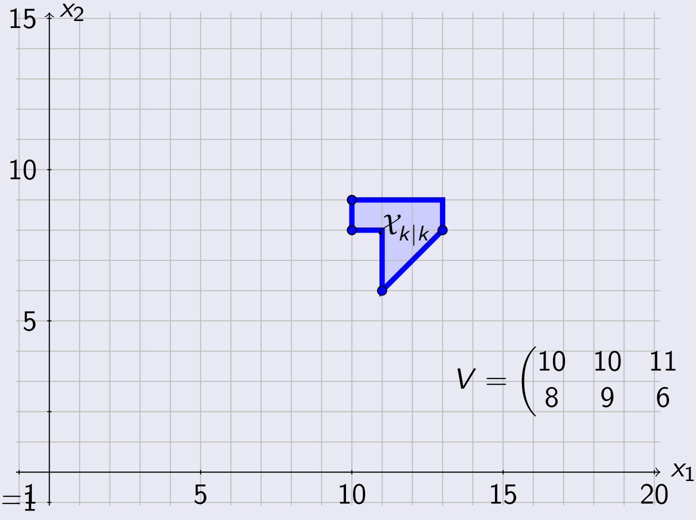
State Estimation : Set-membership approach

Q3 : Is it possible to obtain the intersection
$\mathcal{X}_{k|k} = [C]^{-1}(y(k)) \cap \mathcal{X}_{k|k-1}$



State Estimation : Set-membership approach

Q3 : Is it possible to obtain the intersection $\mathcal{X}_{k|k} = [C]^{-1}(y(k)) \cap \mathcal{X}_{k|k-1}$



$$V = \begin{pmatrix} 10 & 10 & 11 & 13 \\ 8 & 9 & 6 & 8 \end{pmatrix}$$

State Estimation : Set-membership approach

Filtering algorithm :

Require : $\mathcal{X}_{k-1|k-1}, y(k)$

Ensure : $\mathcal{X}_{k|k}$

$$\mathcal{X}_{k|k-1} = [\underline{A}, \overline{A}]\mathcal{X}_{k-1|k-1} \qquad \textit{(prediction)}$$

$$\underline{\mathcal{X}} = \{x | x \preceq \underline{C} \diamond\!\!\backslash y(k)\}$$

$$\overline{\mathcal{X}} = \{x | y(k) \preceq \overline{C} x\}$$

$$[C]^{-1}(y(k)) = \underline{\mathcal{X}} \cap \overline{\mathcal{X}} \qquad \textit{(likelihood)}$$

$$\mathcal{X}_{k|k} = \mathcal{X}_{k|k-1} \cap [C]^{-1}(y(k)) \qquad \textit{(estimation)}$$

Filtering algorithm :

Require : $\mathcal{X}_{k-1|k-1}, y(k)$ $\quad n, N, q$

Ensure : $\mathcal{X}_{k|k}$

$$\mathcal{X}_{k|k-1} = [\underline{A}, \overline{A}]\mathcal{X}_{k-1|k-1} \qquad \mathcal{O}(2Nn^2)$$

$$\underline{\mathcal{X}} = \{x | x \preceq \underline{C} \backslash\!\!\circ\, y(k)\} \qquad \mathcal{O}(nq)$$

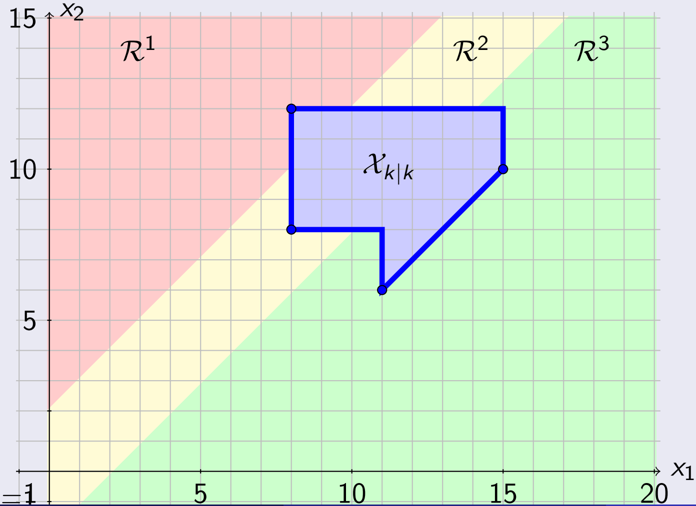
$$\overline{\mathcal{X}} = \{x | y(k) \preceq \overline{C}x\} \qquad \mathcal{O}(nq)$$

$$[C]^{-1}(y(k)) = \underline{\mathcal{X}} \cap \overline{\mathcal{X}}$$

$$\mathcal{X}_{k|k} = \mathcal{X}_{k|k-1} \cap [C]^{-1}(y(k)) \qquad \mathcal{O}(n^n)$$

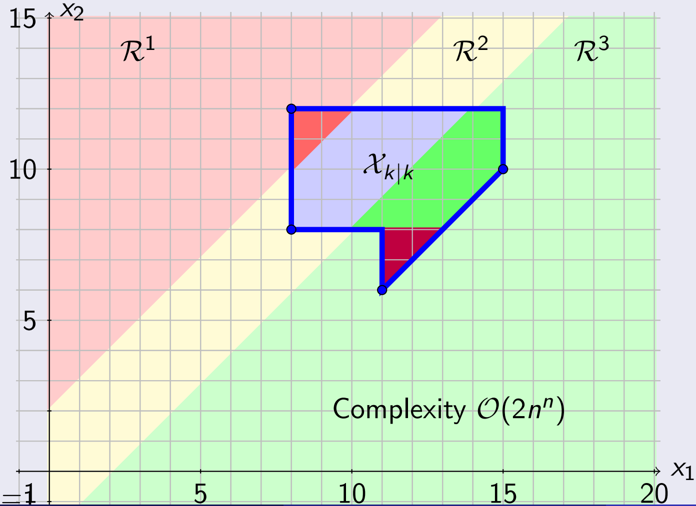
State Estimation : Set-membership approach

Alternative approaches : Decomposition in PWA (Adzkiya et al. Automatica 2015)



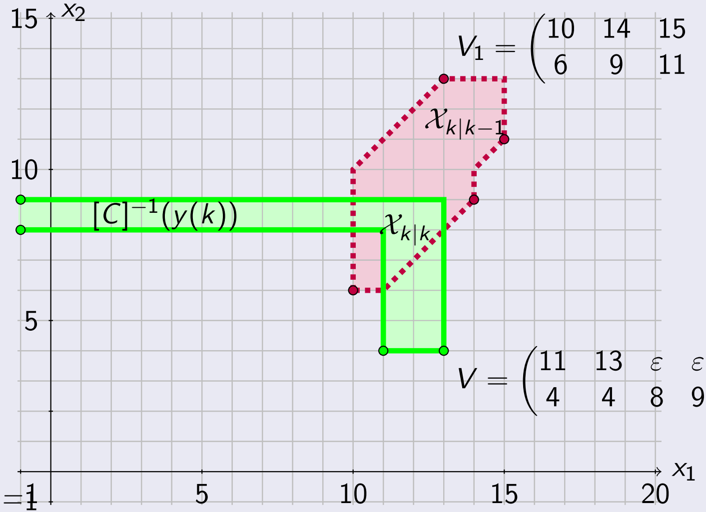
State Estimation : Set-membership approach

Alternative approaches : Decomposition in PWA (Adzkiya et al. Automatica 2015)



State Estimation : Set-membership approach

Alternative approaches : Interval analysis (Winck PhD 2022)

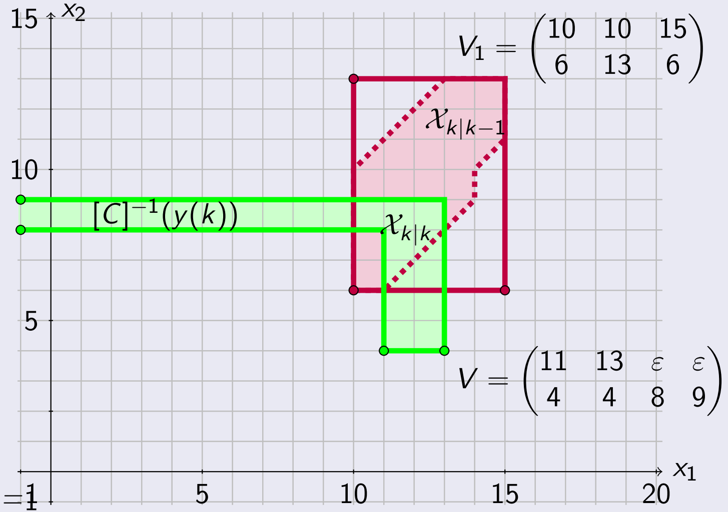


$$V_1 = \begin{pmatrix} 10 & 14 & 15 & 13 \\ 6 & 9 & 11 & 13 \end{pmatrix}$$

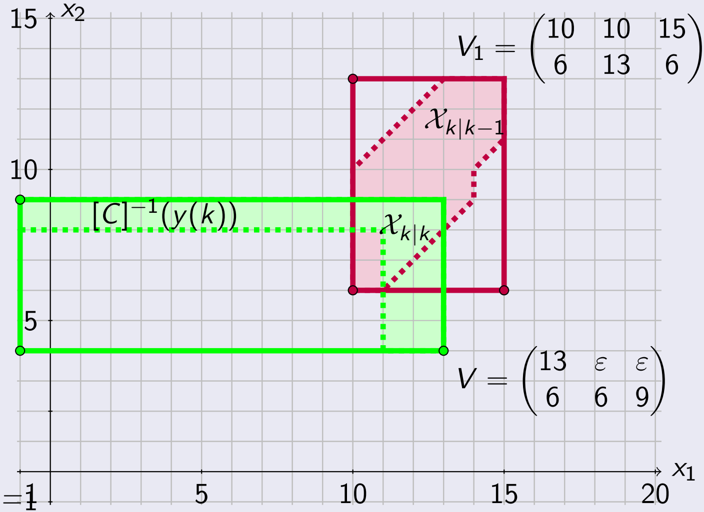
$$V = \begin{pmatrix} 11 & 13 & \varepsilon & \varepsilon \\ 4 & 4 & 8 & 9 \end{pmatrix}$$

State Estimation : Set-membership approach

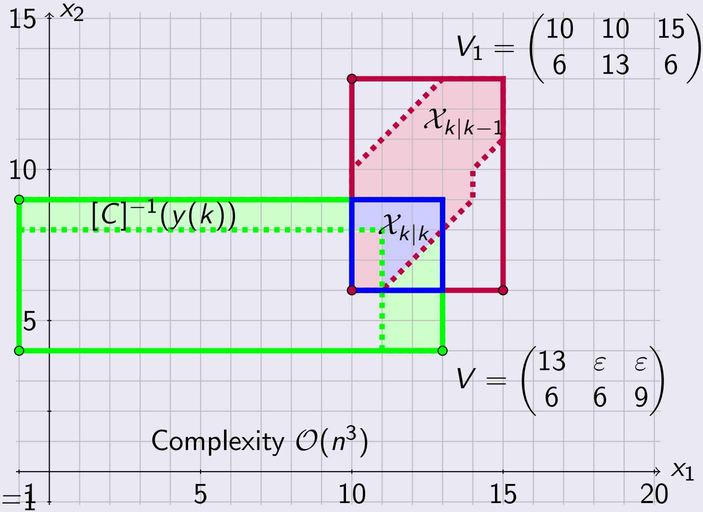
Alternative approaches : Interval analysis (Winck PhD 2022)



Alternative approaches : Interval analysis (Winck PhD 2022)



Alternative approaches : Interval analysis (Winck PhD 2022)



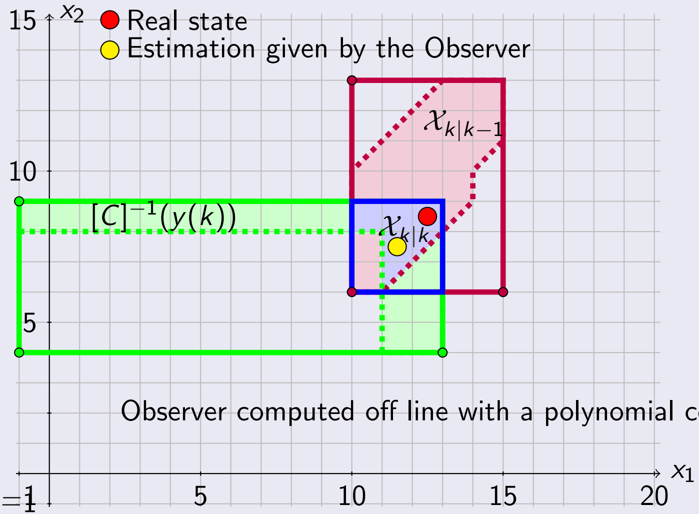
State Estimation : Set-membership approach

Performances comparison

- Using tropical polytope approach, the set of all possible solution is obtained, the complexity is $\mathcal{O}(n^n)$.
- Using DBM the same set is obtained (Adzkiya et al. Automatica 2015),with an exponential complexity also, but practically worst.
- Using Box an overapproximation is obtained with a polynomial complexity (Winck, PhD 2022).
- Using SMT (Satisfability Modulo theory) solver (e.g., z3 solver) (Mufid et al. IEEE TAC, 2022) is equivalent to keep the $\mathcal{H}$-form of the tropical polytope. This is suitable when a point included in the estimation set is desired (check a solution). But needs to keep all the constraints on the horizon of estimation, which growth at each step.

State Estimation : Set-membership approach

Where is the estimation given by the observer ?



Observer computed off line with a polynomial complexity

Conclusion

State Estimation

- An efficient observer exists, the greatest possible solution is obtained
- A set-membership approach based on max-plus polytope is the most efficient to obtain the set of all possible solutions, even if the complexity is still exponential.
- Interval analysis yields an over estimation of the solution set with a polynomial complexity

Conclusion

Open problems to address

- Developing an interval observer to compute on-line an upper bound
- Developing more efficient algorithms to compute intersection of max-plus polytope
- Developing method to obtain underestimation set (set included in the solution set), (Barnhill et al., arxiv.org, 2023).
- Selecting a point in the solution set (support) by considering stochastic approach (Santos-Mendes et al. IEEE TAC, 2019, Winck et al. IEEE TAC 2022).
- Considering only $\mathcal{H}$-form to avoid the costly transposition to $\mathcal{V}$-form.
- Developing state estimation method for systems involving resource sharing (Schafaschek et al. 2020).

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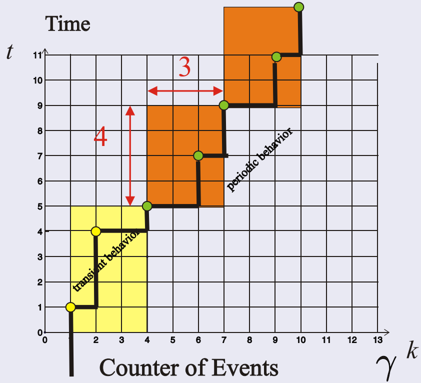
Semiring of periodic series (Cohen, Quadrat et al. IEEE TAC 89)

A periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

◂ Back

$s = p \oplus q(\tau\gamma^{\nu})^*$ where $p = \bigoplus_i t_i \gamma^{n_i}$ and $q = \bigoplus_j t_j \gamma^{n_j}$ are polynomials and $\sigma_\infty(s) = \nu/\tau$ is the asymptotic slope (the throughput).

$s = (1\gamma \oplus 4\gamma^2) \oplus (5\gamma^4 \oplus 7\gamma^6)(4\gamma^3)^*$ and $\sigma_\infty(s) = 3/4$



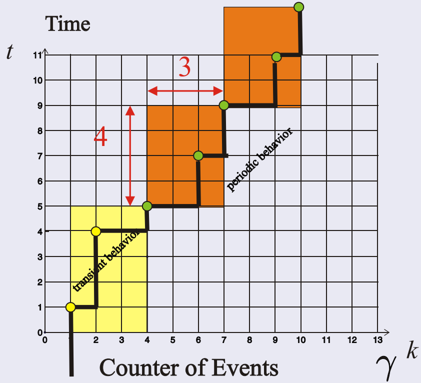
Semiring of periodic series (Cohen, Quadrat et al. IEEE TAC 89)

A periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

Back

$s = p \oplus q(\tau\gamma^{\nu})^*$ where $p = \bigoplus_i t_i\gamma^{n_i}$ and $q = \bigoplus_j t_j\gamma^{n_j}$ are polynomials and $\sigma_\infty(s) = \nu/\tau$ is the asymptotic slope (the throughput).

$s = (1\gamma \oplus 4\gamma^2) \oplus (5\gamma^4 \oplus 7\gamma^6)(4\gamma^3)^*$ and $\sigma_\infty(s) = 3/4$



Semiring of periodic series (Cohen, Quadrat et al. IEEE TAC 89)

Operations over semiring of periodic series over $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

- $s = s_1 \oplus s_2$ is a periodic series, asymptotic slope $\sigma_\infty(s) = min(\sigma_\infty(s_1), \sigma_\infty(s_2))$
- $s = s_1 \otimes s_2$ is a periodic series, asymptotic slope $\sigma_\infty(s) = min(\sigma_\infty(s_1), \sigma_\infty(s_2))$
- $s = s_1 \wedge s_2$ is a periodic series, asymptotic slope $\sigma_\infty(s) = max(\sigma_\infty(s_1), \sigma_\infty(s_2))$
- $s = s_1 \diamond\!\!\!\backslash s_2$ is a periodic series, $\sigma_\infty(s) = \sigma_\infty(s_2)$ if $\sigma_\infty(s_2) \leq \sigma_\infty(s_1)$ else $s = \varepsilon$.

Software Tools

Software to handle periodic series is available on :
http://perso-laris.univ-angers.fr/~hardouin/outils.html

Second order theory (MAXPLUS, IEEE CDC 91)

Counter associated to a series

Let $s = \bigoplus_{k \in \mathbb{Z}} s(k)\gamma^k$ be a series and $\mathcal{C}_s$ the counter function associated to s defined by $s = \bigoplus_{t \in \mathbb{Z}} t\gamma^{\mathcal{C}_s(t)}$.

Distance in the event domain between 2 series (Santos Mendes et al. ETFA 05)

Let s_1 and s_2 be two series, the distance in the event domain is denoted $\Delta_{s_1 s_2}$ and defined by

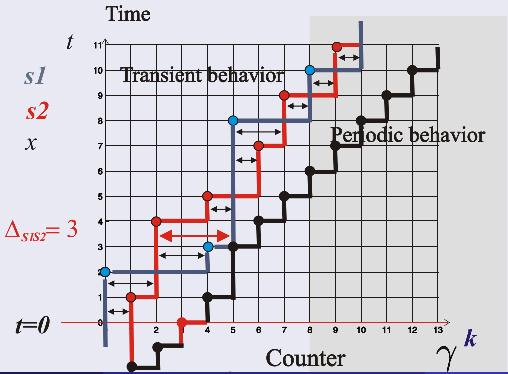
$$\Delta_{s_1 s_2} = max\{|\mathcal{C}_{s_1}(t) - \mathcal{C}_{s_2}(t)| \text{ s.t. } t \in \mathbb{Z}\}$$

it can be evaluated by considering
$\Delta_{s_1 s_2} = \mathcal{C}_{d_{12}}(0)$ where $d_{12} = (s_1 \wedge s_2) \phi (s_1 \oplus s_2)$

Distance in the event domain (Santos Mendes et al. ETFA 05)

Illustration : practical computation of the distance

Let $s_1 = 2 \oplus 3\gamma^4 \oplus 8\gamma^5 \oplus 10\gamma^8(2\gamma^2)^*$ and $s_2 = 1\gamma \oplus 4\gamma^2 \oplus 5\gamma^4 \oplus 7\gamma^6 \oplus 9\gamma^7(2\gamma^2)^*$ be two series. Series $d_{12} = (s_1 \wedge s_2) \oslash (s_1 \oplus s_2) = -2\gamma \oplus -1\gamma \oplus 0\gamma^3 \oplus 1\gamma^4 \oplus 3\gamma^5(1\gamma)^*$ and the associated counter $\Delta_{s_1 s_2} = \mathcal{C}_{d_{12}}(0) = 3$.



Distance in the event domain (Santos Mendes et al. ETFA 05)

Application : bound computation for the difference between firing of two transitions subject to the same inputs

Let $x_1 = s_1 u$ and $x_2 = s_2 u$ two series describing the behavior of two states. The distance between these trajectories can be computed by considering $du_{12} = (s_1 u \wedge s_2 u) \not\phi (s_1 u \oplus s_2 u) = (x_1 \wedge x_2) \not\phi (x_1 \oplus x_2)$. We can prove that $du_{12} \succeq (s_1 \wedge s_2) \not\phi (s_1 \oplus s_2) = d_{12}$, hence $\mathcal{C}_{du_{12}}(0) \leq \mathcal{C}_{d_{12}}(0)\ \forall u$ and consequently

$$\Delta_{x_1 x_2} \leq \mathcal{C}_{d_{12}}(0)\ \forall u.$$

Corollary

Extension in the matrix case is straight forward, the bound for the event difference between two transitions will be the maximum for each entry.

Idempotent semi-ring and (max,+) algebra

Sum of matrices $A \oplus B = C$ with entries in $\overline{\mathbb{Z}}_{\max}$

Back

$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \oplus \begin{pmatrix} e & 8 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & 7 \end{pmatrix}$$

Product of matrices $A \otimes B = C$ with entries in $\overline{\mathbb{Z}}_{\max}$

Back

$$\begin{pmatrix} 2 & 5 \\ \varepsilon & 3 \\ 1 & 8 \end{pmatrix} \otimes \begin{pmatrix} e \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \otimes e \oplus 5 \otimes 1 \\ \varepsilon \otimes e \oplus 3 \otimes 1 \\ 1 \otimes e \oplus 8 \otimes 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$$

Residuation of matrices $A \circ\!\!\!\backslash B$ is the greatest solution of $A \otimes X \preceq B$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \circ\!\!\!\backslash \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} = \begin{pmatrix} (1 \circ\!\!\!\backslash 8) \wedge (3 \circ\!\!\!\backslash 9) \wedge (5 \circ\!\!\!\backslash 10) \\ (2 \circ\!\!\!\backslash 8) \wedge (4 \circ\!\!\!\backslash 9) \wedge (6 \circ\!\!\!\backslash 10) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

TEG Model in $\overline{\mathbb{Z}}_{max}$

$\overline{\mathbb{Z}}_{max}$ is the (max,+) algebra

$\overline{\mathbb{Z}}_{max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, +)$ is an idempotent semiring, *i.e.*, sum $\oplus$ is the operator *max*, product $\otimes$ is classical sum $+$, the neutral element of the sum is denoted $\varepsilon = -\infty$ and the neutral element of the product is de noted $e = 0$. The sum is idempotent :

$$(\forall a \in \overline{\mathbb{Z}}_{max}, a \oplus a = a)$$

Example :, $\overline{\mathbb{Z}}_{max}$

Back

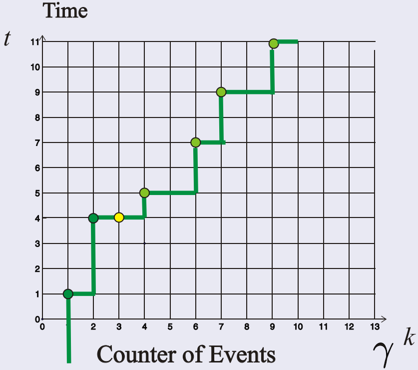
$$1 \oplus 1 = 1 = max(1,1),$$
$$2 \otimes 1 = 3 = 2 + 1.$$
$$a \oplus \varepsilon = a = max(a, -\infty),$$
$$a \otimes e = a = a + 0,$$
$$\varepsilon \otimes a = \varepsilon = -\infty + a = -\infty.$$

Series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

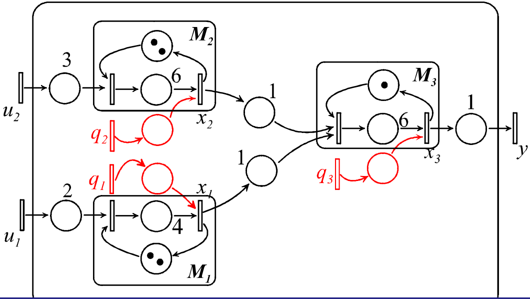
◂ Back

A series : $s = \bigoplus_{k \in \mathbb{Z}} s(k)\gamma^k$ codes a non decreasing trajectory. The set of series is a semiring denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$. A series with a finite support is called a polynomial, and a monomial if there is only one element.

$s = 1\gamma \oplus 4\gamma^2 \oplus 5\gamma^4 \oplus 7\gamma^6 \oplus \ldots$



State Estimation : Observer Synthesis

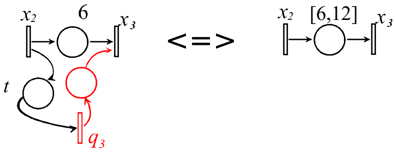


Matrix S and input q :

Back

- vector q represents a vector of exogenous uncontrollable inputs (disturbances, disabling the firing) which act on the system through matrix S.
- When matrix S is equal to identity matrix and $q = x_0$ they may represent the initial state of the system.

State Estimation : Observer Synthesis



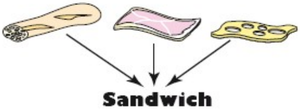
Matrix S and input q :

Back

- vector q can depict uncertain delay. E.g., by choosing $q_3 = t\ x_2$ with $t \in [6, 12]$

Idempotent semi-ring and (max,+) algebra

Back



Sandwiches Algebra [Cohen et al.]

1 piece of Bread + 1 slice of ham + 1 slice of cheese is equal to 1 sandwich. Another way of counting !

TEG Model in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

A periodic series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

Back

$s = p \oplus q(\tau\gamma^{\nu})^*$ where $p = \bigoplus_i t_i\gamma^{n_i}$ and $q = \bigoplus_j t_j\gamma^{n_j}$ are polynomials and $\sigma_\infty(s) = \nu/\tau$ is the throughput.

$s = (1\gamma \oplus 4\gamma^2) \oplus (5\gamma^4 \oplus 7\gamma^6)(4\gamma^3)^*$ and $\sigma_\infty(s) = 3/4$