Controllers Synthesis for Max-Plus Linear Systems under State Constraints

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***Abstract*— This paper deals with the synthesis of controllers for (max,+) linear systems. The objective of the controllers is to guarantee that the state trajectories are maintained within a sub-semi-module. Among the possible controllers, the greatest is calculated in order to achieve an output as close as possible to that of the unconstrained system.**

I. Introduction

The max-plus algebraic framework is a powerful tool for modeling and analyzing discrete event systems such as Timed Event Graphs (TEGs) which are used in communication networks, genetic regulatory networks and queueing systems (see [Baccelli et al., 1992]). The fundamental problems for max-plus linear systems have been studied by researchers for the past three decades, for example, controllability [MaxPlus, 1991], [Hardouin et al., 2017], observability [Hardouin et al., 2010], and the model reference control problem [Hardouin et al., 2018]. There are some existing research results on generalizing fundamental concepts such as the computation of different controlled invariant sets ([Animobono et al., 2023], [Hardouin et al., 2011]) and the disturbance decoupling problem [Shang et al., 2016]. This paper reports upon new investigations on the controller synthesis for max-plus linear systems in order to ensure that the state trajectories belong to a given subspace which can represent a set of constraints on the state. Due to the monotone non-decreasing property of max-plus linear systems, the controller can only increase the inputs. In other words, only the delay of tokens entries is possible in the corresponding timed event graph. Hence, it would be interesting to obtain a controller respecting the set of constraints while maintaining system performances as close as possible to that of the unconstrained system. This problem is a generalization of the one introduced in [Maia et al., 2005]. Here a semiring of periodic series is considered, hence the constraints are applied on infinite trajectories. The remainder of this paper is organized as follows. Section II presents the mathematical preliminaries in max-plus algebra. Section III presents the literature on max-plus linear system models. Section IV presents the control problems, first the synthesis of optimal controllers ensuring to preserve the performance of the system while delaying as much as possible the input (just-in-time strategy) is recalled, and then the controllers ensuring that the constraints on the state are introduced. Section V shows the efficiency of the method on a manufacturing system subject to sojourn time constraint.

II. Algebraic Setting

Definition 1: A *semiring* is a set $\mathcal{S}$, equipped with two operations, denoted as $\oplus$ and $\otimes$, such that $(\mathcal{S},\oplus)$ is a commutative monoid (the zero element will be denoted ε), $(\mathcal{S},\otimes)$ is a monoid (the unit element will be denoted e), operation $\otimes$ is right and left distributive over $\oplus$, and ε is absorbing for the product (*i.e.* $\varepsilon\otimes a = a\otimes\varepsilon = \varepsilon, \forall a$).

A semiring $\mathcal{S}$ is *idempotent* if $a\oplus a = a$ for all $a\in\mathcal{S}$. As in classical algebra, the operator $\otimes$ will often be omitted, moreover, $a^i = a\otimes a^{i-1}$ and $a^0 = e$. In this algebraic structure, a partial order relation is defined by $a\succeq b \Leftrightarrow a = a\oplus b \Leftrightarrow\ b = a\wedge b$ (where $a\wedge b$ is the greatest lower bound of a and b). Therefore, an idempotent semiring is a partially ordered set (see [Baccelli et al., 1992] for a more detailed introduction). An idempotent semiring is said to be complete if it is closed for infinite $\oplus$-sums and if $\otimes$ distributes over infinite $\oplus$-sums. In particular, $\top = \bigoplus_{x\in\mathcal{S}} x$ is the greatest element of $\mathcal{S}$ ($\top$ is called the top element of $\mathcal{S}$).

Example 1 ($\overline{\mathbb{Z}}_{\max}$): The set $\overline{\mathbb{Z}}_{\max} = \mathbb{Z}\cup\{-\infty,+\infty\}$ endowed with the $\max$ operator as sum and the classical sum $+$ as product is a complete idempotent semiring, usually denoted $\overline{\mathbb{Z}}_{\max}$, of which $\varepsilon = -\infty$ and $e = 0$.

Theorem 1 (Kleene star operator): The implicit inequality $x\succeq ax\oplus b$ as well as $x = ax\oplus b$ defined on $\mathcal{S}$, admit $x = a^*b$ as the least solution, where $a^* = \bigoplus\limits_{i\in\mathbb{N}} a^i$. (see [Baccelli et al., 1992], Th. 4.75).

Properties 1: Kleene star operator admits the following useful property

$$(ab)^*a = a(ba)^*. \tag{1}$$

Definition 2 (Residual and residuated mapping): Let $\mathcal{D}$, $\mathcal{C}$ be two complete idempotent semirings and $f:\mathcal{D}\to\mathcal{C}$ be an order preserving mapping, f is a *residuated mapping* if for all $y\in\mathcal{C}$ there exists a greatest solution to the inequality $f(x)\preceq y$ (hereafter denoted $f^\sharp(y)$). Obviously, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ yields the greatest solution. The mapping $f^\sharp$ is called the *residual* of f and $f^\sharp(y)$ is the optimal solution of the inequality.

Theorem 2: Let $\mathcal{D}$, $\mathcal{C}$ be two complete idempotent semirings and $f:\mathcal{D}\to\mathcal{C}$ be an order-preserving mapping, the following statements are equivalent (see [Baccelli et al., 1992]):

(i) f is residuated.

(ii) there exists an unique order-preserving mapping $f^\sharp : \mathcal{C}\to\mathcal{D}$ such that $f\circ f^\sharp\preceq \mathsf{Id}_{\mathcal{C}}$ and $f^\sharp\circ f\succeq\mathsf{Id}_{\mathcal{D}}$.

Example 2: The mappings $\Lambda_a : x\mapsto a\otimes x$ and $\Psi_a : x\mapsto x\otimes a$ defined on $\mathcal{S}$ are both residuated (see [Baccelli et al., 1992], Section 4.4.4). Their residuals are order-preserving mappings, denoted respectively by $\Lambda_a^\sharp(x) = a\backslash\!\!\circ x$ and $\Psi_a^\sharp(x) = x/\!\!\circ a$. This means that $a\backslash\!\!\circ b$ (resp. $b/\!\!\circ a$) is the greatest solution of inequality $a\otimes x\preceq b$ (resp. $x\otimes a\preceq b$).

Example 3: Mapping $K:\mathcal{S}\to\mathcal{S}, x\mapsto x^*$ is residuated (see [Cottenceau et al., 2001]). Actually it is shown in [Cottenceau et al., 2001] that $x = a^*$ is the greatest solution of inequality $x^* \preceq a$ if $a = a^*$, that is $x \preceq a^* \Leftrightarrow x^* \preceq a^*$.

The set of $n \times n$ matrices with entries in $\mathcal{S}$ is an idempotent semiring. The sum, the product and the residuation of matrices are defined after the sum, the product and the residuation of scalars in $\mathcal{S}$, *i.e.*,

$$(A \otimes B)_{ik} = \bigoplus_{j=1}^{n} (a_{ij} \otimes b_{jk}) \tag{2}$$

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \tag{3}$$

$$(A \backslash B)_{ij} = \bigwedge_{k=1}^{n} (a_{ki} \backslash b_{kj})\ ,\ (B \phi A)_{ij} = \bigwedge_{k=1}^{n} (b_{ik} \phi a_{jk}). \tag{4}$$

The identity matrix of $\mathcal{S}^{n\times n}$ is a matrix with entries equal to e on the diagonal and to ε elsewhere. This identity matrix will be denoted I_n, and the matrix with all its entries equal to ε will also be denoted ε.

Properties 2: ([MaxPlus, 1991]) Given a complete semiring $\mathcal{S}$, and two matrices $A \in \mathcal{S}^{p\times n}$, $B \in \mathcal{S}^{n\times p}$, the following equations hold :

$$A \backslash A \quad = \quad (A \backslash A)^*,\ B \phi B = (B \phi B)^*. \tag{5}$$

A. Fixed point of isotone mapping

Whereas residuation theory provides optimal solutions to inequalities $f(x) \preceq b$, the fixed point theory enables to find the greatest finite solutions to equations $f(x) = x$, where f is an isotone mapping defined over a complete idempotent semiring $\mathcal{S}$.

Definition 3: Let $\mathcal{F}_f = \{x \in \mathcal{S} \mid f(x) = x\}$ be the set of fixed points of an isotone mapping f defined over $\mathcal{S}$. Respectively, let $\mathcal{P}_f = \{x \in \mathcal{S} \mid f(x) \succeq x\}$ be the set of post-fixed points which can be interpreted in $\mathcal{F}_f$ as the following equivalence : $f(x) \succeq x \Leftrightarrow f(x) \wedge x = x$.

Theorem 3: (Knaster-Tarski) Let $\mathcal{F}_f$ be a complete lattice. The greatest fixed solution $\hat{y}$ of $\mathcal{F}_f$ is given by:

$$\hat{y} = \lim_{n\to\infty} f^n(\top)$$

where $f^{n+1} = f \circ f^n$ and $f^0 = \mathsf{Id}_{\mathcal{D}}$.

In order to obtain this solution, the following theorem presents a method to compute it in a recurrent way.

Theorem 4: Let f be an isotone mapping defined over $\mathcal{D}$ and recall that $\mathcal{F}_f$ is the set of fixed points of f. Now consider the following iterative scheme:

Let $x_0 = \top$,
do $x_{n+1} = f(x_n)$,
until $x_{m+1} = x_m$ for $m \in \mathbb{N}$.

If function f admits a finite fixed point $x \in \mathcal{F}_f$ then the previous algorithm converges toward the greatest fixed point $\hat{y} = x_m$.

Proof: Firstly, as $x_{m+1} = x_m$, $x_m = f(x_m)$ and so x_m belongs to the set $\mathcal{F}_f$. Secondly, it is necessary to show that x_m is the greatest solution of $\mathcal{F}_f$. Let $x' \in \mathcal{F}_f$, since $x_0 = \top$, $x_0 \succeq x'$. Finally, if $x_m \succeq x'$ then $x_{m+1} \succeq x'$: $x_{m+1} = f(x_m) \succeq f(x') = x'$ (thanks to the isotony of f and knowing that $x' \in \mathcal{F}_f$). ■

It must be noted that the number of steps for the convergence can be infinite. It is also possible to use this algorithm to find the greatest fixed point smaller than or equal to a given value of $\mathcal{S}$. In this case, the following corollary is given.

Corollary 1: Let $h : \mathcal{S} \mapsto \mathcal{S}$ be an isotone mapping and $val \in \mathcal{S}$; let f be defined by $f : \mathcal{S} \mapsto \mathcal{S}, x \mapsto h(x) \wedge val$. If f admits a fixed point $x \in \mathcal{F}_f$, then the algorithm of Theorem 4 converges toward the greatest fixed point of f less than or equal to val, that is, $\hat{y} = x_m \preceq val$.

Corollary 2: Let D and C be two complete idempotent semirings and let $h : D \to C$ and $g : D \to C$ be two residuated mappings. The greatest solution of the equality $h(x) = g(x)$ is equal to the greatest fixed point of the isotone mapping $f : D \to D$, defined by

$$f(x) = x \wedge g^\sharp(h(x)) \wedge h^\sharp(g(x)).$$

Proof: The following equivalences hold:

$$\begin{aligned} h(x) = g(x) &\Leftrightarrow h(x) \preceq g(x) \text{ and } g(x) \preceq h(x) \\ &\Leftrightarrow x \preceq h^\sharp(g(x)) \text{ and } x \preceq g^\sharp(h(x)) \\ &\Leftrightarrow x \preceq h^\sharp(g(x)) \wedge g^\sharp(h(x)) \\ &\Leftrightarrow x = x \wedge h^\sharp(g(x)) \wedge g^\sharp(h(x)). \end{aligned}$$

Hence, the greatest fixed point of $f(x) = x \wedge h^\sharp(g(x)) \wedge g^\sharp(h(x))$ is the greatest solution of the equation $h(x) = g(x)$. Furthermore, since the operation $\wedge$ and the mappings h, $h^\sharp$, g, and $g^\sharp$ are isotone, the mapping f is isotone. ■

As an immediate consequence, the greatest solution of equation $h(x) = g(x)$ smaller than a given x_0 can be obtained by using Corollary 1 and Theorem 4.

Corollary 3: Given a complete semiring $\mathcal{S}$, and two matrices $A \in \mathcal{S}^{p\times n}$, $B \in \mathcal{S}^{p\times n}$, the greatest element in the solution set $\{x \in \mathcal{S}^n | Ax = Bx\}$ can be obtained by considering the mapping $f(x) = x \wedge (B \backslash (Ax)) \wedge (A \backslash (Bx))$.

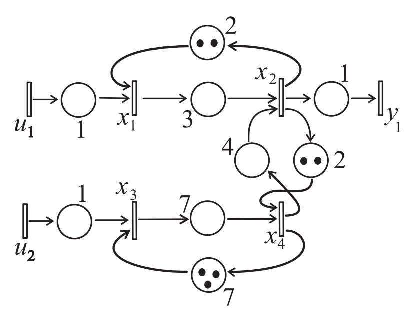

Fig. 1. TEG with 2 controllable transitions (u_1, u_2) and 1 measurable transition (y_1).

III. The TEG description in an idempotent semiring

TEGs constitute a subclass of timed Petri nets in which each place has one upstream and one downstream transition. A TEG description can be transformed into a max-plus or a min-plus linear model and *vice versa*. To obtain an algebraic model in $\overline{\mathbb{Z}}_{\max}$, a "dater" function is associated to each transition. For transition labeled x_i, $x_i(k)$ represents the date of the k^{th} firing (see [Baccelli et al., 1992]). In this paper, without loss of generality, TEGs are described by the following max-plus linear system:

$$\begin{aligned} x(k) &= \bigoplus_{j=0}^{N_a} A_j x(k-j) \oplus \bigoplus_{l=0}^{N_b} B_l u(k-l), \\ y(k) &= C_0 x(k), \end{aligned} \tag{6}$$

where $u(k) \in (\overline{\mathbb{Z}}_{\max})^p$, $y(k) \in (\overline{\mathbb{Z}}_{\max})^m$, and $x(k) \in (\overline{\mathbb{Z}}_{\max})^n$ are the controllable inputs, outputs and state vectors, respectively. These vectors collect the firing dates of the corresponding transition. The integer number N_a (resp. N_b) is equal to a maximum number of tokens initially available in internal places (resp. in the places between input and internal transitions). Matrices $A_j \in (\overline{\mathbb{Z}}_{\max})^{n\times n}$, $B_l \in (\overline{\mathbb{Z}}_{\max})^{n\times p}$, and $C_0 \in (\overline{\mathbb{Z}}_{\max})^{m\times n}$ represent the links between each transition and then describe the structure of the graph. Without loss of generality, it is assumed that each output transition is linked to one and only one internal transition, and no token is initially located in the place between this internal transition and the output transition, *i.e.* only matrix C_0 is needed to model the connection between the internal states and the output, and one entry is different from ε on each row of matrix C_0 and at most one entry is different from ε on each column.

Example 4: In Fig. 1, a TEG with $p = 2$ controllable inputs, and $m = 1$ measurable outputs, is depicted. Clearly, $N_a = 3$ and $N_b = 0$, therefore, the TEG model can be represented as the max-plus linear system in Eq. (6), where system matrices are

$$A_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 7 & \varepsilon \end{bmatrix}, \quad A_1 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix};$$

$$A_2 = \begin{bmatrix} \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon \end{bmatrix}, \quad A_3 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 7 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix};$$

$$B_0 = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & 1 \\ \varepsilon & \varepsilon \end{bmatrix}, \quad C_0 = \begin{bmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \end{bmatrix}.$$

A trajectory of a TEG transition is then a sequence of firing dates. This collection of dates can be represented by a formal series $x_i(\gamma) = \bigoplus_{k\in\mathbb{Z}} \left(x_i(k) \otimes \gamma^k\right)$ where $x_i(k) \in \overline{\mathbb{Z}}_{\max}$ and γ is a backward shift operator[1] in the event domain (formally $\gamma x_i(k) = x_i(k-1)$). The set of formal series in γ is denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and constitutes a complete idempotent semiring. The support of series $x_i(\gamma)$ is defined by $Supp(x_i) = \{k \in \mathbb{Z} | x_i(k) \neq \varepsilon\}$. The valuation in γ of $x_i(k)$ is defined as: $val(x_i) = min\{k | k \in Supp(x_i)\}$. A series $x_i \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is said to be a polynomial if $Supp(x_i)$ is finite. Furthermore, a polynomial is said to be a monomial if there is only one element.

[1]Operator γ plays a role similar to operator z^{-1} in the $\mathcal{Z}$-transform for the conventional linear systems theory.

The TEGs can then be described by the following model:

$$\begin{aligned} x &= Ax \oplus Bu, \\ y &= Cx, \end{aligned} \tag{7}$$

where $u \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^p$, $y \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^m$ and $x \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^n$ are series of inputs, outputs, states and disturbances, respectively. Matrices $A \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times p}$, and $C \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times n}$ represent the links between each transition, and are defined as follows:

$$A = \bigoplus_{j=0}^{N_a} \gamma^j A_j, \; B = \bigoplus_{l=0}^{N_b} \gamma^l B_l, \; C = C_0.$$

Therefore, the γ-domain representation describes the same structure of the TEG model as the event domain equation in Eq. (6). By considering Theorem 1, for this system, the state and the output trajectories can be rewritten as:

$$\begin{aligned} x &= A^* Bu \\ y &= CA^* Bu, \end{aligned} \tag{8}$$

where $CA^*B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times p}$ is the input/output transfer matrix.

Example 5: The model of the system depicted in Fig. 1 can be described concisely thanks to the representation in Eq. (7) where the system matrices are given as

$$A = \begin{bmatrix} \varepsilon & 2\gamma^2 & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & 4 \\ \varepsilon & \varepsilon & \varepsilon & 7\gamma^3 \\ \varepsilon & 2\gamma^2 & 7 & \varepsilon \end{bmatrix}, \; B = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & 1 \\ \varepsilon & \varepsilon \end{bmatrix},$$

$$C = \begin{bmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \end{bmatrix}.$$

The entry $A(3,4) = 7\gamma^3$ represents the place between the transition x_4 and x_3, which means that there is 3 tokens in this place and a minimal sojourn time equal to 7 time units. The implicit model as given in Eq. (8) can be easily computed by using the toolbox `MinMaxGD`, a C++ library developed in order to handle periodic series. For example the transfer matrix between input u and state x (see [Cottenceau et al., 2006]) for the source code) is given below (where $r = [14\gamma^3]$) :

$$A^*B = \begin{bmatrix} 1\gamma^0 \oplus 6\gamma^2 \oplus (12\gamma^4 \oplus 18\gamma^6)r^* & (14\gamma^2 \oplus 20\gamma^4)r^* \\ 4\gamma^0 \oplus (10\gamma^2 \oplus 16\gamma^4)r^* & (12\gamma^0 \oplus 18\gamma^2)r^* \\ (13\gamma^5 \oplus 19\gamma^7)r^* & 1\gamma^0 \oplus (15\gamma^3 \oplus 21\gamma^5)r^* \\ (6\gamma^2 \oplus 12\gamma^4)r^* & (8\gamma^0 \oplus 14\gamma^2)r^* \end{bmatrix} \tag{9}$$

The entries of transfer matrices are actually periodic and causal series. Below, we recall the definition and properties of such series.

Definition 4 (Periodicity): A series $s \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is said to be periodic if it can be written as $s = p \oplus qr^*$, with $p = \bigoplus_{i=1}^{m} t_i\gamma^{n_i}$, $q = \bigoplus_{j=1}^{n} T_j\gamma^{N_j}$ are polynomials and $r = \tau\gamma^\nu$, with ν,$\tau \in \mathbb{N}$, is a monomial depicting the asymptotic slope of the series. Polynomial p depicts the transient behavior of the series, polynomial q represents a pattern which is repeated each ν events and τ time units. The ratio $\sigma_\infty(s) = \nu/\tau$ is the production rate of the series. A matrix is said to be periodic if its entries are periodic.

Sum, product and residuation of periodic series are well defined and algorithms and software toolboxes are available in ([Cottenceau et al., 2006]).

Theorem 5: ([MaxPlus, 1991] [LeCorronc et al., 2009]) Let s_1 and s_2 be two series of $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ and $s_1 \preceq s_2$. The series $s_2 \circ\!\!\!\backslash s_1$ is called the correlation of s_1 with s_2 and contains the maximum distances between s_1 and s_2 in the event domain and in time domain. More precisely, monomial $0\gamma^{\nu}$ of series $s_2 \circ\!\!\!\backslash s_1$ gives the maximal event distance ν between s_1 and s_2 also denoted by:

$$\nu = \Delta_\gamma(s_1, s_2) = \min\{n \mid \gamma^n v \preceq s_2\}, \tag{10}$$

whereas $-\tau\gamma^0$ provides the maximal time distance τ denoted by:

$$\tau = \Delta_\delta(s_1, s_2) = \min\{t \mid ts_1 \succeq s_2\}. \tag{11}$$

It is possible that $s_2 \circ\!\!\!\backslash s_1 = \varepsilon$. In such a case, the distances $\Delta_\gamma(s_1, s_2)$ and $\Delta_\delta(s_1, s_2)$ are infinite.

Example 6: Let $u = 1\gamma^1 \oplus 3\gamma^3(2\gamma^3)^*$ and $v = 1\gamma^1 \oplus 4\gamma^2 \oplus 6\gamma^4(2\gamma^3)^*$ be two series of $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ (see Fig. 2). The correlation of u with v is the following series (see Fig. 2): $v \circ\!\!\!\backslash u = \mathbf{-3\gamma^0} \oplus -1\gamma^2 \oplus \mathbf{1\gamma^5}(2\gamma^3)^*$. The maximum event distance is $\Delta_\gamma(u, v) = 5$, while the maximum time distance is $\Delta_\delta(u, v) = 3$.

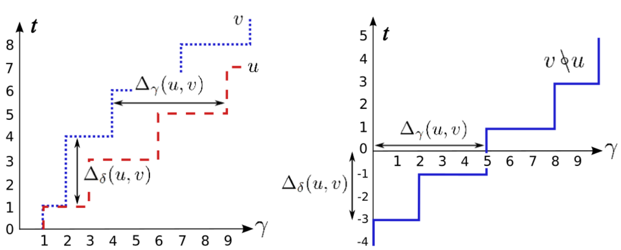

Fig. 2. Maximal distances in event and time domains between two series of $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$, u and v are given on the left and $v \circ\!\!\!\backslash u$ on the right.

Definition 5 (Causality): A series $s \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ is causal if $s = \varepsilon$ or if both $val(s) \geq 0$ and $s \succeq \gamma^{val(s)}$. A matrix is causal if its entries are causal.

Definition 6 (Realizability): A series $s \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ is said to be realizable if there exists three matrices A, B and C with entries in $\mathbb{N} \cup \{-\infty, +\infty\}$ such that $s = C(\gamma A)^* B$. A matrix is said to be realizable if its entries are realizable.
In other words, a series s is realizable if it corresponds to the transfer relation of a timed event graph. A transfer matrix H is realizable if each entry is realizable.

Theorem 6: Let $H \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^{q\times p}$ be a matrix with entries in $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$. The following statement holds (see [Baccelli et al., 1992], [Hardouin et al., 2017]) :

$$H \text{ is realizable } \Leftrightarrow H \text{ is periodic and causal.}$$

Definition 7 (Semiring $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+$): The set of causal elements of $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ is a semiring denoted $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+$.
It must be noted that $\varepsilon, e \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+ \subset \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ and that $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+$ is closed for laws $\oplus$ and $\otimes$, and also for infinite sums too. Hence $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+$ is a complete subsemiring of $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$.

Theorem 7: The canonical injection $\mathsf{Id}_{|\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+} : \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+ \to \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ is residuated and its residual is denoted $\mathsf{Pr}_+ : \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket \to \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+$.
$\mathsf{Pr}_+(s)$ is the greatest causal series less than or equal to the series $s \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$. From a practical point of view, for all series $s \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$, the computation of $\mathsf{Pr}_+(s)$ is obtained by the following:

$$\mathsf{Pr}_+(s) = \mathsf{Pr}_+\left(\bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k\right) = \bigoplus_{k\in\mathbb{Z}} s_+(k)\gamma^k$$

where

$$s_+(k) = \begin{cases} s(k) & \text{if } (k, s(k)) \geq (0, 0), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Example 7: Let $s = -5\gamma^{-1}(3\gamma^2)^* \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket$ be a periodic series. It can be written $s = -5\gamma^{-1} \oplus -2\gamma^1 \oplus 1\gamma^3 \oplus 4\gamma^5 \oplus \ldots$, hence the causal projection $s_+ = \mathsf{Pr}_+(s) = 1\gamma^3 \oplus 4\gamma^5 \oplus \ldots = 1\gamma^3(3\gamma^2)^*$ is the greatest series in $\overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^+$ such that $s_+ \preceq s$.

IV. Control Problem

TEG control consists in computing the firing of input transitions. A common control objective is the just-in-time control of system inputs, which aims to minimize the internal stock of tokens in the TEG while reaching a given desired behavior.

An interesting control strategy is known as the neutral control which aims at preserving the best system's input-output performance while delaying as much as possible the inputs. An example is the design of two neutral controllers $P \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^{p\times p}$ and $F \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^{p\times m}$, such that $u = P(v \oplus Fy)$, where $v \in \overline{\mathbb{Z}}_{\max}\llbracket\gamma\rrbracket^p$. Formally, the problem can be expressed as computing the greatest controllers such that:

$y \preceq CA^*Bv$ with $y = Cx$ and $x = Ax \oplus BP(v \oplus Fy) = A^*BP(v \oplus Fy) = A^*BPv \oplus A^*BPFy$ (owing to Theorem 1), leading to $y = CA^*BPv \oplus CA^*BPFy = H_{yv}v$. By considering again Theorem 1 and property 1, the following equalities hold

$$\begin{aligned} y &= (CA^*BPF)^*CA^*BPv \\ &= CA^*BP(FCA^*BP)^*v = H_{yv}v \end{aligned}$$

where H_{yv} is the input/output transfer matrix.

Therefore, the problem can be formulated as: $CA^*BP(FCA^*BP)^* \preceq CA^*B$ which implies that $CA^*BP \preceq CA^*B$ (since $(FCA^*BP)^* \succeq e$). Therefore, considering the residuation of the mapping Λ_{CA^*B} we get $P \preceq (CA^*B) \circ\!\!\!\backslash (CA^*B) = P_{opt}$, where P_{opt} is the upper bound for the controller P. Then the control strategy has to respect the inequality $CA^*BP_{opt}(FCA^*BP_{opt})^* \preceq CA^*B$ and the optimal F is obtained by considering the following equivalences :

$$\begin{aligned}
& CA^*BP_{opt}(FCA^*BP_{opt})^* \preceq CA^*B \\
\Leftrightarrow\ & P_{opt}(FCA^*BP_{opt})^* \preceq (CA^*B)\backslash\!\!\circ(CA^*B) = P_{opt} \\
\Leftrightarrow\ & (FCA^*BP_{opt})^* \preceq P_{opt}\backslash\!\!\circ P_{opt} \\
\Leftrightarrow\ & (FCA^*BP_{opt})^* \preceq (P_{opt}\backslash\!\!\circ P_{opt})^* \text{(see Properties 2)} \\
\Leftrightarrow\ & (FCA^*BP_{opt}) \preceq P_{opt}\backslash\!\!\circ P_{opt} \text{(see Example 3)} \\
\Leftrightarrow\ & F \preceq (P_{opt}\backslash\!\!\circ P_{opt})/\!\!\circ(CA^*BP_{opt}) = F_{opt}.
\end{aligned}$$

The matrices P_{opt}, F_{opt} are not necessarily causal (hence not realizable), to be realized, they have to be projected in the causal set (see Theorem 7). The matrices $P^+_{opt} = \mathsf{Pr}_+(P^+_{opt})$ and $F^+_{opt} = \mathsf{Pr}_+(F_{opt})$ are called optimal neutral controllers.

In this paper, another control strategy is considered. Its purpose is to design a control law to ensure that the state trajectories satisfy a two-sided equation $Nx = Mx$ where matrices $N, M \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times n}$ characterize some constraints between internal transitions of the TEG. The set of solutions $\mathcal{X} = \{x|Mx = Nx\}$ defined in the semiring $\overline{\mathbb{Z}}_{\max}$ is studied in [Butkovič and Hegedus, 1984], [Allamigeon et al., 2010], where it is proven to be finitely generated and algorithms are given to obtain its generators. In semiring $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, if all entries in matrices M and N are monomials with a same exponent[2] these algorithms can be adapted to obtain the generators.

Alternatively, Corollary 3 can be invoked to find the greatest solution of the set $\mathcal{X}$. Formally, the problem can be formulated as computing the greatest causal P and F in order to ensure that the state $x = A^*BP(FCA^*BP)^*v$ satisfy $Nx = Mx$. The constraint expresses the desired relation between internal transitions. Furthermore, P and F have to be causal to be realizable, that is, $\mathsf{Pr}_+(P) = P$ and $\mathsf{Pr}_+(F) = F$. We first consider the open-loop problem, that is, the computation of the greatest matrix P which satisfies the following constraints $NA^*BP = MA^*BP$ (since $x = A^*BPv$). According to Corollary 3, solution can be obtained by considering the following implicit function :

$$\begin{aligned}
f(P) = \quad & \mathsf{Pr}_+(P) \wedge (NA^*B)\backslash\!\!\circ(MA^*BP) \\
& \wedge(MA^*B)\backslash\!\!\circ(NA^*BP) \qquad (12)
\end{aligned}$$

The algorithm proposed in Theorem 4 can be considered to find the greatest solution. Initial value P_0 can be chosen as an expected upper bound for example $P_0 = P_{0M} \oplus P_{0N}$ with $P_{0M} = (MA^*B)\backslash\!\!\circ(MA^*B)$ the greatest neutral controller such that $MA^*BP_{0M} = MA^*B$ and $P_{0N} = (MA^*B)\backslash\!\!\circ(MA^*B)$ the greatest neutral controller such that $NA^*BP_{0N} = NA^*B$. This algorithm converges if there is a finite solution, but unfortunately, the number of steps required to achieve convergence is not necessarily finite, even though this is the case for most practical applications. It is still an open question to establish conditions which ensure convergence in a finite number of steps.

The optimal solution P^+_{opt} is the greatest such that $P^+_{opt} \preceq P_0$ and $NA^*BP^+_{opt} = MA^*BP^+_{opt}$ and $P^+_{opt} = \mathsf{Pr}_+(P^+_{opt})$.

The closed-loop control $u = P^+_{opt}(v \oplus Fy)$ can then be considered. This control strategy yields the following state expression

$$x = A^*BP^+_{opt}(FCA^*BP^+_{opt})^*v.$$

The control problem is then to find the optimal feedback F^+_{opt} such that :

$$\begin{aligned}
& MA^*BP^+_{opt}(F^+_{opt}CA^*BP^+_{opt})^* \\
& = NA^*BP^+_{opt}(F^+_{opt}CA^*BP^+_{opt})^*.
\end{aligned}$$

Before we address this problem, we consider the following proposition.

Lemma 1: All feedback controllers F such that

$$F \preceq (A^*BP^+_{opt})\backslash\!\!\circ(A^*BP^+_{opt})/\!\!\circ(CA^*BP^+_{opt}) = F_{opt}$$

are such that $A^*BP^+_{opt}(FCA^*BP^+_{opt})^* = A^*BP^+_{opt}$.

Proof: First, let us note that $\forall F$, $A^*BP^+_{opt}(FCA^*BP^+_{opt})^* \succeq A^*BP^+_{opt}$.

Second the following equivalences hold :

$$\begin{aligned}
& A^*BP^+_{opt}(FCA^*BP^+_{opt})^* \preceq A^*BP^+_{opt} \\
\Leftrightarrow\ & (FCA^*BP^+_{opt})^* \preceq (A^*BP^+_{opt})\backslash\!\!\circ(A^*BP^+_{opt}) \\
\Leftrightarrow\ & (FCA^*BP^+_{opt}) \preceq (A^*BP^+_{opt})\backslash\!\!\circ(A^*BP^+_{opt}) \\
\Leftrightarrow\ & F \preceq (A^*BP^+_{opt})\backslash\!\!\circ(A^*BP^+_{opt})/\!\!\circ(CA^*BP^+_{opt}) = F_{opt}
\end{aligned}$$

Hence all $F \preceq F_{opt}$ achieves the equality. ■

Proposition 1: The optimal filter P^+_{opt} associated with the feedback controller $F^+_{opt} = \mathsf{Pr}_+(F_{opt})$ ensures that :

$$Nx = Mx.$$

Proof: According to Lemma 1, $F^+_{opt} = \mathsf{Pr}_+(F_{opt})$ ensures equality $A^*BP^+_{opt}(F^+_{opt}CA^*BP^+_{opt})^* = A^*BP^+_{opt}$. Furthermore, P^+_{opt}, the greatest solution of Eq.12, is such that $NA^*BP^+_{opt} = MA^*BP^+_{opt}$, which implies $Nx = Mx, \quad \forall v$ since $x = A^*BP^+_{opt}(F^+_{opt}CA^*BP^+_{opt})^*v$. ■

V. Application

To illustrate the approach, we consider the system represented in Fig. 1, with transfer relations provided in Example 5. The constraint matrix $N \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{4\times 4}$ is chosen as the identity matrix and

$$M = \begin{bmatrix} (5\gamma^1)^* & -6 & \varepsilon & \varepsilon \\ \varepsilon & (5\gamma^1)^* & \varepsilon & \varepsilon \\ \varepsilon & -12 & (5\gamma^1)^* & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & (5\gamma^1)^* \end{bmatrix}.$$

This matrix of periodic series represents an elementary set of constraints that the controlled system must satisfy. These constraints can be explained as follows. The diagonal entries are equal to $(5\gamma^1)^*$, which means that the asymptotic slope of series associated to each transition must be such that the production rate be equal to one event each 5 time units. In the first row, entry $M_{12} = -6$ requires that $x_1 \succeq -6 \otimes x_2$. This means that the difference between the firing trajectories of transition x_2 and x_1 must be less than or equal to 6 time units. Similarly, the entry $M_{32} = -12$ requires that the difference between x_2 and x_3 must be less than or equal to 12 time units. By iterating the implicit Eq. (12) and by initializing the algorithm with (where $r^* = [5\gamma^1]^*$):

[2] This latest assumption is not restrictive, as it is always possible to increase the size of the state space to comply with it.

$$P_0 = ((MA^*B) \mathbin{\circ\!\!\!\!\backslash} (MA^*B)) \oplus ((NA^*B) \mathbin{\circ\!\!\!\!\backslash} (NA^*B)) =$$

$$\begin{bmatrix} (0\gamma^0 \oplus 6\gamma^2 \oplus 12\gamma^4) \oplus (21\gamma^5)r^* & (0\gamma^0 \oplus 5\gamma^1) \oplus (13\gamma^2)r^* \\ (3\gamma^4) \oplus (12\gamma^5)r^* & (0\gamma^0 \oplus 8\gamma^2 \oplus 14\gamma^3) \oplus (18\gamma^4)r^* \end{bmatrix},$$

the convergence in two steps yields :

$$P^+_{opt} = \begin{bmatrix} (12\gamma^4) \oplus (21\gamma^5)r^* & (5\gamma^1) \oplus (13\gamma^2)r^* \\ (3\gamma^4) \oplus (12\gamma^5)r^* & (0\gamma^1) \oplus (8\gamma^2)r^* \end{bmatrix},$$

then the feedback $F^+_{opt} = \mathsf{Pr}_+(P_{opt} \mathbin{\circ\!\!\!\!\backslash} P_{opt} \mathbin{\circ\!\!\!/} (CA^*BP_{opt}))$ is :

$$F^+_{opt} = \begin{bmatrix} (1\gamma^1)r^* \\ (3\gamma^3)r^* \end{bmatrix}.$$

The transfer relation between v and x is then given by $H_{xv} = A^*BP_{opt}(F^+_{opt}CA^*BP_{opt})^*$ with

$$H_{xv} = \begin{bmatrix} (13\gamma^4) \oplus (22\gamma^5)r^* & (6\gamma^1) \oplus (14\gamma^2)r^* \\ (16\gamma^4) \oplus (25\gamma^5)r^* & (12\gamma^1) \oplus (20\gamma^2)r^* \\ (4\gamma^4) \oplus (13\gamma^5)r^* & (1\gamma^1) \oplus (9\gamma^2)r^* \\ (11\gamma^4) \oplus (20\gamma^5)r^* & (8\gamma^1 \oplus (16\gamma^2)r^* \end{bmatrix}$$

To illustrate the influence of the controllers, we can focus on transitions x_2 and x_3, the constraint M_{32} implies that each firing x_2 must not occur later than 12 time units later than the corresponding firing of x_3. Without controller the state are $x_2 = (A^*B)_{2,1}u_1 \oplus (A^*B)_{2,2}u_2$ and $x_3 = (A^*B)_{3,1}u_1 \oplus (A^*B)_{3,2}u_2$. In order to obtain an upper bound for the duration a token can spend in the place, we can compute $(A^*B)_{3,1} \mathbin{\circ\!\!\!\!\backslash} (A^*B)_{2,1} = (9\gamma^5 \oplus 15\gamma^7)[14\gamma^3]^*$ leading to a maximal delay $\tau_1 = +\infty$ and $(A^*B)_{3,2} \mathbin{\circ\!\!\!\!\backslash} (A^*B)_{2,2} = (-17\gamma^0 \oplus -11\gamma^1) \oplus (3\gamma^3 \oplus 9\gamma^5)[14\gamma^3]^*$ leading to $\tau_2 = 17$, that is the difference in firing date between x_3 and x_2 can be infinite, and does not respect the constraint. With controllers the state are $x_2 = H_{xv_{2,1}}u_1 \oplus H_{xv_{2,2}}u_2$, and $x_3 = (H_{xv})_{3,1}u_1 \oplus (H_{xv})_{3,2}u_2$. Then $(H_{xv})_{2,1} \mathbin{\circ\!\!\!\!\backslash} (H_{xv})_{3,1} = (-12\gamma^0)[5\gamma^1]^*$ leads to a maximal delay $\tau_1 = 12$ and $H_{xv_{2,2}} \mathbin{\circ\!\!\!\!\backslash} H_{xv_{3,2}} = (-11\gamma^0)[5\gamma^1]^*$ leads to $\tau_2 = 11$, that is the difference in firing date between x_3 and x_2 is smaller than or equal to 12, *i.e.*, the constraint is respected. This example illustrates the effectiveness of the proposed approach. The constraint given in this example can be useful in case of systems with strong sojourn time constraints such as parts crossing an oven (a bakery example is given in [Zorzenon et al., 2023]). More complex constraints may obviously be considered, the only limit is the existence of a solution in order to ensure convergence of the fixed-point algorithm.

VI. Conclusion and Future Work

This paper reports upon recent investigations on the control for max-plus linear systems for respecting constraints on the subspace of the state space depicted by an equation $Mx = Nx$ where each entry is given by a periodic series. The open- and closed-loop controllers working on infinite horizon are provided. In a future work design method based on sufficient condition ensuring the existence of controllers should be addressed.

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