Disturbance Decoupling of Timed Event Graphs by Output Feedback Controller

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Abstract

This paper deals with the closed loop controller design for (max,+)-linear systems (Timed Event Graphs in the Petri nets formalism) when some exogenous and uncontrollable inputs disturb the system. The control law ($u = Fx \oplus v$) is designed in order to take into account the disturbance effects in an optimal manner with regards to the just-in-time criterion. This problem is reminiscent of the disturbance decoupling control problem for classical linear systems.

Index Terms

Discrete event systems ; Timed event graphs ; Semiring ; Max-plus algebra ; Disturbance decoupling

I. Introduction

Discrete Event Systems (DES) appear in many applications in manufacturing systems [1], computer and communication networks [14] and are often described by the Petri Net formalism. Timed-Event Graphs (TEG) are Timed Petri Nets in which all places have single upstream and single downstream transitions and appropriately model DES characterized by delay and synchronization phenomena. Twenty-five years ago, TEG behavior was described by linear model in some idempotent semiring (see [5] and also [2], [9], [12]). In the sequel many achievements on the control of TEG arose in [9], [10], [17]. The control strategies of TEG are very reminiscent of the control of classical linear systems, the main difference being due to the algebraic setting. In [2, §5.6] the optimal open-loop control is introduced. In [10], [16] closed-loop control strategies are introduced, the controller design is given in order to achieve model matching problem. This paper proposes to consider control of $(\max,+)$-linear system when some disturbances act on the system. Disturbances are uncontrollable inputs which disable the firing of internal transitions of the TEG and lead to an useless accumulation of tokens inside the graph. The controller aims to avoid, as much as possible, this useless token accumulation[1] without altering the system performances. This is the best that we can do from the just-in-time point of view when these systems are disturbed.

II. Algebraic tools

A. Ordered sets, residuation and idempotent semirings

In this section, we recall some basic notions about partially ordered sets, residuation and idempotent semirings. See [2]–[4] for more details. By *ordered set*, we will mean throughout the paper a set equipped with a partial order. We say that an ordered set $(\mathcal{X}, \preceq)$ is *complete* if any subset $A \subset \mathcal{X}$ has a least upper bound (denoted by $\bigvee A$). In particular, $\mathcal{X}$ has both a minimal (bottom) element $\bot\mathcal{X} = \bigvee \emptyset$, and a maximal (top) element $\top\mathcal{X} = \bigvee \mathcal{X}$. Since the greatest lower bound of a subset $A \subset \mathcal{X}$ can be defined by $\bigwedge A = \bigvee \{x \in \mathcal{X} \mid x \preceq a, \forall a \in A\}$, $\mathcal{X}$ is a complete lattice.

If $(\mathcal{X}, \preceq)$ and $(\mathcal{Y}, \preceq)$ are ordered sets, we say that a map $f : \mathcal{X} \to \mathcal{Y}$ is isotone if $x \preceq x' \Rightarrow f(x) \preceq f(x')$. Map $f : \mathcal{X} \to \mathcal{Y}$ is *residuated* if there exists a map $f^\sharp : \mathcal{Y} \to \mathcal{X}$ such that

$$\forall x \in \mathcal{X}, \forall y \in \mathcal{Y}, \quad f(x) \preceq y \quad \Longleftrightarrow \quad x \preceq f^\sharp(y),$$

which means that for all $y \in \mathcal{Y}$, the set $\{x \in \mathcal{X} \mid f(x) \preceq y\}$ has a maximal element, $f^\sharp(y)$. Let $(\mathcal{X}, \preceq)$ and $(\mathcal{Y}, \preceq)$ be complete ordered sets, map $f : \mathcal{X} \to \mathcal{Y}$ is lower semicontinuous (for short l.s.c.) if for all $A \subset \mathcal{X}, f(\bigvee A) = \bigvee f(A)$, where $f(A) = \{f(a) \mid a \in A\}$. In particular, when $A = \emptyset$, we get $f(\bot\mathcal{X}) = \bot\mathcal{Y}$.

When $(\mathcal{X}, \preceq)$ and $(\mathcal{Y}, \preceq)$ are complete ordered sets, there is a simple characterization of residuated map.

Lemma 1: Let $(\mathcal{X}, \preceq)$ and $(\mathcal{Y}, \preceq)$ be complete ordered sets. A map $f : \mathcal{X} \to \mathcal{Y}$ is residuated if, and only if, it is lower semicontinuous.

Proof: See [4, Th. 5.2] or [2, Th. 4.50]. ■

We now apply these notions to idempotent semirings. Recall that a *semiring* is a set $\mathcal{S}$, equipped with two operations $\oplus, \otimes$, such that $(\mathcal{S}, \oplus)$ is a commutative monoid (the zero element will be

[1]In the manufacturing framework, this corresponds to reduce the work-in-process and to satisfy the just-in-time criterion.

denoted ε), $(\mathcal{S}, \otimes)$ is a monoid (the unit element will be denoted e), operation $\otimes$ is right and left distributive over $\oplus$, and ε is absorbing for the product (*i.e.* $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a$). A semiring $\mathcal{S}$ is *idempotent* if $a \oplus a = a$ for all $a \in \mathcal{S}$.

A non empty subset B of a semiring $\mathcal{S}$ is a subsemiring of $\mathcal{S}$ if for all $a, b \in B$ we have $a \oplus b \in B$ and $a \otimes b \in B$.

In an idempotent semiring $\mathcal{S}$, operation $\oplus$ induces a partial order relation

$$a \succeq b \iff a = a \oplus b, \quad \forall a, b \in \mathcal{S}. \tag{1}$$

Then, $a \vee b = a \oplus b$. We say that an idempotent semiring $\mathcal{S}$ is *complete* if it is complete as an ordered set, and if for all $a \in \mathcal{S}$, the left and right multiplications[2] by a, $L_a : \mathcal{S} \to \mathcal{S}, x \mapsto ax$ and $R_a : \mathcal{S} \to \mathcal{S}, x \mapsto xa$ are l.s.c.. These maps are residuated, then the following notation are considered :

$$L_a^\sharp(b) = a \backslash b = \bigoplus \{x | ax \preceq b\} \quad \text{and} \quad R_a^\sharp(b) = b / a = \bigoplus \{x | xa \preceq b\}, \quad \forall a, b \in \mathcal{S}.$$

The set of $n \times n$ matrices with entries in $\mathcal{S}$ is an idempotent semiring. The sum and product of matrices are defined conventionally after the sum and product of scalars in $\mathcal{S}$, *i.e.*,

$$(A \otimes B)_{ik} = \bigoplus_{j=1 \ldots n} \{A_{ij} \otimes B_{jk}\} \quad \text{and} \quad (A \oplus B)_{ij} = \{A_{ij} \oplus B_{ij}\}.$$

The identity matrix of $\mathcal{S}^{n \times n}$ is the matrix with entries equal to e on the diagonal and to ε elsewhere. This identity matrix will also be denoted e, and the matrix with all its entries equal to ε will also be denoted ε.

The map $L_A : \mathcal{S}^p \to \mathcal{S}^n, x \mapsto Ax$, with A a $n \times p$ matrix, is residuated, the maximal element of set $\{x \in \mathcal{S}^p \mid Ax \preceq b\}$ is denoted $L_A^\sharp(b) = A \backslash b$ with $(A \backslash b)_j = \bigwedge_i A_{ij} \backslash b_i$. Useful results concerning residuation are given in Appendix B.

B. Equivalence kernel and closure residuation

Definition 1 (Kernel [6]–[8]): Let $\mathcal{S}$ be a complete idempotent semiring and let C be a $n \times p$ matrix with entries in $\mathcal{S}$. We call *kernel* of L_C (denoted by $\ker C$), the subset of all pairs of

[2]The symbol $\otimes$ is often omitted.

elements of $\mathcal{S}^p$ whose components are both mapped by L_C to the same element in $\mathcal{S}^n$, *i.e.*, the following definition

$$\ker C \ := \ \left\{(s,s') \in (\mathcal{S}^p)^2 \ \mid Cs = Cs'\right\}. \qquad (2)$$

Clearly $\ker C$, is an equivalence relation on $\mathcal{X}$, $i.e.$, $Cs = Cs' \iff s' \equiv s \ (\operatorname{mod} \ker C)$ and furthermore it is a congruence and then we can define the quotient $\mathcal{S}/\ker C$.

Notation 1: The subset of elements $s' \in \mathcal{S}^p$ that are equivalent to s modulo $\ker C$ is denoted $[s]_C$, $i.e.$,

$$[s]_C = \{s' \in \mathcal{S}^p \mid s' \equiv s(\operatorname{mod} \ker C)\} \subset \mathcal{S}^p.$$

The problem of map restriction and its connection with the residuation theory is now addressed.

Definition 2 (Restricted map): Let $f : \mathcal{S}^p \to \mathcal{S}^n$ be a map and $\mathcal{A} \subseteq \mathcal{S}^p$. We will denote[3] $f_{|\mathcal{A}} : \mathcal{A} \to \mathcal{S}^n$ the map defined by $f_{|\mathcal{A}} = f \circ \mathsf{Id}_{|\mathcal{A}}$ where $\mathsf{Id}_{|\mathcal{A}} : \mathcal{A} \to \mathcal{S}^p, x \mapsto x$ be the canonical injection. Identically, let $\mathcal{B} \subseteq \mathcal{S}^n$ with $\mathsf{Im} f \subseteq \mathcal{B}$. Map ${}_{\mathcal{B}|}f : \mathcal{S}^p \to \mathcal{B}$ is defined by $f = \mathsf{Id}_{|\mathcal{B}} \circ {}_{\mathcal{B}|}f$, where $\mathsf{Id}_{|\mathcal{B}} : \mathcal{B} \to \mathcal{S}^n, x \mapsto x$ be the canonical injection.

Definition 3 (Closure map): An isotone map $f : \mathcal{S}^p \to \mathcal{S}^p$ is a *closure map* if $f \succeq \mathsf{Id}_{\mathcal{S}}$ and $f \circ f = f$.

Proposition 1 ([10]): A closure map $f : \mathcal{S}^p \to \mathcal{S}^p$ restricted to its image ${}_{\mathsf{Im} f|}f$ is a residuated map whose residual is the canonical injection $\mathsf{Id}_{|\mathsf{Im} f} : \mathsf{Im} f \to \mathcal{S}^p$, $s \mapsto s$.

Corollary 1: Let $\mathcal{K} : \mathcal{S}^p \to \mathcal{S}^p, s \mapsto s^*$ be a map, where $s^* = \bigoplus_{i \in \mathbb{N}} s^i$ (see Appendix A for complementary results on map $\mathcal{K}$). The map ${}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}$ is a residuated map whose residual is $\left({}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}\right)^\sharp = \mathsf{Id}_{|\mathsf{Im}\mathcal{K}}$. This means that $x = s^*$ is the greatest solution to inequality $x^* \preceq s^*$. Actually, the greatest solution achieves equality.

III. Application to Timed-Event Graphs control

A. TEG model in idempotent semirings

A trajectory of a TEG transition x is a firing date sequence $\{x(k)\} \in \mathbb{Z}$. For each increasing sequence $\{x(k)\}$, it is possible to define the transformation $x(\gamma) = \bigoplus\limits_{k \in \mathbb{Z}} x(k)\gamma^k$ where γ is a backward shift operator in event domain (*i.e.*, $y(\gamma) = \gamma x(\gamma) \iff \{y(k)\} = \{x(k-1)\}$, (see [2], p. 228). This transformation is analogous to the Z-transform used in discrete-time classical

[3] These notations are borrowed from classical linear system theory see [22].

control theory and the formal series $x(\gamma)$ is a synthetic representation of the trajectory $x(k)$. The set of the formal power series in γ is denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and constitutes an idempotent semiring.

The model considered is given by

$$\begin{cases} x &= Ax \oplus Bu \oplus Sq \\ y &= Cx \end{cases} \tag{3}$$

Where $x \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^n$ is the state vector and each entry $x_i(\gamma)$ represents the behavior of the transition labelled x_i, it is a series ($i.e.$ $x_i(\gamma) = \bigoplus_{k\in\mathbb{Z}} x_i(k)\gamma^k$) which depicts the firing trajectory of the internal transition x_i (with $x_i(k)$ the date of the firing numbered k). Vector $y \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^q$ is the output and each component $y_i(\gamma)$ $(i \in [1, q])$ represents the behavior of the transition labelled y_i, the series $y_i(\gamma)$ depicts the date of the tokens output from this transition. Vector $u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^p$ is the controllable inputs and each component $u_i(\gamma)$ $(i \in [1, p])$ represents the behavior of the transition labelled u_i, the series $u_i(\gamma)$ depicts the date of the tokens input in the TEG. Matrices $A \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n\times n}$, $B \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n\times p}$, $C \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q\times n}$ and $S \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n\times r}$ represent the link between transitions. The trajectories u and y can be related ([2], p. 243) by the equation $y = Hu$, where $H = CA^*B \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q\times p}$ is called the transfer matrix of the TEG. Entries of matrix H are periodic series ([2], p. 260) in the idempotent semiring, usually represented by[4] $p(\gamma) \oplus q(\gamma)(\tau\gamma^\nu)^*$.

The control of a transition u_i means that the firing may be enable or disable, that means, the input date is controlled. Therefore, a control law aims to control the input date of tokens in order to achieve some specifications. A classical specification is to track a trajectory (a reference output sequence) while delaying as much as possible the token input, this strategy consists in computing the optimal control with regard to the well-known just-in-time criterion. Formally, let $z \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^q$ be a given reference output, the problem is to compute the greatest control, denoted $u_{opt} \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^p$ such that $y \preceq z$. Among the controls which respect the constraint $y \preceq z$, u_{opt} is the greatest, *i.e.*, the one which delays as much as possible the input of the tokens in the graph, *i.e.*, this control minimizes in an optimal manner the sojourn time of tokens.

[4] $p(\gamma) = \bigoplus_{i=0}^{n-1} p_i\gamma^i, p_i \in \mathbb{N}$, is a polynomial that represents the transient and $q(\gamma) = \bigoplus_{j=0}^{\nu-1} q_j\gamma^j, q_j \in \mathbb{N}$ is a polynomial that represents a pattern which is repeated each τ time units and each ν firings of the transition.

In [10], [15] closed-loop controllers synthesis, in order to achieve the model matching problem, is addressed. The objective is to compute the greatest closed-loop controller $F \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{p\times n}$ (with $u = Fx \oplus v$) which ensures that output $y \preceq G_{ref}v$, where $G_{ref} \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q\times p}$ is a model to track. This controller leads to a exact model matching if possible and delays as much as possible the input of token while ensuring the constraint ($y \preceq G_{ref}v$).

In this paper a specific design goal is to compute a closed-loop controller F (*i.e.*, $u = Fx \oplus v$) in order to take into account the influence of the uncontrollable input q. An uncontrollable input q_i may disable the firing of the internal transitions bind to q_i through matrix S. Therefore, this uncontrollable input q_i may decreased the performance of the system, *i.e.*, the token output may be delayed, and some tokens may needlessly wait in the graph since the system is blocked. Therefore, the controller design aims to obtain the greatest F which avoid the input of useless tokens. This means that controller F must be the greatest such that the output y, (*i.e.*, with the control $u = Fx \oplus v$) be equal to the output without controller (*i.e.*, with $u = v$), in other words the control must be neutral with regard to the output, *i.e.*, it must not disturb the system more than disturbance q does it. From the just-in-time point of view it is the best that we can do. Formally, thanks to Theorem 1 (see Appendix A), system (3) may be written

$$\left\{\begin{array}{lcl} x &=& A^*Bu \oplus A^*Sq = A^*\left[B \mid S\right]\begin{pmatrix} u \\ q \end{pmatrix} = A^*\overline{B}\begin{pmatrix} u \\ q \end{pmatrix} \\ y &=& CA^*Bu \oplus CA^*Sq = CA^*\overline{B}\begin{pmatrix} u \\ q \end{pmatrix} \end{array}\right. . \tag{4}$$

The objective is to compute the greatest feedback controller F such that the output be the same than with $u = v$, *i.e.*,

$$C(A \oplus BF)^*\overline{B}\begin{pmatrix} v \\ q \end{pmatrix} = CA^*\overline{B}\begin{pmatrix} v \\ q \end{pmatrix}, \ \forall \begin{pmatrix} v \\ q \end{pmatrix}.$$

This equation is equivalent to

$$C(A \oplus BF)^*\overline{B} = CA^*\overline{B} \iff \left((A \oplus BF)^*\,\overline{B}, A^*\overline{B}\right) \in \ker C.$$

The right side of the equivalence shows that F must be such that the transfer between state x and control input $\begin{pmatrix} v & q \end{pmatrix}^t$ be equivalent to $A^*\overline{B}$ modulo $\ker C$. This is very reminiscent to the disturbance decoupling problem for the classical linear system (see [22]), which leads to keep the state in the kernel of output matrix C. Let us note that in [11], [13] the classical concept of (A,B)-invariant space is extended to linear dynamical systems over max-plus semiring (see also [20]).

Proposition 2: The greatest controller F such that, $(A \oplus BF)^* \overline{B} \in \left[A^*\overline{B}\right]_C$ is given by

$$F \;=\; CA^*B \backslash CA^*\overline{B} / A^*\overline{B}. \tag{5}$$

Proof: Let us note that $B = \overline{B}\begin{pmatrix} e \\ \varepsilon \end{pmatrix}$, therefore the problem is to compute the greatest F such that

$$C\left(A \oplus \overline{B}\begin{pmatrix} e \\ \varepsilon \end{pmatrix} F\right)^* \overline{B} \;=\; CA^*\overline{B}. \tag{6}$$

Obviously, $F = \varepsilon$ is solution, then the greatest solution of

$$C\left(A \oplus \overline{B}\begin{pmatrix} e \\ \varepsilon \end{pmatrix} F\right)^* \overline{B} \;\preceq\; CA^*\overline{B}, \tag{7}$$

leads to equality. From (f.1) and (f.3) we have that

$$C\left(A \oplus \overline{B}\begin{pmatrix} e \\ \varepsilon \end{pmatrix} F\right)^* \overline{B} \preceq CA^*\overline{B} \iff CA^*\overline{B}\left(\begin{pmatrix} e \\ \varepsilon \end{pmatrix} FA^*\overline{B}\right)^* \preceq CA^*\overline{B}. \tag{8}$$

By applying the residuation theory (see Section II), we have the following equivalence

$$CA^*\overline{B}\left(\begin{pmatrix} e \\ \varepsilon \end{pmatrix} FA^*\overline{B}\right)^* \preceq CA^*\overline{B} \iff \left(\begin{pmatrix} e \\ \varepsilon \end{pmatrix} FA^*\overline{B}\right)^* \preceq \left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right).$$

Relation (f.7) yields $\left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right) = \left(\left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right)\right)^*$, then $\left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right)$ belongs to image of map $\mathcal{K}$ (see Corollary 1). Since ${}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}$ is residuated (see Corollary 1), we get

$$\left(\begin{pmatrix} e \\ \varepsilon \end{pmatrix} FA^*\overline{B}\right)^* \preceq \left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right) \iff \begin{pmatrix} e \\ \varepsilon \end{pmatrix} FA^*\overline{B} \preceq \left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right).$$

Finally, by using residuation theory and (f.5), we obtain

$$F = \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \backslash \left(CA^*\overline{B}\right) \backslash \left(CA^*\overline{B}\right) / \left(A^*\overline{B}\right) = \left(CA^*\overline{B}\begin{pmatrix} e \\ \varepsilon \end{pmatrix}\right) \backslash \left(CA^*\overline{B}\right) / \left(A^*\overline{B}\right)$$

$$= \left(CA^*B\right) \backslash \left(CA^*\overline{B}\right) / \left(A^*\overline{B}\right). \tag{9}$$

■

Remark 1: As in classical control linear system theory, controller synthesis, when disturbance acts on the system, may be seen as a particular model matching problem [18], [21].

Remark 2: Controller F is the greatest such that

$$(A \oplus BF)^* \overline{B} \in \mathsf{Im} A^*\overline{B} \cap \left[A^*\overline{B}\right]_C,$$

where $\mathsf{Im} A^*\overline{B} = \left\{A^*\overline{B}u \mid u \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^p\right\}$. Indeed, it is sufficient to note, thanks to (f.1) and (f.3), that $(A \oplus BF)^*\overline{B} = \left(A^*\overline{B}\begin{pmatrix} e \\ \varepsilon \end{pmatrix} F\right)^* A^*\overline{B} = A^*\overline{B}\left(\begin{pmatrix} e \\ \varepsilon \end{pmatrix} FA^*\overline{B}\right)^*$, clearly $(A \oplus BF)^* \overline{B} \in \mathsf{Im} A^*\overline{B}$.

IV. Illustration

The example considered here is borrowed to the manufacturing setting but may be transposed easily to transport system [12] or network system [14]. The TEG depicted figure 1 may represent a workshop with 3 machines (M_1 to M_3). Machine M_1 processes 2 parts simultaneously, each processing lasts 4 times units. Machine M_3 processes the parts released by machines M_1 and M_2. Transitions q_1, q_2 and q_3 are uncontrollable inputs (disturbances), which delay the parts output of machines M_1, M_2 and M_3. In a manufacturing context, inputs q may represent machine breakdowns, uncontrollable supplies of raw materials, $\ldots$. For this example, the matrices of model (3) are given by

$$A = \begin{pmatrix} 4\gamma^2 & \varepsilon & \varepsilon \\ \varepsilon & 6\gamma^2 & \varepsilon \\ 7 & 7 & 6\gamma \end{pmatrix}, B = \begin{pmatrix} 6 & \varepsilon \\ \varepsilon & 9 \\ \varepsilon & \varepsilon \end{pmatrix}, S = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix} \text{ and } C = \begin{pmatrix} \varepsilon & \varepsilon & 1 \end{pmatrix},$$

where, for each entry, the exponent in γ denotes the tokens number in the place and the coefficient depicts the processing time. This yields to the following transfer between output y and disturbance q (respectively input u)

$$CA^*S = \begin{pmatrix} 8(6\gamma)^* & 8(6\gamma)^* & 1(6\gamma)^* \end{pmatrix} \quad \text{and} \quad CA^*B = \begin{pmatrix} 14(6\gamma)^* & 17(6\gamma)^* \end{pmatrix},$$

each component of these matrices is a periodic series. The example has been computed by using toolbox `MinMaxGD` which runs with Scilab (see [19]).

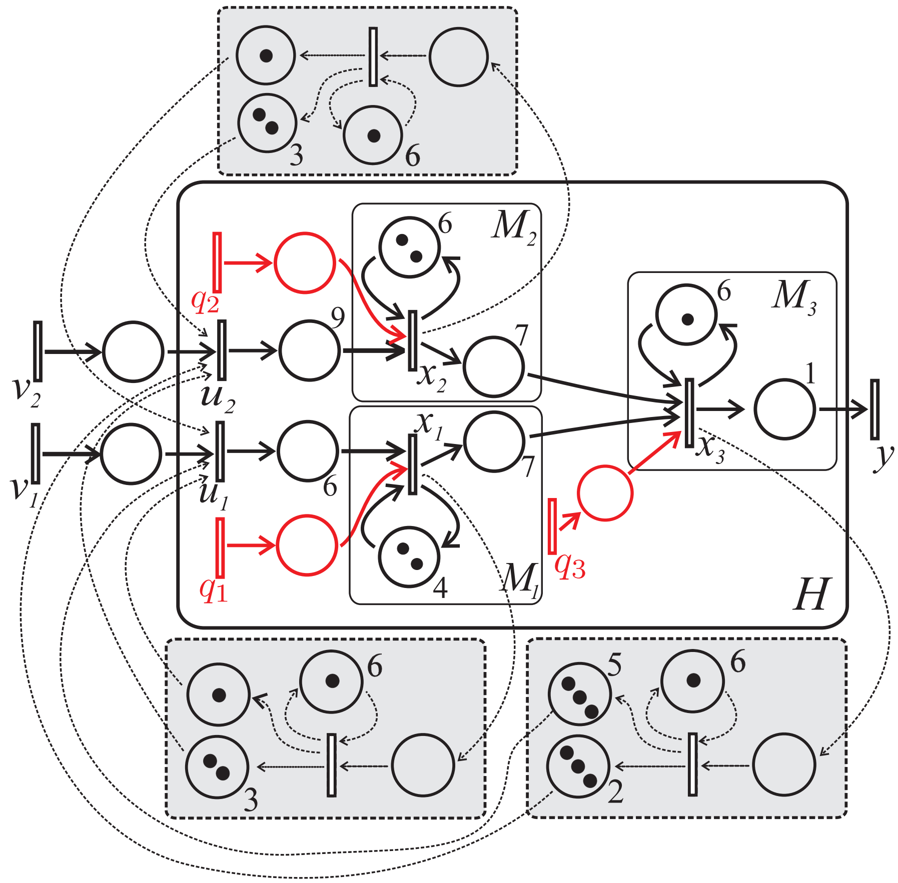

Fig. 1: System in bold lines and controller in dotted lines

According to Proposition 2 and solution (5), the controller is obtained by computing $CA^*B \circ\!\!\!\!\backslash\, CA^*\overline{B} \circ\!\!\!\!/\, A^*\overline{B}$.

Therefore, we obtain

$$F \quad = \quad CA^*B \lrcorner CA^*\overline{B} \not\!/ A^*\overline{B} = \begin{pmatrix} -6(6\gamma)^* & -6(6\gamma)^* & -13(6\gamma)^* \\ -9(6\gamma)^* & -9(6\gamma)^* & -16(6\gamma)^* \end{pmatrix}.$$

This feedback is not causal because there are negative coefficients in the matrix (see [2, Def. 5.35] for a strict definition of causality). The canonical injection from the causal elements of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ (denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^+$) in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is also residuated (see [10] for details). Its residual is given by $\mathsf{Pr} = \left(\bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k\right) = \bigoplus_{k\in\mathbb{Z}} s_+(k)\gamma^k$ where

$$s_+(k) = \begin{cases} s(k) & \text{if} \quad (k, s(k)) \geq (0,0), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Therefore, the greatest causal feedback is

$$F_+ \quad = \quad \mathsf{Pr}(F) = \begin{pmatrix} \gamma(6\gamma)^* & \gamma(6\gamma)^* & 5\gamma^3(6\gamma)^* \\ 3\gamma^2(6\gamma)^* & 3\gamma^2(6\gamma)^* & 2\gamma^3(6\gamma)^* \end{pmatrix}. \tag{10}$$

Figure 1 shows a realization of the controller (bold dotted lines).

In order to simulate the system, following input v is considered

$$v \quad = \quad \begin{pmatrix} 20 \oplus +\infty\gamma^6 \\ 20 \oplus +\infty\gamma^6 \end{pmatrix}.$$

It means that 6 tokens are available at time $t = 20$. First the system is assumed to be not disturbed, *i.e.*, $q = \varepsilon$. The system trajectories without controller ($F = \varepsilon$, then $u = v$, *i.e.*, the open-loop behavior), denoted u_{ol}, x_{ol} and y_{ol}, are given by

$$\begin{aligned} u_{ol} \quad &= \quad v, \\ x_{ol} \quad &= \quad A^*Bv = \begin{pmatrix} 26 \oplus 30\gamma^2 \oplus 34\gamma^4 \oplus +\infty\gamma^6 \\ 29 \oplus 35\gamma^2 \oplus 41\gamma^4 \oplus +\infty\gamma^6 \\ 36 \oplus 42\gamma \oplus 48\gamma^2 \oplus 54\gamma^3 \oplus 60\gamma^4 \oplus 66\gamma^5 \oplus +\infty\gamma^6 \end{pmatrix}, \\ \text{and } y_{ol} \quad &= \quad CA^*Bv = 37 \oplus 43\gamma \oplus 49\gamma^2 \oplus 55\gamma^3 \oplus 61\gamma^4 \oplus 67\gamma^5 \oplus +\infty\gamma^6. \end{aligned}$$

With controller F_+ (*i.e.* $u = Fx \oplus v$) these trajectories, denoted u_{cl}, x_{cl} and y_{cl}, become

$$\begin{aligned} u_{cl} \quad &= \quad (F_+A^*B)^*v = \begin{pmatrix} 20 \oplus 29\gamma \oplus 35\gamma^2 \oplus 41\gamma^3 \oplus 47\gamma^4 \oplus 53\gamma^5 \oplus +\infty\gamma^6 \\ 20 \oplus \oplus 32\gamma^2 \oplus 38\gamma^3 \oplus 44\gamma^4 \oplus 50\gamma^5 \oplus +\infty\gamma^6 \end{pmatrix}, \\ x_{cl} \quad &= \quad (A \oplus BF_+)^*Bv = \begin{pmatrix} 26 \oplus 35\gamma \oplus 41\gamma^2 \oplus 47\gamma^3 \oplus 53\gamma^4 \oplus 59\gamma^5 \oplus +\infty\gamma^6 \\ 29 \oplus 41\gamma^2 \oplus 47\gamma^3 \oplus 53\gamma^4 \oplus 59\gamma^5 \oplus +\infty\gamma^6 \\ 36 \oplus 42\gamma \oplus 48\gamma^2 \oplus 54\gamma^3 \oplus 60\gamma^4 \oplus 66\gamma^5 \oplus +\infty\gamma^6 \end{pmatrix} \\ \text{and } y_{cl} \quad &= \quad C(A \oplus BF_+)^*Bv = 37 \oplus 43\gamma \oplus 49\gamma^2 \oplus 55\gamma^3 \oplus 61\gamma^4 \oplus 67\gamma^5 \oplus +\infty\gamma^6. \end{aligned}$$

Clearly, the output trajectories are equal $y_{cl} = y_{ol}$ and $u_{cl} \succeq u_{ol}, x_{cl} \succeq x_{ol}$, *i.e.*, controller F_+ is neutral in regards to the output, but delay as much as possible the tokens input.

In a second step, the system is assumed to be disturbed, with $q = \begin{pmatrix} \varepsilon & 85\gamma^3 & \varepsilon \end{pmatrix}^t$. Entry $q_2 = 85\gamma^3$ means that the fourth firing occurs at time 85. This may represents a machine breakdown occurring after the third part be processed and this breakdown lasts until time 85. The system trajectories without controller ($u = v$), denoted u_{olq}, x_{olq} and y_{olq}, become

$$\begin{aligned}
u_{olq} &= v \\
x_{olq} &= A^*Bv \oplus A^*Sq = \begin{pmatrix} 26 \oplus 30\gamma^2 \oplus 34\gamma^4 \oplus +\infty\gamma^6 \\ 29 \oplus 35\gamma^2 \oplus 85\gamma^3 \oplus 91\gamma^5 \oplus +\infty\gamma^6 \\ 36 \oplus 42\gamma \oplus 48\gamma^2 \oplus 92\gamma^3 \oplus 98\gamma^4 \oplus 104\gamma^5 \oplus +\infty\gamma^6 \end{pmatrix}, \\
\text{and } y_{olq} &= CA^*Bv \oplus CA^*Sq = 37 \oplus 43\gamma \oplus 49\gamma^2 \oplus 93\gamma^3 \oplus 99\gamma^4 \oplus 105\gamma^5 \oplus +\infty\gamma^6.
\end{aligned}$$

Obviously, this machine breakdown delay the firing of transitions x_2 and x_3 (see Figure 2), indeed $x_{olq} \succeq x_{ol}$ and $y_{olq} \succeq y_{ol}$. With controller F_+, these trajectories, denoted u_{clq}, x_{clq} and y_{clq} become

$$\begin{aligned}
u_{clq} &= (F_+A^*B)^*v \oplus F_+(A \oplus BF_+)^*Sq = \begin{pmatrix} 20 \oplus 29\gamma \oplus 35\gamma^2 \oplus 41\gamma^3 \oplus 85\gamma^4 \oplus 91\gamma^5 \oplus +\infty\gamma^6 \\ 20 \oplus 32\gamma^2 \oplus 38\gamma^3 \oplus 44\gamma^4 \oplus 88\gamma^5 \oplus +\infty\gamma^6 \end{pmatrix}, \\
x_{clq} &= (A \oplus BF_+)^*Bv \oplus (A \oplus BF_+)^*Sq = \begin{pmatrix} 26 \oplus 35\gamma \oplus 41\gamma^2 \oplus 47\gamma^3 \oplus 91\gamma^4 \oplus 97\gamma^5 \oplus +\infty\gamma^6 \\ 29 \oplus 41\gamma^2 \oplus 85\gamma^3 \oplus 97\gamma^5 \oplus +\infty\gamma^6 \\ 36 \oplus 42\gamma \oplus 48\gamma^2 \oplus 92\gamma^3 \oplus 98\gamma^4 \oplus 104\gamma^5 \oplus +\infty\gamma^6 \end{pmatrix}, \\
\text{and } y_{clq} &= C(A \oplus BF_+)^*Bv \oplus C(A \oplus BF_+)^*Sq = 37 \oplus 43\gamma \oplus 49\gamma^2 \oplus 93\gamma^3 \oplus 99\gamma^4 \oplus 105\gamma^5 \oplus +\infty\gamma^6.
\end{aligned}$$

The output $y_{clq} = y_{olq}$, *i.e.*, the controller F_+ does not disturb the system, nevertheless $x_{clq} \succeq x_{olq}$ and $u_{clq} \succeq u_{olq}$ this means that the tokens input is delayed. Furthermore this is done in an optimal manner, then the input of useless tokens is avoid.

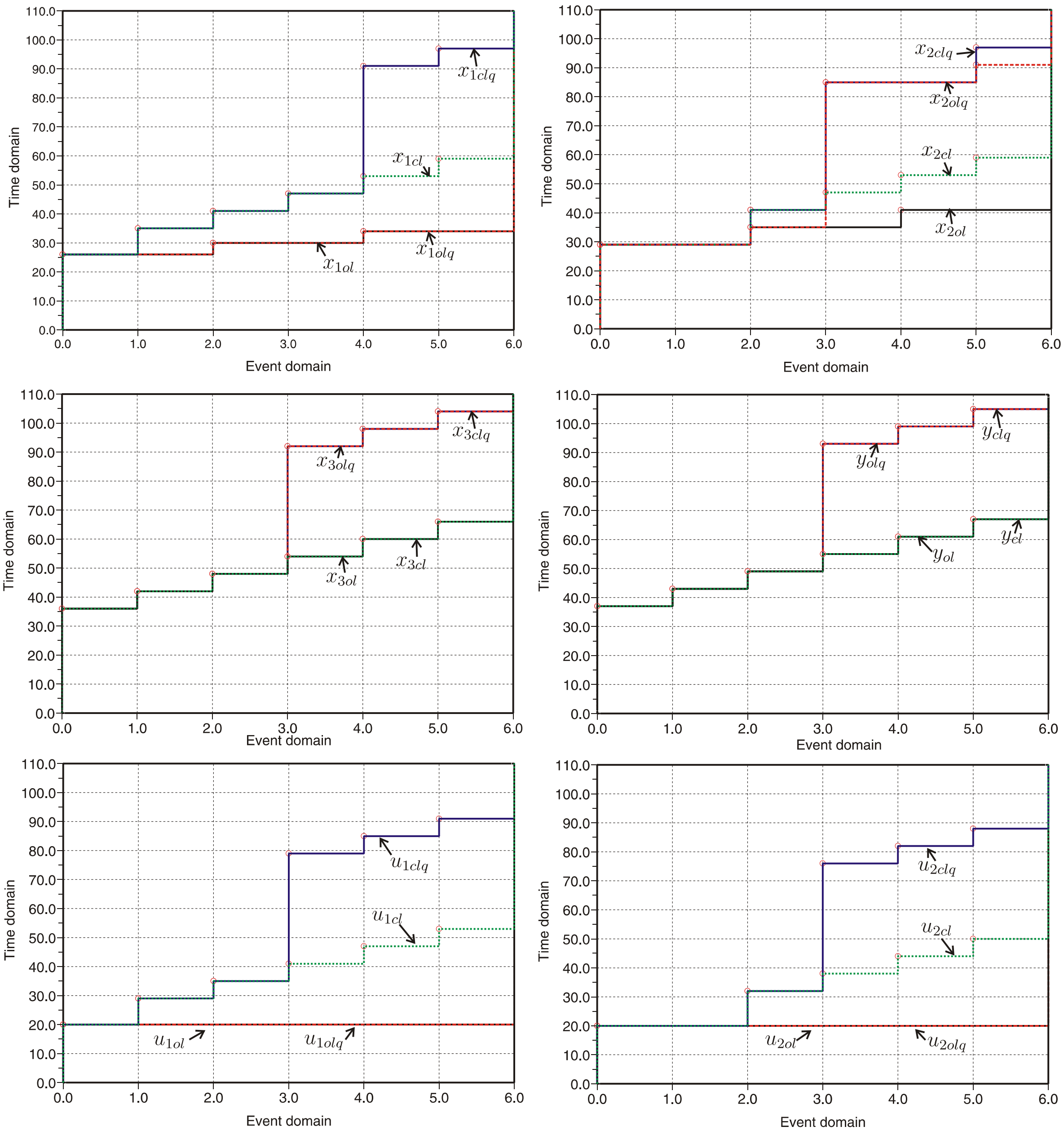
Time domain
Event domain
x_{1clq}
x_{1cl}
x_{1olq}
x_{1ol}
x_{2clq}
x_{2olq}
x_{2cl}
x_{2ol}
x_{3clq}
x_{3olq}
x_{3cl}
x_{3ol}
y_{clq}
y_{olq}
y_{cl}
y_{ol}
u_{1clq}
u_{1cl}
u_{1ol}
u_{1olq}
u_{2clq}
u_{2cl}
u_{2ol}
u_{2olq}

Fig 2: Responses of the system without disturbance $\Big($ where $x_{ol} = (x_{1ol}\ x_{2ol}\ x_{3ol})^t$ (resp. $x_{cl} = (x_{1cl}\ x_{2cl}\ x_{3cl})^t$) is the open-loop state (resp. closed-loop state) $\Big)$ and the system with disturbance $\Big($where $x_{olq} = (x_{1olq}\ x_{2olq}\ x_{3olq})^t$ (resp. $x_{clq} = (x_{1clq}\ x_{2clq}\ x_{3clq})^t$) is the open-loop state (resp. closed-loop state) $\Big)$. The trajectory y_{ol} (resp. y_{olq}) correspond to the output of the open-loop system without disturbance (resp. with disturbance) and the trajectory y_{cl} (resp. y_{clq}) correspond to the output of the closed-loop system without disturbance (resp. with disturbance). The control trajectory $u_{ol} = (u_{1ol}\ u_{2ol})^t$ (resp. $u_{olq} = (u_{1olq}\ u_{2olq})^t$ is equal to the v and the trajectory $u_{cl} = (u_{1cl}\ u_{2cl})^t$ (resp. $u_{clq} = (u_{1clq}\ u_{2clq})^t$) is the control provided by the controller when the system is not disturbed (resp. when the system is disturbed).

V. Conclusion

In this paper, the closed-loop controller design for max-plus linear systems subjected to disturbances is given. The control law obtained (*i.e.* $u = Fx \oplus v$) is neutral with regard to the output but leads to the greatest input achieving our objective, it is optimal in regards to just-in-time point of view. The design method may now be applied for $(\max, +)$-linear systems in order to transpose the analogous of the disturbance decoupling problem for the classical linear system.

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Appendix A
Formulae involving star operator

Theorem 1 ([2, Th. 4.75]): In a complete idempotent semiring S, the least solution of $x = ax \oplus b$ is $x = a^*b$, where $a^* = \bigoplus_{i\in\mathbb{N}} a^i$ (Kleene star operator) with $a^0 = e$. Similarly, the least solution of $x = Ax \oplus b$ in $\mathcal{D}^n$ is $x = A^*b$.

$a^*(ba^*)^* = (a \oplus b)^* = (a^*b)^*a^*$	(f.1)
$(a^*)^* = a^*$	(f.2)
$(ab)^*a = a(ba)^*$	(f.3)
$a^*a^* = a^*$	(f.4)
$aa^* = a^*a$	(f.5)

$(ab) \backslash x = b \backslash (a \backslash x)$	(f.5)
$a(a \backslash x) \preceq x$	(f.6)
$a \backslash a = (a \backslash a)^*$	(f.7)
$a \backslash (ax) \succeq x$	(f.8)
$(x / a)a \preceq x$	(f.9)

APPENDIX B

FORMULAE INVOLVING DIVISION

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