Efficient state-estimation of Uncertain Max-Plus linear systems with high observation noise ⋆

Guilherme Espindola-Winck [*]
Renato Markele Ferreira Cândido [*] **Laurent Hardouin** [*]
Mehdi Lhommeau [*]

[*] *Laboratoire Angevin de Recherche en Ingénierie des Systèmes, Université d'Angers, LARIS, Polytech Angers, 49000 Angers, France (e-mails: guilherme.espindolawinck@univ-angers.fr; renatomarkele@hotmail.com; laurent.hardouin@univ-angers.fr; mehdi.lhommeau@univ-angers.fr),*

Abstract: This paper presents a new approach to bounded error state-estimation for uncertain max-plus linear systems. This method yields the smallest interval vector including the real state in a guaranteed way. The parameters of the max-plus linear systems are assumed to be bounded, the nondeterministic measurement of the system output is assumed to be given as well as the interval vector including the state at the preceding step. The observation matrix is assumed to be with high noise, i.e., the width of its interval components is large enough. The computation makes intensive use of the residuation theory (Baccelli et al., 1992) over the intervals. This method is worth of interest because it gives a guaranteed over-approximation of the support of the state and could be used to improve the probabilistic approach considered in the literature with an overall complexity of computation lower than existing methods.

Keywords: Timed event graphs, Idempotent semirings, Max-plus algebra, Estimation and filtering, Reachability analysis.

1. INTRODUCTION

This paper presents a set-membership method for the state-estimation of a Max-Plus Linear (MPL) dynamical systems which are Discrete Event Dynamic Systems (DEDS) involving only *delay and synchronization phenomena*, i.e., the starting of a task waits for a previous set of tasks to be completed.

Taking advantage of the linearity property over dioids, several authors have developed methods to estimate the system states (Hardouin et al., 2010; Loreto et al., 2010; Cândido et al., 2013), which is an essential problem to address applications such as fault detection and diagnosis (Paya et al., 2020) or state feedback control (Hardouin et al., 2018). The state estimation can be achieved by considering an observer as proposed in Hardouin et al. (2010), this leads to an estimation of the state as close as possible from below, i.e., the estimation is smaller than the real state. This estimator is efficient to deal with deterministic systems and useful to design observer-based controller (Hardouin et al., 2017) focusing on just-in-time control strategies.

However, if the system is with uncertain parameters, some alternative methods can be considered in order to take advantage of the knowledge about the characteristics of this uncertainty. Two ways have been considered: the stochastic approaches which focus on the probability density (Xu et al., 2019; Farahani et al., 2017; van den Boom and De Schutter, 2014) of the MPL system parameters and the set-membership approaches focusing on the reachable set (Brunsch et al., 2012b). More precisely, to deal with state estimation the two existing approaches are:

- **The *stochastic filtering* approaches:** In Cândido et al. (2013) a Particle Filter for MPL is proposed, it uses a particle representation of the probability density of the system state to perform a Sequential Monte-Carlo estimation of the state. This approach is limited by the numerical difficulties due to the generation of the particles and by the fact that the lower dimension of the measurements with respect to the state, introduces an imprecise generation of particles in the state space. In Mendes et al. (2019) an alternative Bayesian method is proposed, it is based on an algorithm leading to compute the inverse of the conditional expectation measurement $= \mathbb{E}[\text{observation}|\text{state}]$, by taking available measurements and the prediction into account in order to compute a state estimate. This procedure is based on a Constraint Satisfaction Problem (CSP) (Jaulin et al., 2001), but unfortunately it is over-optimistic since the estimation must respect the condition that measurement $= \mathbb{E}[\text{observation}|\text{estimation}]$. Moreover, as another drawback, this procedure does not consider the trade-off mechanism between the noise in the measurement versus the noise in the prediction as it is efficiently done in classical Bayesian methods.

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- **The *set-membership* estimation approaches:** In Cândido et al. (2018); Mufid et al. (2021) the authors consider uncertain Max Plus Linear (uMPL) systems, which are non-deterministic MPL systems whose parameters can take arbitrary values in a given interval. The state-estimation computation can be carried out by considering Difference-Bound Matrices (DBM) (Adzkiya et al., 2015) or more efficiently via interval analysis (Candido et al., 2020), this latest method is called an Interval Filter (IF) in the sequel. These estimation methods yield the set of possible states and can then be used to compute the support of the posterior density function PDF. Even though the support of the PDF is known, it is assumed equal importance for all values inside, which is not desired in the estimation (conservative characteristic). Nevertheless, they are over-pessimistic since, uMPL are expansive, i.e., the hyper-volume of the intervals is *increasing at each step of computation.*

Our contribution In this paper, we consider a set-membership approach in order to design an improved IF, i.e., with a good enough accuracy for high noise observation matrices, as it is shown in the numerical results section, and a lower computational complexity than the existing methods (Mendes et al. (2019); Cândido et al. (2013)), since it uses only trivial matrix operations over dioid. It can be defined as the intersection of the interval representing the *a priori* information (can be associated to the prediction stage of the Bayesian approach) and the one calculated thanks to the given measurement (can be seen as the correction stage of the Bayesian approach). This method can be seen as the analogous for uMPL to the one proposed to compute the robot trajectories in Rohou et al. (2017).

This paper is organized as follows. In Section 2 algebraic background on max-plus algebra, interval arithmetic and MPL systems are given. Section 3 defines the over-approximation of the direct image of an interval and recalls the inverse image of the measurement w.r.t. uMPL system. Section 4 presents the tools necessary to design the new IF scheme, which is faster but less precise than considering exact computations. Section 5 is dedicated to show the correctness of the proposed procedure. Section 6 presents the conclusions and final remarks.

2. MATHEMATICAL BACKGROUND

2.1 Algebraic framework

A set $\mathcal{S}$ endowed with two internal operations, *sum* ($\oplus$) and *product* ($\otimes$), is an idempotent semiring $\mathcal{D}$ (Baccelli et al., 1992, Chapter 4), (Heidergott et al., 2006) if the sum is associative, commutative and idempotent (i.e., $a \oplus a = a$) and the product is associative and left and right distributive w.r.t. the sum [1]. The null (or zero) element, denoted by ε, is such that $\forall a \in \mathcal{D}, a \oplus \varepsilon = a$ and the identity element, denoted by e, is such that $\forall a \in \mathcal{D}, a \otimes e = a$. Besides, the zero element is absorbing for the $\otimes$ operation (i.e., $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon$). As in classical algebra, the operator $\otimes$ will usually be omitted in expressions, $a^i = a \otimes a^{i-1}$ and $a^0 = e$. In this algebraic structure, a partial ordering is defined by $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow b = a \wedge b$ (where $a \wedge b$ is the greatest lower bound). Therefore, $\mathcal{D}$ is a partially ordered set. Furthermore, $\mathcal{D}$ is complete if it is closed for infinite sum and if the product distributes with the infinite sum. Particularly, $\top = \bigoplus_{x \in \mathcal{D}} x$ is the top element of $\mathcal{D}$, it respects the absorbing rule, i.e., $\varepsilon \otimes \top = \varepsilon$ and $\top \otimes \varepsilon = \varepsilon$. A dioid $\mathcal{D}$ is complete if it is closed w.r.t. the addition of an infinite number of elements and distributive w.r.t. the addition of an infinite number of elements.

[1] The $\otimes$-product is not necessarily commutative.

The set $\mathcal{D}^n$ refers to the n-th fold Cartesian product of the idempotent semiring. Its elements can be thought of as points of an affine space, or as vectors. They are denoted by bold symbols, for instance $\mathbf{x} = (x_1, \dots, x_n)^{\mathrm{t}}$. The element $\boldsymbol{\varepsilon}$, $\top$, and $\boldsymbol{e}$ refer to the vectors whose coordinates are all equal to ε, $\top$ and e respectively. The $\oplus$ and $\otimes$ operations can be extended to matrices as follows. If $A, B \in \mathcal{D}^{n\times p}$ and $C \in \mathcal{D}^{p\times q}$, then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes C)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj}$.

The inequality $A\mathbf{x} \preceq \mathbf{y}$ with matrix $A \in \mathcal{D}^{n\times p}$, vectors $\mathbf{x} \in \mathcal{D}^p$ and $\mathbf{y} \in \mathcal{D}^n$ admits a greatest solution denoted $\hat{\mathbf{x}} = A \backslash\!\!\!\circ\, \mathbf{y}$, with

$$\hat{x}_i = \bigwedge_{k=1}^{n} a_{ki} \backslash\!\!\!\circ\, y_k, \text{ for all } i \in \{1, \dots, p\}, \tag{1}$$

where $a_{ki} \backslash\!\!\!\circ\, y_k$ is the greatest solution of the scalar inequality $a_{ki} \otimes x \preceq y_k$.

In the same way, the inequality $X\mathbf{a} \preceq \mathbf{y}$ with matrix $X \in \mathcal{D}^{n\times p}$, vectors $\mathbf{a} \in \mathcal{D}^p$ and $\mathbf{y} \in \mathcal{D}^n$ admits a greatest solution denoted $\hat{X} = \mathbf{y} /\!\!\!\circ\, \mathbf{a}$, with

$$\hat{X}_{ij} = y_i /\!\!\!\circ\, a_j, \text{ for all } i \in \{1, \dots, n\}, \text{ and } j \in \{1, \dots, p\}, \tag{2}$$

where $y_i /\!\!\!\circ\, a_j$ is the greatest solution of the scalar inequality $x \otimes a_j \preceq y_i$.

Example 1. The set $\overline{\mathbb{R}}_{\max} = \mathbb{R} \cup \{-\infty, +\infty\}$ endowed with the max operator as $\oplus$ and the classical sum $+$ as $\otimes$ is a complete idempotent semiring with $\varepsilon = -\infty$, $\top = +\infty$ and $e = 0$, and with the convention that $+\infty - \infty = -\infty$. Furthermore, in this semiring the product is commutative, hence $x \otimes a = a \otimes x \preceq y$ admits $x = a \backslash\!\!\!\circ\, y = y /\!\!\!\circ\, a$ as greatest solution where operators $\backslash\!\!\!\circ$ and $/\!\!\!\circ$ are the classical subtraction $-$, i.e., $x = y - a$.

Lemma 1. Given $\mathbf{x} \in \overline{\mathbb{R}}^p_{\max}$ and $\mathbf{y} \in \overline{\mathbb{R}}^n_{\max}$, the following equality holds $(\mathbf{y} /\!\!\!\circ\, \mathbf{x}) \backslash\!\!\!\circ\, \mathbf{y} = \mathbf{x}$.

Proof. From (2), for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, p\}$,

$$(\mathbf{y} /\!\!\!\circ\, \mathbf{x})_{ij} = y_i /\!\!\!\circ\, x_j = y_i - x_j,$$

and from (1) we have for all $j \in \{1, \dots, p\}$,

$$((\mathbf{y} /\!\!\!\circ\, \mathbf{x}) \backslash\!\!\!\circ\, \mathbf{y}))_j = \bigwedge_{i=1}^{n} (\mathbf{y} /\!\!\!\circ\, \mathbf{x})_{ij} \backslash\!\!\!\circ\, y_i = \bigwedge_{i=1}^{n} y_i - (y_i - x_j) = x_j.$$

$\square$

Lemma 2. Let $a, b, c, d \in \overline{\mathbb{R}}_{\max}$. If $c \prec a$ then the following equivalence holds

$$a \oplus b \preceq c \oplus d \Leftrightarrow a \oplus b \preceq d.$$

Proof. First, $a \oplus b \preceq c \oplus d \Rightarrow a \oplus b \preceq d$ since by assumption $c \prec a$ which implies $c \prec a \oplus b$, hence $c \prec a \oplus b \preceq c \oplus d$ and $c \prec c \oplus d \Leftrightarrow c \prec d \Leftrightarrow c \oplus d = d$. Similarly, $c \prec a$ and $a \oplus b \preceq d$ imply $c \prec a \oplus b \preceq d \Rightarrow c \oplus c \prec a \oplus b \oplus c = a \oplus$

$b \preceq c \oplus d$, i.e., we have $a \oplus b \preceq d \Rightarrow a \oplus b \preceq c \oplus d$ which concludes the proof. □

2.2 Interval arithmetic over semiring $\overline{\mathbb{R}}_{\max}$

Interval arithmetic is presented in Moore and Bierbaum (1979). An interval of $\overline{\mathbb{R}}_{\max}$ is defined as $[x] = [\underline{x}, \overline{x}] = \{x \in \overline{\mathbb{R}}_{\max} : \underline{x} \preceq x\} \cap \{x \in \overline{\mathbb{R}}_{\max} : x \preceq \overline{x}\} = \{x \in \overline{\mathbb{R}}_{\max} : \underline{x} \preceq x \preceq \overline{x}\}$. An interval $[x]$ is empty if $\underline{x} \succ \overline{x}$. The width of an interval $[x]$ of $\overline{\mathbb{R}}_{\max}$ is defined as $\mathtt{w}([x]) = \overline{x} \phi \underline{x} = \underline{x} \diamond \overline{x}$.

The max-plus operations can be, therefore, extended to intervals as follows: (Brunsch et al., 2012a; Hardouin et al., 2009; Litvinov and Sobolevskiī, 2001; Lhommeau et al., 2005):

$$[x] \oplus [y] = \{x \oplus y : x \in [x], y \in [y]\} = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]\,, \quad (3)$$

$$[x] \otimes [y] = \{x \otimes y : x \in [x], y \in [y]\} = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]\,. \quad (4)$$

Two set-theoretic operations are important in this paper to properly handle intervals. First, the intersection between the intervals $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$ is defined as the set $\mathcal{Z} = \{z \in \overline{\mathbb{R}}_{\max} : z \in [x] \texttt{ and } z \in [y]\}$ and coincides with $[z] = [x] \cap [y]$, i.e., $\mathcal{Z} = [z]$. Therefore,

$$[z] = [\max\{\underline{x}, \underline{y}\}, \min\{\overline{x}, \overline{y}\}]. \quad (5)$$

Secondly, the union of the same two intervals is defined as the set $\mathcal{Z} = \{z \in \overline{\mathbb{R}}_{\max} : z \in [x] \texttt{ or } z \in [y]\}$ but, in order to make the set of intervals closed w.r.t. this operation, we define the *interval union*, i.e., the *interval hull* [2] of $\mathcal{Z}$ as:

$$[z] = [x] \sqcup [y] = [\min\{\underline{x}, \underline{y}\}, \max\{\overline{x}, \overline{y}\}], \text{ such that } \mathcal{Z} \subseteq [z]. \quad (6)$$

The $\cap$ and $\sqcup$ operations of two interval vectors can be computed as the element-wise operation of the corresponding entries.

The $\oplus$ and $\otimes$ are extended to interval matrices as follows: if $[A]$, $[B]$ and $[C]$ are, respectively, $(n \times p)$, $(n \times p)$ and $(p \times q)$-dimensional interval matrices, then $([A] \oplus [B])_{ij} = [a_{ij}] \oplus [b_{ij}]$ and $([A] \otimes [C])_{ij} = \bigoplus_{k=1}^{p}([a_{ik}] \otimes [c_{kj}])$.

Remark 1. Any matrix $A \in \overline{\mathbb{R}}_{\max}^{n \times p}$ can be represented by a deprecated interval matrix $[A]$, in which $\underline{a}_{ij} = \overline{a}_{ij}$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, p\}$.

Remark 2. An interval matrix is considered with *high noise* if the width of its elements is considerable large.

2.3 Max-Plus Linear (MPL) Systems

The nonautonomous model of an MPL system, considering the earliest firing rule, is given by:

$$\mathbf{x}(k) = A\mathbf{x}(k-1) \oplus B\mathbf{u}(k), \quad (7a)$$

$$\mathbf{z}(k) = C\mathbf{x}(k), \quad (7b)$$

where the entries of matrices $A \in \overline{\mathbb{R}}_{\max}^{n \times n}$, $B \in \overline{\mathbb{R}}_{\max}^{n \times p}$ and $C \in \overline{\mathbb{R}}_{\max}^{q \times n}$ represent the process times. The variable $k \in \mathbb{N}$ is an event-number and $\mathbf{x} \in \overline{\mathbb{R}}_{\max}^{n}$ is a dater, i.e., $\mathbf{x}(k)$ contains the k-th date of occurrence of each event of the system. The vector $\mathbf{z} \in \overline{\mathbb{R}}_{\max}^{q}$ is the output and the input (or control) vector $\mathbf{u} \in \overline{\mathbb{R}}_{\max}^{p}$.

[2] The interval hull of a set $\mathbb{X} \subseteq \mathbb{R}$ is the smallest interval $[\mathbb{X}]$ such that $\mathbb{X} \subseteq [\mathbb{X}]$.

The matrix entries of the equations above are considered to be *bounded noisy*, i.e., it is assumed that at each event k these entries can take an arbitrary value within a real interval. Hence, it is possible to model Uncertain Max-Plus Linear (uMPL) systems as defined in Cândido et al. (2018); Candido et al. (2020) from (7) considering that $A \doteq A(k) \in [\underline{A}, \overline{A}]$, $B \doteq B(k) \in [\underline{B}, \overline{B}]$ and $C \doteq C(k) \in [\underline{C}, \overline{C}]$ are matrices of independent random variables with finite support and whose entries are mutually independent. [3] For instance, matrices $\underline{A}$ and $\overline{A}$ are respectively the lower and upper bounds of $[A]$, such that $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$. The same reasoning is applied to the lower and upper bounds of $[B]$ and $[C]$.

Remark 3. According to Remark 1, any MPL system can be seen as a uMPL system in which its matrix entries are a deprecated interval.

Remark 4. Any nonautonomous uMPL system can be transformed into an augmented autonomous uMPL model as $\mathbf{x}(k) = \mathcal{M}\mathbf{r}(k)$, where $\mathcal{M} \in ([A]\ [B])$, and $\mathbf{r}(k) = (\mathbf{x}(k-1)^{\mathrm{t}}\ \mathbf{u}(k)^{\mathrm{t}})^{\mathrm{t}}$.

Remark 5. The autonomous system $\mathbf{x}(k) = A\mathbf{x}(k-1)$ is assumed to be FIFO (first in, first out). In view of this assumption, it is always true that $\mathbf{x}(k) \succeq \mathbf{x}(k-1)$, such that the elements of the main diagonal of A can be assumed to be greater or equal to e at each event k.

In this work, we therefore consider, without loss of generality, only autonomous systems, i.e., we drop $B\mathbf{u}(k)$ in (7a), and only uMPL systems, i.e., we assume that the entries of the system matrices are intervals in (7).

In the sequel, for the sake of readability, we use the following notation: $\mathbf{x} \doteq \mathbf{x}(k)$, $\mathbf{z} \doteq \mathbf{z}(k)$ and $\mathbf{x}_0 \doteq \mathbf{x}(k-1)$.

3. DIRECT IMAGE OF AN INTERVAL VECTOR AND INVERSE IMAGE OF A POINT W.R.T. THE UNCERTAIN MPL SYSTEM

This section presents an approach to compute an over-approximation of the direct image of an interval vector w.r.t. the autonomous uMPL dynamical equation $\mathbf{x} = A\mathbf{x}_0$, and it recalls the inverse image of the measurement w.r.t. the observation equation $\mathbf{z} = C\mathbf{x}$.

3.1 Over-approximation of the direct image of an interval vector w.r.t the nonautonomous uMPL dynamical equation

Let $[A]$ be an $(n \times n)$-dimensional interval matrix and $\mathscr{X}_0$ be a set that is contained in $\overline{\mathbb{R}}_{\max}^{n}$, the direct image of $\mathscr{X}_0$ is called the *reach set* $\mathscr{X}$ which is defined as:

$$\mathscr{X} = \mathcal{I}_{[A]}\{\mathscr{X}_0\} = \{A\mathbf{x} \in \overline{\mathbb{R}}_{\max}^{n} : \mathbf{x} \in \mathscr{X}_0, A \in [A]\}. \quad (8)$$

In (Cândido et al., 2018, Sec 4.1), a method based on Difference-Bound Matrices (DBM) is presented in order to compute exactly $\mathscr{X}$ (Cândido et al., 2018, Algorithm 1) when $\mathscr{X}_0$ is the union of d_0 DBM. The set $\mathscr{X}$ is then

[3] This assumption of statistical independence between the matrix entries means that the minimum task duration or transportation time are independent of each other. This assumption is reasonable for practical problems, *e.g.*, in the field of transport systems, a failure of one train does not affect the potential efficiency of the others, even if they are blocked due to precedence constraint.

the union of DBM obtained thanks to a procedure with a complexity equal to $\mathcal{O}(d_0 n^{n+3})$.

In order to avoid this computational effort, we consider $[\mathbf{x}_0]$ an n-dimensional interval vector such that $\mathscr{X}_0 \subseteq [\mathbf{x}_0]$. Hence, the reach set $\mathscr{X}$ is over-approximated by the following interval vector (i.e., $\mathscr{X} \subseteq [\mathbf{x}]$):

$$[\mathbf{x}] = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{A}\ \underline{\mathbf{x}}_0 \preceq \mathbf{x} \preceq \overline{A}\overline{\mathbf{x}}_0\}. \tag{9}$$

3.2 Interval hull of the inverse image of the measurement w.r.t. the uMPL observation equation

We are interested in characterizing the inverse image of $\mathbf{z}$ w.r.t. the observation equation, i.e., the set of all states $\mathbf{x}$ that may lead to $\mathbf{z}$. This set is defined in Candido et al. (2020) as:

$$\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\} = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \exists C \in [C] : C\mathbf{x} = \mathbf{z}\}, \tag{10}$$
$$\iff \mathcal{I}^{-1}_{[C]}\{\mathbf{z}\} = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{C}\mathbf{x} \preceq \mathbf{z} \preceq \overline{C}\mathbf{x}\},$$

with $\underline{C}, \overline{C} \in \overline{\mathbb{R}}^{q\times n}_{\max}$ and $\mathbf{z} \in \overline{\mathbb{R}}^q_{\max}$.

The max-plus mapping is generally residuated but not dually residuated, i.e., given $\mathbf{z}$, there is a unique greatest $\mathbf{x}$ given by (1) such that $\underline{C}\mathbf{x} \preceq \mathbf{z}$, but not a unique least $\mathbf{x}$ such that $\overline{C}\mathbf{x} \succeq \mathbf{z}$. Hence, we split $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ into two sets

$$L = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{z} \preceq \overline{C}\mathbf{x}\} \text{ and } U = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{C}\mathbf{x} \preceq \mathbf{z}\}, \tag{11}$$

which are equivalent to:

$$L \equiv \bigcap_{i=1}^{q} L_i,\ U \equiv \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{X}}\},\ \text{with } \overline{\mathbf{X}} = \underline{C} \diamond\!\!\!\setminus \mathbf{z}, \tag{12}$$

with $L_i = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : z_i \preceq (\overline{C}\mathbf{x})_i\} \equiv \bigcup_{j=1}^{n}\{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : x_j \succeq \overline{c}_{ij} \diamond\!\!\!\setminus z_i\}$. Then,

$$\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\} = L \cap U = \left(\bigcap_{i=1}^{q} L_i\right) \cap U = \bigcap_{i=1}^{q} L_i \cap U, \tag{13}$$

is a set of cardinality bounded by n^q and $L_i \cap U = \bigcup_{j=1}^{n} set^i_j\{\overline{\mathbf{X}}\}$ where

$$set^i_j\{\overline{\mathbf{X}}\} = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : x_j \succeq \overline{c}_{ij} \diamond\!\!\!\setminus z_i\} \cap \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{X}}\},$$

such that $set^i_j\{\overline{\mathbf{X}}\} = \emptyset$ if $\overline{c}_{ij} \diamond\!\!\!\setminus z_i \succ \overline{X}_j$. In addition, this set can be represented with the same expressiveness as an interval vector, i.e.,

$$set^i_j\{\overline{\mathbf{X}}\} \equiv [(\varepsilon, \ldots, \overline{c}_{ij} \diamond\!\!\!\setminus z_i, \ldots, \varepsilon)^{\mathsf{t}}, \overline{\mathbf{X}}]. \tag{14}$$

Remark 6. As already mentioned in Subsection 2.2, in order to make the set of intervals closed w.r.t. $\cup$, we can use $\sqcup$ to compute the interval hull of $L_i \cap U$ as follows:

$$\left([L_i \cap U] = \bigsqcup_{j=1}^{n} set^i_j\{\overline{\mathbf{X}}\}\right) \supseteq \left(L_i \cap U = \bigcup_{j=1}^{n} set^i_j\{\overline{\mathbf{X}}\}\right).$$

Moreover, it is straightforward to remark if $L_i \cap U$ has cardinality greater than 1 then $[L_i \cap U] = [(\varepsilon, \ldots, \varepsilon)^{\mathsf{t}}, \underline{C} \diamond\!\!\!\setminus \mathbf{z}]$.

Remark 7. In (Candido et al., 2020, Algorithm 1) a general procedure is described to compute $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ The worst-case complexity of this procedure is $\mathcal{O}(qn^{q+1})$ where $\underline{C}, \overline{C} \in \overline{\mathbb{R}}^{q\times n}_{\max}$ and $\mathbf{z} \in \overline{\mathbb{R}}^q_{\max}$. .

4. AN OVER-APPROXIMATION FOR THE CONDITIONAL REACHABILITY PROBLEM

The *conditional reachability problem* consists in computing the following set $\chi = [\mathbf{x}] \cap \mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ which is the intersection between the over-approximation of the direct image $[\mathbf{x}]$ (see (9)), i.e., the *a priori* information computed thanks to the dynamic equation over $[\mathbf{x}_0]$, and the inverse image $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ (see (10)), i.e., the *a posteriori* information obtained thanks to the observation equation.

This problem is addressed in Cândido et al. (2018) and is slightly different since it uses the exact direct image instead of $[\mathbf{x}]$. However, the overall complexity of computing χ is exponential for both representations of the direct image (either the exact one or its over-approximation). In order to avoid this computational burden, we propose to compute the smallest interval $[\chi]$ that enclosures χ, which is called *over-approximation of the conditional reachability problem*.

Let

$$\chi = [\mathbf{x}] \cap \mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}, \tag{15}$$
$$= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\mathbf{x}} \preceq \mathbf{x} \preceq \overline{\mathbf{x}}\} \cap L \cap U,$$
$$= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\mathbf{x}} \preceq \mathbf{x}\} \cap L \cap U \cap \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{x}}\},$$
$$U' = U \cap \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{x}}\}, \tag{16}$$
$$= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \underline{C} \diamond\!\!\!\setminus \mathbf{z}\} \cap \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{x}}\},$$
$$= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{X}}'\}, \text{with } \overline{\mathbf{X}}' = \min\{\underline{C} \diamond\!\!\!\setminus \mathbf{z}, \overline{\mathbf{x}}\},$$
$$S = L \cap U' = \bigcap_{i=1}^{q} L_i \cap U',\ \text{(see (13))} \tag{17}$$
$$[S] = \bigcap_{i=1}^{q} [L_i \cap U'] \supseteq S,\ \text{(see Remark 6)} \tag{18}$$
$$\chi = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\mathbf{x}} \preceq \mathbf{x}\} \cap S, \tag{19}$$
$$[\chi] = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\mathbf{x}} \preceq \mathbf{x}\} \cap [S]. \tag{20}$$

Clearly, $[\chi]$ is calculated in polynomial-time whereas χ is computed in exponential-time, as already pointed-out. Nevertheless, in the sequel this result can purely be reinterpreted working only with matrix operations with a direct impact in the corresponding TEG's behavior.

4.1 On the reinterpretation of the set S

Below, we propose results in order to reinterpret the set $S = \bigcap_{i=1}^{q} L_i \cap U'$.

Lemma 3. The term $\overline{\mathbf{X}}' = \min\{\underline{C} \diamond\!\!\!\setminus \mathbf{z}, \overline{\mathbf{x}}\}$ is also given by $\overline{\mathbf{X}}' = \hat{\underline{C}} \diamond\!\!\!\setminus \mathbf{z}$ where $\hat{\underline{C}} = \mathbf{z} \phi \overline{\mathbf{X}}'$.

Proof. From Lemma 1, the following holds: $(\mathbf{z} \phi \overline{\mathbf{X}}') \diamond\!\!\!\setminus \mathbf{z} = \overline{\mathbf{X}}'$, hence $\overline{\mathbf{X}}' = \hat{\underline{C}} \diamond\!\!\!\setminus \mathbf{z}$. □

According to Lemma 3, set U' can be expressed equivalently as: $U' = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{X}}'\} \equiv \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \hat{\underline{C}} \diamond\!\!\!\setminus \mathbf{z}\} \equiv \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \hat{\underline{C}}\mathbf{x} \preceq \mathbf{z}\}$. Simultaneously, set $S = L \cap U'$ can be characterized, in analogy with (11), as

$$\begin{aligned} S &= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{z} \preceq \overline{C}\mathbf{x}\} \cap \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\} \\ &= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{z} \preceq \overline{C}\mathbf{x}\} \cap \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\hat{C}}\mathbf{x} \preceq \mathbf{z}\} \\ &= \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\hat{C}}\mathbf{x} \preceq \mathbf{z} \preceq \overline{C}\mathbf{x}\} = \bigcap_{i=1}^{q} L_i \cap U'. \end{aligned} \tag{21}$$

Proposition 1. Set S can be expressed equivalently as: $S = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\hat{C}}\mathbf{x} \preceq \mathbf{z} \preceq \overline{C}\mathbf{x}\} \equiv \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\hat{C}}\mathbf{x} \preceq \mathbf{z} \preceq \hat{\overline{C}}\mathbf{x}\}$, with $\hat{\overline{C}}$ defined as

$$\hat{\overline{c}}_{ij} = \begin{cases} \varepsilon & \text{if } \underline{\hat{c}}_{ij} \succ \overline{c}_{ij}, \\ \overline{c}_{ij} & \text{otherwise.} \end{cases} \tag{22}$$

for all $i \in \{1, \ldots, q\}$ and all $j \in \{1, \ldots, n\}$.

Proof. First, we consider $(\underline{\hat{C}}\mathbf{x})_i \preceq z_i \preceq (\overline{C}\mathbf{x})_i$ for all $i \in \{1, \ldots, q\}$, this implies $(\underline{\hat{C}}\mathbf{x})_i \preceq (\overline{C}\mathbf{x})_i$, i.e.,

$$\bigoplus_{k=1}^{n} \underline{\hat{c}}_{ik} \otimes x_k = \underline{\hat{c}}_{i1} \otimes x_1 \oplus \cdots \oplus \underline{\hat{c}}_{in} \otimes x_n$$
$$\preceq \overline{c}_{i1} \otimes x_1 \oplus \cdots \oplus \overline{c}_{in} \otimes x_n = \bigoplus_{k=1}^{n} \overline{c}_{ik} \otimes x_k.$$

Let us define $a = \underline{\hat{c}}_{ij} \otimes x_j$, $c = \overline{c}_{ij} \otimes x_j$,

$$b = \bigoplus_{\substack{k=1,\\ k\neq j}}^{n} \underline{\hat{c}}_{ik} \otimes x_k \text{ and } d = \bigoplus_{\substack{k=1,\\ k\neq j}}^{n} \overline{c}_{ik} \otimes x_k,$$

then it is straightforward to apply the Lemma 2 if $c = \overline{c}_{ij} \otimes x_j \prec a = \underline{\hat{c}}_{ij} \otimes x_j$ as shown below:

$$a \oplus b = \underline{\hat{c}}_{ij} \otimes x_j \oplus \bigoplus_{\substack{k=1,\\ k\neq j}}^{n} \underline{\hat{c}}_{ik} \otimes x_k = \bigoplus_{k=1}^{n} \underline{\hat{c}}_{ik} \otimes x_k$$
$$\preceq c \oplus d = \bigoplus_{k=1}^{n} \overline{c}_{ik} \otimes x_k = \bigoplus_{\substack{k=1,\\ k\neq j}}^{n} \overline{c}_{ik} \otimes x_k = \varepsilon \oplus d.$$

Furthermore, the following equivalence holds $\forall x_j$, $\overline{c}_{ij} \otimes x_j \prec \underline{\hat{c}}_{ij} \otimes x_j \Leftrightarrow \overline{c}_{ij} \prec \underline{\hat{c}}_{ij}$, hence, in S definition, matrix $\overline{C}$ can be replaced by matrix $\hat{\overline{C}}$. $\square$

In view of the previous Proposition 1, the observation part of the corresponding TEG is potentially simplified when evaluating its upper process time bound, i.e., some places can be neglected, without loss of information.

4.2 On the lower bound of $S = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \underline{\hat{C}}\mathbf{x} \preceq \mathbf{z} \preceq \hat{\overline{C}}\mathbf{x}\}$

As already pointed out, the inequality $\underline{\hat{C}}\mathbf{x} \preceq \mathbf{z}$ has a unique greatest solution given by $\overline{\mathbf{X}}' = \underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}$ but, in general, the inequality $\mathbf{z} \preceq \hat{\overline{C}}\mathbf{x}$ does not admit a unique least solution. Nevertheless, we present below some assumptions ensuring the existence of such solution.

Assumption 1. The matrix $\hat{\overline{C}}$ is considered to be row *G-astic*, i.e., it has no row with only ε elements (see (Cuninghame-Green and Butkovic, 2003)).

In view of the previous assumption, it is straightforward to see that $S \neq \emptyset$.

Assumption 2. The i-th row of $\hat{\overline{C}}$ has one and only one index $j' \in \{1, \ldots, n\}$, such that $\hat{\overline{c}}_{ij'} \neq \varepsilon$.

Lemma 4. If Assumption 2 holds for a particular $i \in \{1, \ldots, n\}$, then the inequality $z_i \preceq (\hat{\overline{C}}\mathbf{x})_i$ admits a unique least solution $\underline{\hat{x}}^{(i)}_{j'} = \hat{\overline{c}}_{ij'} \circ\!\!\!\!\backslash z_i$.

Proof. From Assumption 2, we have $(\hat{\overline{c}}_{i1}, \ldots, \hat{\overline{c}}_{in}) = (\varepsilon, \ldots, \hat{\overline{c}}_{ij'}, \ldots, \varepsilon)$, and hence $(\hat{\overline{C}}\mathbf{x})_i = \hat{\overline{c}}_{ij'} \otimes x_{j'}$. Thus, $z_i \preceq \hat{\overline{c}}_{ij'} \otimes x_{j'} \Leftrightarrow \underline{\hat{x}}^{(i)}_{j'} = \hat{\overline{c}}_{ij'} \circ\!\!\!\!\backslash z_i \preceq x_{j'}$. $\square$

Considering Lemma 4 we have that the inequality $\mathbf{z} \preceq \hat{\overline{C}}\mathbf{x}$ admits a solution given by $\underline{\hat{\mathbf{X}}} = \bigoplus_{i=1}^{q} \underline{\hat{\mathbf{x}}}^{(i)}$ where

$$\underline{\hat{\mathbf{x}}}^{(i)} = \begin{cases} (\varepsilon, \ldots, \hat{\overline{c}}_{ij'} \circ\!\!\!\!\backslash z_i, \ldots, \varepsilon)^{\mathrm{t}} & \text{if Assumption 2 holds,} \\ (\varepsilon, \ldots, \varepsilon)^{\mathrm{t}} & \text{otherwise.} \end{cases} \tag{23}$$

Moreover, for any $j \in \{1, \ldots, n\}$ if $\underline{\hat{X}}_j \neq \varepsilon$ then we say that $\underline{\hat{X}}_j$ is the least finite solution such that $x_j \succeq \underline{\hat{X}}_j$.

4.3 Retrieving $[S]$

Now, we show that $\underline{\hat{\mathbf{X}}}$ is indeed the lower bound of $[S]$.

Proposition 2. $\underline{\hat{\mathbf{x}}}^{(i)}$ is equivalent to the lower bound of $set^i_j\{\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\}$ given in (14) and $\underline{\hat{\mathbf{X}}}$ is equivalent to the lower bound of $[L_i \cap U']$ given in (18).

Proof. Regarding $set^i_j\{\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\} = [(\varepsilon, \ldots, \overline{c}_{ij} \circ\!\!\!\!\backslash z_i, \ldots, \varepsilon)^{\mathrm{t}}, \underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}]$ we have that if $\hat{\overline{c}}_{ij} = \varepsilon$ for all $j \in \{1, \ldots, n\} \backslash \{j'\}$ then $\hat{\overline{c}}_{ij} \circ\!\!\!\!\backslash z_i = \varepsilon \circ\!\!\!\!\backslash z_i = \top \succ (\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z})_j$ and hence $set^i_j\{\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\} = \emptyset$. In addition, for this $j' \in \{1, \ldots, n\}$ we have $\hat{\overline{c}}_{ij'} \circ\!\!\!\!\backslash z_i = \varepsilon \circ\!\!\!\!\backslash z_i \preceq (\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z})_j$ and hence $set^i_{j'}\{\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\} \neq \emptyset$ which can be associated to the interval vector $[\underline{\hat{\mathbf{x}}}^{(i)}, \underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}]$, where $\underline{\hat{\mathbf{x}}}^{(i)} = (\varepsilon, \ldots, \hat{\overline{c}}_{ij'} \circ\!\!\!\!\backslash z_i, \ldots, \varepsilon)^{\mathrm{t}}$.
Summing-up:

- if Assumption 2 holds, we have that $L_i \cap U' = \bigcup_{j=1}^{n} set^i_j\{\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\} = set^i_{j'}\{\underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}\} = [\underline{\hat{\mathbf{x}}}^{(i)}, \underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}] = [L_i \cap U']$;
- otherwise $L_i \cap U'$ is a collection of finitely many interval vectors, and we therefore are interested in its over-approximation $[(\varepsilon, \ldots, \varepsilon)^{\mathrm{t}}, \underline{\hat{C}} \circ\!\!\!\!\backslash \mathbf{z}]$, i.e., its interval hull $[L_i \cap U']$ (see Remark 6).

Finally, we intersect each interval vector along $i \in \{1, \ldots, q\}$, and we clearly obtain $\underline{\hat{\mathbf{X}}}$ as the lower bound of $[S]$. $\square$

From the previous Proposition 2 we therefore know that $\mathbf{z} \preceq \hat{\overline{C}}\mathbf{x}$ can have unique least solution that is different from ε for a subset of states, which is represented by the subset $\mathcal{J} \subseteq \{1, \ldots, n\}$, and consequently $\forall j \in \mathcal{J}$ we have that the orthogonal projection of S over x_j is equal to the orthogonal projection of $[S]$ over x_j.

Thus, $[S]$ is clearly given as follows:

$$[S] = \begin{cases} \left[\bigoplus_{i=1}^{q} \underline{\hat{\mathbf{x}}}^{(i)}, \overline{\mathbf{X}}'\right] & \text{if Lemma 1 holds,} \\ \emptyset & \text{otherwise} \end{cases} \tag{24}$$

where $\overline{\mathbf{X}}' = \hat{\underline{C}} \backslash\!\!\!\!\circ\, \mathbf{z}$

Finally, if $[S] \neq \emptyset$ then

$$[\chi] = [\max\{\underline{\mathbf{x}}, \underline{S}\}, \overline{\mathbf{X}}'], \tag{25}$$

which is exactly the same as the one obtained using (20) but with a simpler interpretation of the physical meaning of the TEG's behavior. It is easy to interpret that if $\hat{\underline{c}}_{ij} \succ \bar{c}_{ij}$ then the process time $\bar{c}_{ij}$ can be neglected. On the other hand, the physical meaning is not simply interpreted when purely evaluating (18) and (20): from $\overline{\mathbf{X}}' = \min\{\underline{C} \backslash\!\!\!\!\circ\, \mathbf{z}, \overline{\mathbf{x}}\}$ and $\bar{c}_{ij} \backslash\!\!\!\!\circ\, z_i \succ \overline{X}'_j$ we can say that $set^i_j\{\overline{\mathbf{X}}'\}$ is empty and that $\bar{c}_{ij}$ can be neglected.

Example 2. Let $[\mathbf{x}] = ([0,4],[-2,1])^{\mathrm{t}}$, $\mathbf{z} = (5,4)^{\mathrm{t}}$ and $[C] = \begin{pmatrix} [1,4] & [2,3] \\ [1,2] & [e,4] \end{pmatrix}$. Then $\begin{pmatrix} 1 & 2 \\ 1 & e \end{pmatrix} \mathbf{x} \preceq \mathbf{z} \preceq \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix} \mathbf{x}$. From (1) we calculate

$$\begin{pmatrix} 1 & 2 \\ 1 & e \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 5 \\ 4 \end{pmatrix} \Leftrightarrow \mathbf{x} \preceq \overline{\mathbf{X}} = \begin{pmatrix} 1 & 2 \\ 1 & e \end{pmatrix} \backslash\!\!\!\!\circ \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

and we obtain $U = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq (3\ 3)^{\mathrm{t}}\}$. From (16) we obtain $U' = \{\mathbf{x} \in \overline{\mathbb{R}}^n_{\max} : \mathbf{x} \preceq \overline{\mathbf{X}}'\}$ where $\overline{\mathbf{X}}' = \min\{\overline{\mathbf{X}}, \overline{\mathbf{x}}\} = \min\{(3\ 3)^{\mathrm{t}}, (4\ 1)^{\mathrm{t}} = (3\ 1)^{\mathrm{t}}$.

In the Figure below, we use (Candido et al., 2020, Algorithm 1) to compute exactly $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\} = L \cap U$:

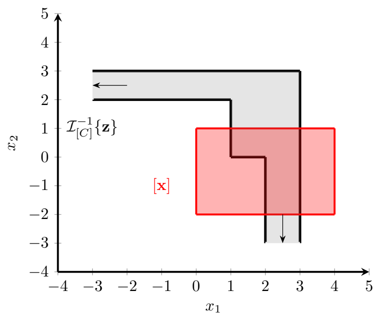

Fig. 1. Sets $\mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ and $[\mathbf{x}]$ of Example 2

In order to compute the smallest interval that enclosures $\chi = [\mathbf{x}] \cap \mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$, we compute $[S]$ using (24) that makes it possible to replace $\overline{C}$ with $\hat{\overline{C}} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & \varepsilon \\ 2 & 4 \end{pmatrix}$ since $\hat{\underline{C}} = \mathbf{z} \phi \overline{\mathbf{X}}' = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \phi \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ is such that $\hat{\underline{c}}_{12} \succ \hat{\bar{c}}_{12}$.

Thus, $\overline{S} = \overline{\mathbf{X}}' = (3,1)^{\mathrm{t}}$ and

$$\begin{aligned} \underline{S} &= \hat{\mathbf{x}}^{(1)} \oplus \hat{\mathbf{x}}^{(2)} = (\hat{\bar{c}}_{11} \backslash\!\!\!\!\circ\, z_1 = 4 \backslash\!\!\!\!\circ\, 5 = 1, \varepsilon)^{\mathrm{t}} \oplus (\varepsilon, \varepsilon)^{\mathrm{t}}, \\ &= (1, \varepsilon)^{\mathrm{t}}. \end{aligned}$$

Finally, $\chi \subseteq [\chi]$ (see (25)) is given by:

$$\begin{aligned} [\chi] &= [\max\{\underline{\mathbf{x}}, \underline{S}\}, \overline{S}] = [\max\{(0,-2)^{\mathrm{t}}, (1,\varepsilon)^{\mathrm{t}}\}, (3,1)^{\mathrm{t}}], \\ &= ([1,3],[-2,1])^{\mathrm{t}}. \end{aligned}$$

5. INTERVAL FILTER

This section deals with the solution of a filtering problem by using an interval approach, herein named Interval Filtering (IF).

The first stage given by (9) can be associated to the prediction stage of the Bayesian approach, and we obtain an interval vector $[\mathbf{x}]$. In the second stage, the new information $\mathbf{z}$ is used to calculate the smallest interval $[\chi]$ (see (25)) that enclosures $\chi = [\mathbf{x}] \cap \mathcal{I}^{-1}_{[C]}\{\mathbf{z}\}$ (see (15)). This phase can be associated to the correction stage of the Bayesian approach and defined as a conditional reachability problem. At the end, our approach is also two-fold as classical filtering algorithms.

Indeed, in a closed-loop system relying on state-estimation (Hardouin et al., 2017), an observer-based controller is expecting an estimated vector $\hat{\mathbf{x}}$ and not an interval vector. Thus, we have to select one point in $[\chi]$. For instance, the estimated state can be chosen as the center of the interval. For all types of choices, the trajectory estimated by the IF is in general less precise than those considering the probabilistic aspects (Cândido et al., 2013; Mendes et al., 2019).

Problem 1. Consider the uMPL system given by (7) with:

$$A(k) \in \begin{pmatrix} [4,6] & [5,7] \\ [2,5] & [1,3] \end{pmatrix}, \qquad C(k) \in \begin{pmatrix} [1,8] & [2,3] \\ [1,3] & [e,4] \end{pmatrix},$$

and $B = \boldsymbol{\varepsilon}$ (autonomous system). In addition, consider that the nondeterministic matrices entries are random variables uniformly distributed in the given intervals, e.g. the element $a_{12}(k)$ of $A(k)$ is uniformly distributed between 5 and 7, and $\hat{\mathbf{x}}(0|0) \in [\mathbf{x}](0|0) = ([0,2],[0,2])^{\mathrm{t}}$.

The following procedure describes a general method for computing an estimated vector $\hat{\mathbf{x}}(k|k) \in [\chi](k|k)$:

Interval Filtering:

(1) From $\mathbf{x}(k-1|k-1) \in [\mathbf{x}](k-1|k-1)$ compute $[\mathbf{x}](k|k-1)$ as $[\underline{A}\ \underline{\mathbf{x}}(k-1|k-1), \overline{A}\overline{\mathbf{x}}(k-1|k-1)]$ (see (9));
(2) Compute $\overline{\mathbf{X}}(k|k) = \min\{\overline{\mathbf{x}}(k|k-1), \underline{C} \backslash\!\!\!\!\circ\, \mathbf{z}(k)\}$ according to (16);
(3) Compute $[S](k|k)$ according to (24);
(4) Compute $[\chi](k|k) = [\max\{\underline{\mathbf{x}}(k|k-1), \underline{S}(k|k)\}, \overline{S}(k|k)]$ according to (25);
(5) Compute $\hat{\mathbf{x}}(k|k) = \mathtt{midpoint}([\chi](k|k))$ [4];
(6) Update $[\mathbf{x}](k-1|k-1)$ with $[\chi](k|k)$ (backshift operation);
(7) $k \leftarrow k+1$;

In order to show that the reduction of computational burden of our approach might not affect the precision of the estimation if compared with the probabilistic one, we present some numerical results, comparing accuracy, i.e., the distance from the true value of the state to its estimation, and computational times. For the sake of comparison we use the method presented in Mendes et al. (2019), which computes an estimate $\hat{\mathbf{x}}(k|k)$ as close as possible from $\hat{\mathbf{x}}(k|k-1) = \mathbb{E}[\mathbf{x}(k)|\hat{\mathbf{x}}(k-1|k-1)]$ but subject to the constraint $\mathbf{z}(k) = \mathbb{E}[\mathbf{z}(k)|\hat{\mathbf{x}}(k|k)]$ (assuming $\hat{\mathbf{x}}(0|0)$ is known, and in this case, equal to the mid point of $[\mathbf{x}](0|0)$). We also consider the computational times $\mathtt{T}$ involed in the simulation $\mathbf{x}(k)$ and the computation of $\hat{\mathbf{x}}(k|k)$ for each approach.

[4] The $\mathtt{midpoint}$ of an interval vector $\{[\underline{x}_i, \overline{x}_i)]\}_{i=1}^n$ is defined as $\{(\underline{x}_i + \overline{x}_i)/2\}_{i=1}^n$.

We define a criterion to evaluate the outcome of our approach as the number of times (N) that $\underline{\mathbf{x}}(k|k-1) \neq \underline{\chi}(k|k)$, i.e., if the measurement is capable to reduce the hypervolume of $[\chi](k|k)$ as k evolves.
Table 1 shows the obtained results for simulations up to the occurrence of $k_{\max} = 4000$ firings, i.e., $0 \leq k \leq k_{\max}$. Each position of the table corresponds to mean-absolute-percentage-error[5] $\mathtt{MAPE}(x_i(k), \hat{x}_i(k|k))$. F_1 corresponds to the proposed IF and F_2 to the filter of Mendes et al. (2019).

State i	$\mathtt{MAPE}(x_i(k), \hat{x}_i^{F_1}(k\|k))$	$\mathtt{MAPE}(x_i(k), \hat{x}_i^{F_2}(k\|k))$
1	0.0286%	0.0690%
2	0.0590%	0.0971%

Table 1. IF estimation results of Problem 1

with $N_{score} = 100 \times \frac{N}{k_{\max}} = 48.05\%$ indicating the success rate of contracting the hypervolume of $[\chi](k|k)$.

For the computational times, we have $\mathtt{T}^{F_1} = 0.77\ s$ and $\mathtt{T}^{F_2} = 52.56\ s$.

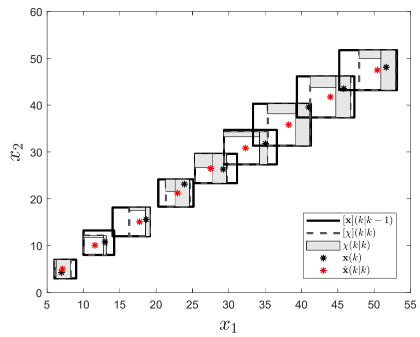

Fig. 2. IF graphical results of Problem 1 for up to occurrence of $k_{\max} = 9$ firings.

Problem 2. Consider the same problem defined in Problem 1 but with $\bar{c}_{11} = 5$ instead of 8.

Table 2 shows the obtained results of Problem 2 with the same $k_{\max}$.

State i	$\mathtt{MAPE}(x_i(k), \hat{x}_i^{F_1}(k\|k))$	$\mathtt{MAPE}(x_i(k), \hat{x}_i^{F_2}(k\|k))$
1	0.0337%	0.0394%
2	0.0600%	0.0849%

Table 2. Estimation comparison of Problem 2

with $N_{score} = 100 \times \frac{N}{k_{\max}} = 34.72\%$.

For the computational times, we have $\mathtt{T}^{F_1} = 0.81\ s$ and $\mathtt{T}^{F_2} = 60.52\ s$.

Simulation results
The analysis of the two tables indicates that the performance of IF is intrinsically linked to the success of the criterion N, i.e., we obtain better results as N increases. Nevertheless, we have shown that our approach has a lower perfomance as N decreases. These results are instrinsly linked to the fact that the noise in $[C]$ of Problem 1 is higher than the noise in $[C]$ of Problem 2. Computationally, it is interesting to use this approach with high noise observation if compared to Mendes et al. (2019).

[5] Notation: $\mathtt{MAPE}(\mathbf{a}, \mathbf{b}) = \frac{100\%}{N} \sum_{i=1}^{N} \left| \frac{\mathbf{a}_i - \mathbf{b}_i}{\mathbf{a}_i} \right|$ where $\mathbf{a}_i$ is the true value and $\mathbf{b}_i$ is its estimated value.

6. CONCLUSIONS

In this work, we have presented an approach based on the residuation theory over interval matrices to compute the guaranteed interval w.r.t. a uMPL system. The procedure presented is computed in polynomial-time and is equivalent to the smallest interval that enclosures the intersection of the inverse image obtained thanks to (Candido et al., 2020, Algorithm 1) with the interval obtained in (9). Although the approach is an over-approximation of the exact intersected region, we have obtained a suitable method for Interval Filtering. As future work the authors aim to combine the probabilistic aspects in order to develop an Interval Stochastic Filtering, which corresponds to a classical Stochastic Filtering (*e.g.* the one proposed in Mendes et al. (2019)) with the aid of an IF: it uses a consistency approach to select the smallest interval in which the real state is included before we consider for instance the probability density function of the variables.

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