Observer Design for $(max, +)$ Linear Systems

Laurent Hardouin, Carlos Andrey Maia , Bertrand Cottenceau, Mehdi Lhommeau

***Abstract*—This paper deals with the state estimation for max-plus linear systems. This estimation is carried out following the ideas of the observer method for classical linear systems. The system matrices are assumed to be known, and the observation of the input and of the output is used to compute the estimated state. The observer design is based on the residuation theory which is suitable to deal with linear mapping inversion in idempotent semiring.**

***Index Terms*—Discrete Event Dynamics Systems, Idempotent Semirings, Max-Plus Algebra, Residuation Theory, Timed Event Graphs, Dioid, Observer, State Estimation.**

I. INTRODUCTION

Many discrete event dynamic systems, such as transportation networks [21], [12], communication networks, manufacturing assembly lines [3], are subject to synchronization phenomena. Timed event graphs (TEGs) are a subclass of timed Petri nets and are suitable tools to model these systems. A timed event graph is a timed Petri net of which all places have exactly one upstream transition and one downstream transition. Its description can be transformed into a $(max, +)$ or a $(min, +)$ linear model and *vice versa* [5], [1]. This property has advantaged the emergence of a specific control theory for these systems, and several control strategies have been proposed, $e.g.$, optimal open loop control [4], [20], [16], [19], and optimal feedback control in order to solve the model matching problem [6], [18], [14], [19] and also [22]. This paper focuses on observer design for $(max, +)$ linear systems. The observer aims at estimating the state for a given plant by using input and output measurements. The state trajectories correspond to the transition firings of the corresponding timed event graph, their estimation is worthy of interest because it provides insight into internal properties of the system. For example these state estimations are sufficient to reconstruct the marking of the graph, as it is done in [10] for Petri nets without temporization. The state estimation has many potential applications, such as fault detection, diagnosis, and state feedback control.

The $(max, +)$ algebra is a particular idempotent semiring, therefore section II reviews some algebraic tools concerning these algebraic structures. Some results about the residuation theory and its applications over semiring are also given. Section III recalls the description of timed event graphs in a semiring of formal series. Section IV presents and develops the proposed observer. It is designed by analogy with the classical Luenberger [17] observer for linear systems. It is done under the assumption that the system behavior is $(max, +)$-linear. This assumption means the model represents the fastest system behavior, in other words it implies that the system is unable to be accelerated, and consequently the disturbances can only reduce the system performances $i.e.$, they can only delay the events occurrence. They can be seen as machine breakdown in a manufacturing system, or delay due to an unexpected crowd of people in a transport network. In the opposite, the disturbances which increase system performances, $i.e.$, which anticipate the events occurrence, could give an upper estimation of the state, in this sense the results obtained are not equivalent to the observer for the classical linear systems. Consequently, it is assumed that the model and the initial state correspond to the fastest behavior (e.g. ideal behavior of the manufacturing system without extra delays or ideal behavior of the transport network without traffic holdup and with the maximal speed) and that disturbances only delay the occurrence of events. Under these assumptions a sufficient condition allowing to ensure equality between the state and the estimated state is given in proposition 4 in spite of possible disturbances, and proposition 3 yields some weaker sufficient conditions allowing to ensure equality between the asymptotic slopes of the state and the one of the estimated state, that means the error between both is always bounded. We invite the reader to consult the following link http://www.istia.univ-angers.fr/~hardouin/Observer.html to discover a dynamic illustration of the observer behavior.

II. ALGEBRAIC SETTING

An idempotent semiring $\mathcal{S}$ is an algebraic structure with two internal operations denoted by $\oplus$ and $\otimes$. The operation $\oplus$ is associative, commutative and idempotent, that is, $a \oplus a = a$. The operation $\otimes$ is associative (but not necessarily commutative) and distributive on the left and on the right with respect to $\oplus$. The neutral elements of $\oplus$ and $\otimes$ are represented by ε and e respectively, and ε is an absorbing element for the law $\otimes$ ($\forall a \in \mathcal{S}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). As in classical algebra, the operator $\otimes$ will be often omitted in the equations, moreover, $a^i = a \otimes a^{i-1}$ and $a^0 = e$. In this algebraic structure, a partial order relation is defined by $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow \ b = a \wedge b$ (where $a \wedge b$ is the greatest lower bound of a and b), therefore an idempotent semiring $\mathcal{S}$ is a partially ordered set (see [1], [12] for an exhaustive introduction). An idempotent semiring $\mathcal{S}$ is said to be complete if it is closed for infinite $\oplus$-sums and if $\otimes$ distributes over infinite $\oplus$-sums. In particular $\top = \bigoplus_{x \in \mathcal{S}} x$ is the greatest element of $\mathcal{S}$ ($\top$ is called the top element of $\mathcal{S}$).

Example 1 ($\overline{\mathbb{Z}}_{\max}$): Set $\overline{\mathbb{Z}}_{\max} = \mathbb{Z} \cup \{-\infty, +\infty\}$ endowed with the max operator as sum and the classical sum $+$ as

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product is a complete idempotent semiring, usually denoted $\overline{\mathbb{Z}}_{\max}$, of which $\varepsilon = -\infty$ and $e = 0$.

Theorem 1 (see [1], th. 4.75): The implicit inequality $x \succeq ax \oplus b$ as well as the equation $x = ax \oplus b$ defined over $\mathcal{S}$, admit $x = a^*b$ as the least solution, where $a^* = \bigoplus_{i\in\mathbb{N}} a^i$ (Kleene star operator).

Properties 1: The Kleene star operator satisfies the following well known properties (see [9] for proofs, and [13] for more general results):

$$a^* = (a^*)^*, \qquad a^*a^* = a^*, \tag{1}$$

$$(a \oplus b)^* = a^*(ba^*)^* = (a^*b)^*a^*, \quad b(ab)^* = (ba)^*b. \tag{2}$$

Thereafter, the operator $a^+ = \bigoplus_{i\in\mathbb{N}^+} a^i = aa^* = a^*a$ is also considered, it satisfies the following properties:

$$a^+ = (a^+)^+, \qquad a^* = e \oplus a^+, \tag{3}$$

$$(a^*)^+ = (a^+)^* = a^*, \quad a^+ \preceq a^*. \tag{4}$$

Definition 1 (Residual and residuated mapping): An order preserving mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are partially ordered sets, is a *residuated mapping* if for all $y \in \mathcal{E}$ there exists a greatest solution for the inequality $f(x) \preceq y$ (hereafter denoted $f^\sharp(y)$). Obviously, if equality $f(x) = y$ is solvable, $f^\sharp(y)$ yields the greatest solution. The mapping $f^\sharp$ is called the *residual* of f and $f^\sharp(y)$ is the optimal solution of the inequality.

Theorem 2 (see [2],[1]): Let $f : (\mathcal{D}, \preceq) \to (\mathcal{C}, \preceq)$ be an order preserving mapping. The following statements are equivalent

(i) f is residuated.

(ii) there exists an unique order preserving mapping $f^\sharp : \mathcal{C} \to \mathcal{D}$ such that $f \circ f^\sharp \preceq \mathsf{Id}_{\mathcal{C}}$ and $f^\sharp \circ f \succeq \mathsf{Id}_{\mathcal{D}}$.

Example 2: Mappings $\Lambda_a : x \mapsto a \otimes x$ and $\Psi_a : x \mapsto x \otimes a$ defined over an idempotent semiring $\mathcal{S}$ are both residuated ([1], p. 181). Their residuals are order preserving mappings denoted respectively by $\Lambda_a^\sharp(x) = a \backslash x$ and $\Psi_a^\sharp(x) = x / a$. This means that $a \backslash b$ (resp. b / a) is the greatest solution of the inequality $a \otimes x \preceq b$ (resp. $x \otimes a \preceq b$).

Definition 2 (Restricted mapping): Let $f : \mathcal{D} \to \mathcal{C}$ be a mapping and $\mathcal{B} \subseteq \mathcal{D}$. We will denote by $f_{|\mathcal{B}} : \mathcal{B} \to \mathcal{C}$ the mapping defined by $f_{|\mathcal{B}} = f \circ \mathsf{Id}_{|\mathcal{B}}$ where $\mathsf{Id}_{|\mathcal{B}} : \mathcal{B} \to \mathcal{D}, x \mapsto x$ is the canonical injection. Identically, let $\mathcal{E} \subseteq \mathcal{C}$ be a set such that $\mathsf{Im} f \subseteq \mathcal{E}$. Mapping ${}_{\mathcal{E}|}f : \mathcal{D} \to \mathcal{E}$ is defined by $f = \mathsf{Id}_{|\mathcal{E}} \circ {}_{\mathcal{E}|}f$, where $\mathsf{Id}_{|\mathcal{E}} : \mathcal{E} \to \mathcal{C}, x \mapsto x$.

Definition 3 (Closure mapping): A closure mapping is an order preserving mapping $f : \mathcal{D} \to \mathcal{D}$ defined on an ordered set $\mathcal{D}$ such that $f \succeq \mathsf{Id}_{\mathcal{D}}$ and $f \circ f = f$.

Proposition 1 (see [6]): Let $f : \mathcal{D} \to \mathcal{D}$ be a closure mapping. Then, ${}_{\mathsf{Im}f|}f$ is a residuated mapping whose residual is the canonical injection $\mathsf{Id}_{|\mathsf{Im}f}$.

Example 3: Mapping $K : \mathcal{S} \to \mathcal{S}, x \mapsto x^*$ is a closure mapping (indeed $a \preceq a^*$ and $a^* = (a^*)^*$ see equation (1)). Then $({}_{\mathsf{Im}K|}K)$ is residuated and its residual is $({}_{\mathsf{Im}K|}K)^\sharp = \mathsf{Id}_{|\mathsf{Im}K}$. In other words, $x = a^*$ is the greatest solution of inequality $x^* \preceq a$ if $a \in \mathsf{Im}K$, that is $x \preceq a^* \Leftrightarrow x^* \preceq a^*$.

Example 4: Mapping $P : \mathcal{S} \to \mathcal{S}, x \mapsto x^+$ is a closure mapping (indeed $a \preceq a^+$ and $a^+ = (a^+)^+$ see equation (3)). Then $({}_{\mathsf{Im}P|}P)$ is residuated and its residual is $({}_{\mathsf{Im}P|}P)^\sharp = \mathsf{Id}_{|\mathsf{Im}P}$. In other words, $x = a^+$ is the greatest solution of inequality $x^+ \preceq a$ if $a \in \mathsf{Im}P$, that is $x \preceq a^+ \Leftrightarrow x^+ \preceq a^+$.

Remark 1: According to equation (4), $(a^*)^+ = a^*$, therefore $\mathsf{Im}K \subset \mathsf{Im}P$.

Properties 2: Some useful results involving these residuals are presented below (see [1] for proofs and more complete results).

$$a \backslash a = (a \backslash a)^* \qquad a / a = (a / a)^* \tag{5}$$

$$a(a \backslash (ax)) = ax \qquad ((xa) / a)a = xa \tag{6}$$

$$b \backslash a \backslash x = (ab) \backslash x \qquad x / a / b = x / (ba) \tag{7}$$

$$a^* \backslash (a^*x) = a^*x \qquad (a^*x) / a^* = a^*x \tag{8}$$

$$(a \backslash x) \wedge (a \backslash y) = a \backslash (x \wedge y) \qquad (x / a) \wedge (y / a) = (x \wedge y) / a \tag{9}$$

The set of $n \times n$ matrices with entries in $\mathcal{S}$ is an idempotent semiring. The sum, the product and the residuation of matrices are defined after the sum, the product and the residuation of scalars in $\mathcal{S}$, *i.e.*,

$$(A \otimes B)_{ik} = \bigoplus_{j=1\ldots n} (a_{ij} \otimes b_{jk}) \tag{10}$$

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, \tag{11}$$

$$(A \backslash B)_{ij} = \bigwedge_{k=1..n} (a_{ki} \backslash b_{kj}) \ , \ (B / A)_{ij} = \bigwedge_{k=1..n} (b_{ik} / a_{jk}). \tag{12}$$

The identity matrix of $\mathcal{S}^{n\times n}$ is the matrix with entries equal to e on the diagonal and to ε elsewhere. This identity matrix will also be denoted e, and the matrix with all its entries equal to ε will also be denoted ε.

Definition 4 (Reducible and irreducible matrices): Let A be a $n \times n$ matrix with entries in a semiring $\mathcal{S}$. Matrix A is said reducible, if and only if for some permutation matrix P, the matrix P^TAP is block upper triangular. If matrix A is not reducible, it is said to be irreducible.

III. TEG DESCRIPTION IN IDEMPOTENT SEMIRING

Timed event graphs constitute a subclass of timed Petri nets $i.e.$ those whose places have one and only one upstream and downstream transition. A timed event graph (TEG) description can be transformed into a $(max, +)$ or a $(min, +)$ linear model and *vice versa*. To obtain an algebraic model in $\overline{\mathbb{Z}}_{\max}$, a "dater" function is associated to each transition. For transition labelled x_i, $x_i(k)$ represents the date of the k^{th} firing (see [1],[12]). A trajectory of a TEG transition is then a firing date sequence of this transition. This collection of dates can be represented by a formal series $x(\gamma) = \bigoplus_{k\in\mathbb{Z}} x_i(k) \otimes \gamma^k$ where $x_i(k) \in \overline{\mathbb{Z}}_{\max}$ and γ is a backward shift operator[1] in the event domain (formally $\gamma x(k) = x(k-1)$). The set of formal series in γ is denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and constitutes a complete idempotent semiring. For instance, considering the TEG in figure 1, daters x_1, x_2 and x_3 are related as follows over $\overline{\mathbb{Z}}_{\max}$: $x_1(k) = 4 \otimes x_1(k-1) \oplus 1 \otimes x_2(k) \oplus 6 \otimes x_3(k)$. Their respective γ-transforms, expressed over $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, are then related as:

$$x_1(\gamma) = 4\gamma x_1(\gamma) \oplus 1x_2(\gamma) \oplus 6x_3(\gamma).$$

[1] Operator γ plays a role similar to operator z^{-1} in the $\mathcal{Z}$ − transform for the conventional linear systems theory.

In this paper TEGs are modelled in this setting, by the following model :

$$\begin{aligned} x &= Ax \oplus Bu \oplus Rw \\ y &= Cx, \end{aligned} \tag{13}$$

where $u \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^p$, $y \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^m$ and $x \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^n$ are respectively the controllable input, output and state vector, $i.e.$, each of their entries is a trajectory which represents the collection of firing dates of the corresponding transition. Matrices $A \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times n}$, $B \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times p}$, $C \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{m\times n}$ represent the links between each transition, and then describe the structure of the graph. Vector $w \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^l$ represents uncontrollable inputs ($i.e.$ disturbances[2]). Each entry of w corresponds to a transition which disables the firing of internal transition of the graph, and then decreases the performance of the system. This vector is bound to the graph through matrix $R \in (\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])^{n\times l}$.

Afterwards, each input transition u_i (respectively w_i) is assumed to be connected to one and only one internal transition x_j, this means that each column of matrix B (resp. R) has one entry equal to e and the others equal to ε and at most one entry equal to e on each row. Furthermore, each output transition y_i is assumed to be linked to one and only one internal transition x_j, $i.e$ each row of matrix C has one entry equal to e and the others equal to ε and at most one entry equal to e on each column. These requirements are satisfied without loss of generality, since it is sufficient to add extra input and output transition. Note that if R is equal to the identity matrix, w can represent initial state of the system $x(0)$ by considering $w = x(0)\gamma^0 \oplus ...$ (see [1], p. 245, for a discussion about compatible initial conditions). By considering theorem 1, this system can be rewritten as :

$$x = A^*Bu \oplus A^*Rw \tag{14}$$
$$y = CA^*Bu \oplus CA^*Rw, \tag{15}$$

where $(CA^*B) \in (\overline{\mathbb{Z}}_{\max})^{m\times p}$ (respectively $(CA^*R) \in (\overline{\mathbb{Z}}_{\max})^{m\times l}$) is the input/output (resp. disturbance/output) transfer matrix. Matrix (CA^*B) represents the earliest behavior of the system, therefore it must be underlined that the uncontrollable inputs vector w (initial conditions or disturbances) is only able to delay the transition firings, $i.e.$, according to the order relation of the semiring, to increase the vectors x and y.

If the TEG is strongly connected, $i.e.$ there exists at least one path between transitions x_i, x_j $\forall i, j$, then matrix A is irreducible. If A is reducible, according to definition 4, there exists a permutation matrix such that :

$$A = \begin{pmatrix} A_{11} & A_{12} & ... & A_{1k} \\ \varepsilon & A_{22} & ... & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & ... & A_{kk} \end{pmatrix} \tag{16}$$

where k is the number of strongly connected components of the TEG, and each matrix A_{ii} is an irreducible matrix associated to the component i. Matrices A_{ij} (with $i \neq j$) represent the links between these strongly connected components. Consequently, for the TEG depicted fig.1, the following

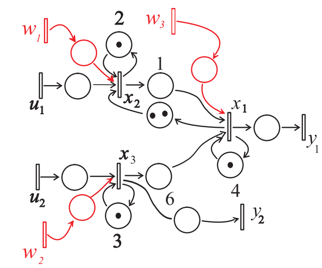

Fig. 1. Timed event graph, u_i controllable and w_i uncontrollable inputs.

matrices are obtained: $A = \begin{pmatrix} 4\gamma & 1 & 6 \\ \gamma^2 & 2\gamma & \varepsilon \\ \varepsilon & \varepsilon & 3\gamma \end{pmatrix}$, $B = \begin{pmatrix} \varepsilon & \varepsilon \\ e & \varepsilon \\ \varepsilon & e \end{pmatrix}$, $C = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix}$, $R = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix}$, and leads to the following A^* matrix:

$$A^* = \begin{pmatrix} (4\gamma)^* & 1(4\gamma)^* & 6(4\gamma)^* \\ \gamma^2(4\gamma)^* & e \oplus 2\gamma \oplus 4\gamma^2 \oplus 6\gamma^3 \oplus 9\gamma^4(4\gamma)^* & 6\gamma^2(4\gamma)^* \\ \varepsilon & \varepsilon & (3\gamma)^* \end{pmatrix}$$

According to assumptions about matrices C, B, and R, the matrices (CA^*B) and (CA^*R) are composed of some entries of matrix A^*. Each entry is a periodic series [1] in the $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ semiring. A periodic series s is usually represented by $s = p \oplus qr^*$, where p (respectively q) is a polynomial depicting the transient (resp. the periodic) behavior, and $r = \tau\gamma^\nu$ is a monomial depicting the periodicity allowing to define the asymptotic slope of the series as $\sigma_\infty(s) = \nu/\tau$ (see figure 2). Sum, product, and residuation of periodic series are well defined (see [9]), and algorithms and software toolboxes are available in order to handle periodic series and compute transfer relations (see [7]). Below, only the rules between monomials and properties concerning asymptotic slope are recalled :

$$\begin{aligned} t_1\gamma^n \oplus t_2\gamma^n &= max(t_1, t_2)\gamma^n, \\ t\gamma^{n_1} \oplus t\gamma^{n_2} &= t\gamma^{min(n_1,n_2)}, \\ t_1\gamma^{n_1} \otimes t_2\gamma^{n_2} &= (t_1 + t_2)\gamma^{(n_1+n_2)}, \\ (t_1\gamma^{n_1}) \phi (t_2\gamma^{n_2}) &= (t_2\gamma^{n_2}) \backslash\!\!\!\circ (t_1\gamma^{n_1}) = (t_1 - t_2)\gamma^{(n_1-n_2)}, \end{aligned}$$

$$\sigma_\infty(s \oplus s') = \min(\sigma_\infty(s), \sigma_\infty(s')), \tag{17}$$
$$\sigma_\infty(s \otimes s') = \min(\sigma_\infty(s), \sigma_\infty(s')), \tag{18}$$
$$\sigma_\infty(s \wedge s') = \max(\sigma_\infty(s), \sigma_\infty(s')), \tag{19}$$

$$\begin{aligned} &\text{if } \sigma_\infty(s) \leq \sigma_\infty(s') \text{ then } \sigma_\infty(s' \backslash\!\!\!\circ s) = \sigma_\infty(s), \\ &\text{else } \; s' \backslash\!\!\!\circ s = \varepsilon. \end{aligned} \tag{20}$$

Let us recall that if matrix A is irreducible then all the entries of matrix A^* have the same asymptotic slope, which will be denoted $\sigma_\infty(A)$. If A is a reducible matrix assumed to be in its block upper triangular representation, then matrix A^* is block upper triangular and matrices $(A^*)_{ii}$ are such that $(A^*)_{ii} = A^*_{ii}$ for each $i \in [1, k]$. Therefore, since A_{ii} is irreducible, all the entries of matrix $(A^*)_{ii}$ have the same asymptotic slope $\sigma_\infty((A^*)_{ii})$. Furthermore, entries of each

[2] In manufacturing setting, w may represent machine breakdowns or failures in component supply.

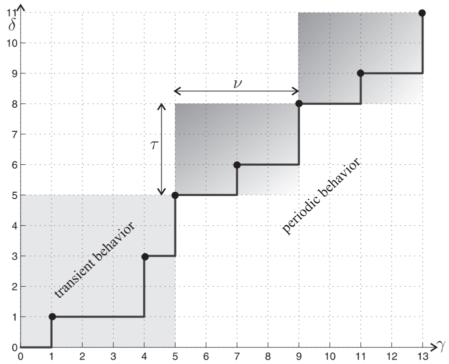

Fig. 2. Periodic series $s = (e \oplus 1\gamma^1 \oplus 3\gamma^4) \oplus (5\gamma^5 \oplus 6\gamma^7)(3\gamma^4)^*$.

matrix $(A^*)_{ij}$ with $i < j$ are such that their asymptotic slope is lower than or equal to $min(\sigma_\infty((A^*)_{ii}), \sigma_\infty((A^*)_{jj}))$.

IV. MAX-PLUS OBSERVER

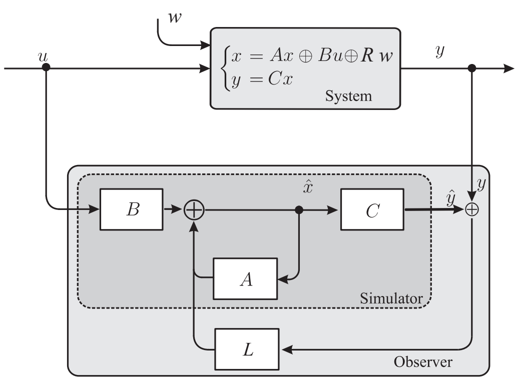

Fig. 3. Observer structure.

Figure 3 depicts the observer structure directly inspired from the classical linear system theory (see [17]). The observer matrix L aims at providing information from the system output into the simulator, in order to take the disturbances w acting on the system into account. The simulator is described by the model[3] (matrices A, B, C) which is assumed to represent the fastest behavior of the real system in a guaranteed way[4], furthermore the simulator is initialized by the canonical initial conditions ($i.e.$ $\hat{x}_i(k) = \varepsilon, \forall k \leq 0$). These assumptions induce that $y \succeq \hat{y}$ since disturbances and initial conditions, depicted by w, are only able to increase the system output. By considering the configuration of figure 3 and these assumptions, the computation of the optimal observer matrix L_x will be proposed in order to achieve the constraint $\hat{x} \preceq x$. Optimality means that the matrix is obtained thanks to the residuation theory and then it is the greatest one (see definition 1), hence the estimated state $\hat{x}$ is the greatest which achieves the objective. Obviously this optimality is only ensured under the assumptions considered ($i.e.$ $\hat{y} \preceq y$). As in the development proposed in conventional linear systems theory, matrices A, B, C and R are assumed to be known, then the system transfer is given by equations (14) and (15). According to figure 3 the observer equations are given by:

$$\begin{aligned} \hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \\ &= A\hat{x} \oplus Bu \oplus LC\hat{x} \oplus LCx \qquad (21) \\ \hat{y} &= C\hat{x}. \end{aligned}$$

By applying Theorem 1 and by considering equation (14), equation (21) becomes :

$$\begin{aligned} \hat{x} = {} & (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Bu \\ & \oplus (A \oplus LC)^*LCA^*Rw. \qquad (22) \end{aligned}$$

By applying equation (2) the following equality is obtained :

$$(A \oplus LC)^* = A^*(LCA^*)^*, \qquad (23)$$

by replacing in equation (22) :

$$\begin{aligned} \hat{x} = {} & A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^*LCA^*Bu \\ & \oplus A^*(LCA^*)^*LCA^*Rw, \end{aligned}$$

and by recalling that $(LCA^*)^*LCA^* = (LCA^*)^+$, this equation may be written as follows :

$$\hat{x} = A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Bu \oplus A^*(LCA^*)^+Rw.$$

Equation (4) yields $(LCA^*)^* \succeq (LCA^*)^+$, then the observer model may be written as follows :

$$\begin{aligned} \hat{x} &= A^*(LCA^*)^*Bu \oplus A^*(LCA^*)^+Rw \\ &= (A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw. \qquad (24) \end{aligned}$$

As said previously the objective considered is to compute the greatest observation matrix L such that the estimated state vector $\hat{x}$ be as close as possible to state x, under the constraint $\hat{x} \preceq x$, formally it can be written :

$$(A \oplus LC)^*Bu \oplus (A \oplus LC)^*LCA^*Rw \preceq A^*Bu \oplus A^*Rw \qquad \forall$$

or equivalently :

$$\begin{aligned} (A \oplus LC)^*B &\preceq A^*B \qquad (25) \\ (A \oplus LC)^*LCA^*R &\preceq A^*R. \qquad (26) \end{aligned}$$

Lemma 1: The greatest matrix L such that $(A \oplus LC)^*B = A^*B$ is given by:

$$L_1 = (A^*B) \phi (CA^*B). \qquad (27)$$

Proof: First let us note that $L = \varepsilon \in \overline{\mathbb{Z}}_{\max}^{n \times m}$ is a solution, indeed $(A \oplus \varepsilon C)^*B = A^*B$. Consequently, the greatest solution of the inequality $(A \oplus LC)^*B \preceq A^*B$ will satisfy the equality. Furthermore, according to equation (2), $(A \oplus LC)^*B = (A^*LC)^*A^*B$. So the objective is given by :

$$\begin{aligned} & (A^*LC)^*A^*B \preceq A^*B \\ \Leftrightarrow \; & (A^*LC)^* \preceq (A^*B)\phi(A^*B) && \text{(see example 2)} \\ \Leftrightarrow \; & (A^*LC)^* \preceq ((A^*B)\phi(A^*B))^* && \text{(see eq.(5))} \\ \Leftrightarrow \; & (A^*LC) \preceq (A^*B)\phi(A^*B) && \text{(see example 3)} \\ \Leftrightarrow \; & L \preceq A^* \backslash\!\!\!\circ (A^*B)\phi(A^*B)\phi C && \text{(see example 2)} \\ \Leftrightarrow \; & L \preceq A^* \backslash\!\!\!\circ (A^*B)\phi(CA^*B) && \text{(see eq.(7))} \\ \Leftrightarrow \; & L \preceq (A^*B)\phi(CA^*B) = L_1 && \text{(see eq.(8))} \end{aligned}$$

[3]Disturbances are uncontrollable and *a priori* unknown, then the simulator does not take them into account.

[4]Unlike in the conventional linear system theory, this assumption means that the fastest behavior of the system is assumed to be known and that the disturbances can only delay its behavior.

■

Lemma 2: The greatest matrix L that satisfies $(A \oplus LC)^*LCA^*R \preceq A^*R$ is given by:

$$L_2 = (A^*R) \phi (CA^*R). \tag{28}$$

Proof:

$$\begin{array}{ll} & (A \oplus LC)^*LCA^*R \preceq A^*R \\ \Leftrightarrow & A^*(LCA^*)^*LCA^*R \preceq A^*R \qquad \text{(see eq.(23))}, \\ \Leftrightarrow & (LCA^*)^*LCA^*R \preceq A^* \backslash\!\!\circ (A^*R) = A^*R \\ & \text{(see example 2 and eq.(8), with } x = R), \\ \Leftrightarrow & (LCA^*)^*LCA^*A^*R = (LCA^*)^+A^*R \preceq A^*R \\ & \text{(see eq.(1) and } a^+ \text{ definition)}, \\ \Leftrightarrow & (LCA^*)^+ \preceq (A^*R) \phi (A^*R) = ((A^*R) \phi (A^*R))^* \\ & \text{(see eq.(5))}, \end{array}$$

according to remark 1 the right member is in $\mathsf{Im}P$, then by applying the result presented in example 4, this inequality may be written as follows :

$$\begin{array}{ll} & LCA^* \preceq (A^*R) \phi (A^*R) \\ \Leftrightarrow & L \preceq (A^*R) \phi (A^*R) \phi (CA^*) = (A^*R) \phi (CA^*A^*R) \\ & \text{(see example 2 and eq. (8))} \\ \Leftrightarrow & L \preceq (A^*R) \phi (CA^*R) = L_2 \\ & \text{(see eq. (1))}. \end{array}$$

■

Proposition 2: $L_x = L_1 \wedge L_2$ is the greatest observer matrix such that:

$$\hat{x} = A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \preceq x = Ax \oplus Bu \oplus Rw \quad \forall (u, w).$$

Proof: Lemma 1 implies $L \preceq L_1$ and lemma 2 implies $L \preceq L_2$, then $L \preceq L_1 \wedge L_2 = L_x$. ■

Corollary 1: The matrix L_x ensures the equality between estimated output $\hat{y}$ and measured output y, *i.e.*

$$C(A \oplus L_xC)^*B = CA^*B, \tag{29}$$

$$C(A \oplus L_xC)^*L_xCA^*R = CA^*R. \tag{30}$$

Proof: Let $\tilde{L} = e \phi C$ be a particular observer matrix. Definition 1 yields $\tilde{L}C \preceq e$ then $(A \oplus \tilde{L}C)^* = A^*$. This equality implies $(A \oplus \tilde{L}C)^*B = A^*B$, therefore according to lemma 1 $\tilde{L} \preceq L_1$, since L_1 is the greatest solution. That implies also that L_1 is solution of equation (29). Equality $(A \oplus \tilde{L}C)^* = A^*$ and inequality $\tilde{L}C \preceq e$ yield $(A \oplus \tilde{L}C)^*\tilde{L}CA^*R = A^*\tilde{L}CA^*R \preceq A^*R$ then according to lemma 2 $\tilde{L} \preceq L_2$ since L_2 is the greatest solution. That implies also that $\tilde{L}$ and L_2 are such that $C(A \oplus \tilde{L}C)^*\tilde{L}CA^*R \preceq C(A \oplus L_2C)^*L_2CA^*R \preceq CA^*R$. The assumption about matrix C (see section III) yields $CC^T = e$ and $\tilde{L} = e \phi C = C^T$, therefore $C(A \oplus \tilde{L}C)^*\tilde{L}CA^*R = CA^*\tilde{L}CA^*R = (C\tilde{L} \oplus CA\tilde{L} \oplus ...)CA^*R \succeq C\tilde{L}CA^*R = CC^TCA^*R = CA^*R$. Therefore, since $\tilde{L} \preceq L_2$, we have $C(A \oplus \tilde{L}C)^*\tilde{L}CA^*R = C(A \oplus L_2C)^*L_2CA^*R = CA^*R$ and both $\tilde{L}$ and L_2 yield equality (30). To conclude $\tilde{L} \preceq L_1 \wedge L_2 = L_x$, hence, $L_x \preceq L_1$ yields the equality (29) and $L_x \preceq L_2$ yields (30). Therefore equality $\hat{y} = y$ is ensured. ■

Remark 2: By considering matrix $\overline{B} = \begin{pmatrix} B & R \end{pmatrix}$, equations (12) and (9), matrix L_x may be written as : $L_x = (A^*\overline{B} \backslash\!\!\circ (CA^*\overline{B})$.

According to the residuation theory (see definition 1), L_x yields $x = \hat{x}$ if possible. Nevertheless, two questions arise, firstly is it possible to ensure equality between the asymptotic slope of each state vector entries ? Secondly is it possible to ensure equality between these vectors ? Below, sufficient conditions allowing to answer positively are given.

Proposition 3: Let k be the number of strongly connected components of the TEG considered. If matrix $C \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{k \times n}$ is defined as in section III and such that each strongly connected component is linked to one and only one output then $\sigma_\infty(x_i) = \sigma_\infty(\hat{x}_i) \forall i \in [1, n]$.

Proof: First, assuming that matrix A is irreducible (i.e., $k = 1$), then all entries of matrix A^* have the same asymptotic slope $\sigma_\infty(A^*)$. As said in section III entries of matrices B, R, and C are equal to ε or e, therefore, according to matrices operation definitions (see equations (10) to (12) and rules (17) to (20)), all the entries of matrices A^*B, A^*R, CA^*B, CA^*R and L_x have the same asymptotic slope which is equal to $\sigma_\infty(A^*)$. Consequently, by considering equation (24), $\sigma_\infty(((A \oplus L_xC)^*B)_{ij}) = \sigma_\infty((A^*B)_{ij})$ and $\sigma_\infty(((A \oplus L_xC)^*L_xCA^*R)_{ij}) = \sigma_\infty((A^*R)_{ij})$, which leads to $\sigma_\infty(x_i) = \sigma_\infty(\hat{x}_i)$ $\forall i \in [1, n]$.
Now the reducible case is considered. To increase the readability, matrices B and R are assumed to be equal to e and the proof is given for a graph with two strongly connected components. The extension for a higher dimension may be obtained in an analogous way. As said in section III, matrix A^* is block upper diagonal :

$$A^* = \begin{pmatrix} (A^*)_{11} & (A^*)_{12} \\ \varepsilon & (A^*)_{22} \end{pmatrix},$$

all the entries of the square matrix $(A^*)_{ii}$ have the same asymptotic slope $\sigma_\infty((A^*)_{ii})$ and all the entries of matrix $(A^*)_{12}$ have the same asymptotic slope, $\sigma_\infty((A^*)_{12}) = min(\sigma_\infty((A^*)_{11}), \sigma_\infty((A^*)_{22}))$. Assumption about matrix $C \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{2 \times n}$, *i.e.* one and only one entry is linked to each strongly connected component, yields the following block upper diagonal matrix :

$$CA^* = \begin{pmatrix} (CA^*)_{11} & (CA^*)_{12} \\ \varepsilon & (CA^*)_{22} \end{pmatrix},$$

where $\begin{pmatrix} (CA^*)_{11} & (CA^*)_{12} \end{pmatrix}$ is one row of matrix $\begin{pmatrix} (A^*)_{11} & (A^*)_{12} \end{pmatrix}$ and $\begin{pmatrix} \varepsilon & (CA^*)_{22} \end{pmatrix}$ is one row of matrix $\begin{pmatrix} \varepsilon & (A^*)_{22} \end{pmatrix}$, hence $\sigma_\infty((CA^*)_{ij}) = \sigma_\infty((A^*)_{ij})$. Matrix L_x is also block upper diagonal :

$$L_x = A^* \phi C = \begin{pmatrix} L_{x11} & L_{x12} \\ \varepsilon & L_{x22} \end{pmatrix},$$

where $\begin{pmatrix} L_{x11} & \varepsilon \end{pmatrix}^T$ is one column of matrix $\begin{pmatrix} (A^*)_{11} & \varepsilon \end{pmatrix}^T$ and $\begin{pmatrix} L_{x12} & L_{x22} \end{pmatrix}^T$ is one column of matrix $\begin{pmatrix} (A^*)_{12} & (A^*)_{22} \end{pmatrix}^T$, hence $\sigma_\infty(L_{xij}) = \sigma_\infty((A^*)_{ij})$. Therefore L_xCA^* is block upper diagonal :

$$\begin{aligned} L_xCA^* &= \begin{pmatrix} L_{x11}(CA^*)_{11} & L_{x11}(CA^*)_{12} \oplus L_{x12}(CA^*)_{22} \\ \varepsilon & L_{x22}(CA^*)_{22} \end{pmatrix} \\ &= \begin{pmatrix} (L_xCA^*)_{11} & (L_xCA^*)_{12} \\ \varepsilon & (L_xCA^*)_{22} \end{pmatrix}, \end{aligned} \tag{31}$$

and by considering rules (17) and (20), the sub matrices are such that $\sigma_\infty((L_xCA^*)_{ij}) = \sigma_\infty((A^*)_{ij})$. By recalling that $(A \oplus L_xC)^* = A^*(L_xCA^*)^*$, we obtain $\sigma_\infty(((A \oplus L_xC)^*)_{ij}) = \sigma_\infty((A^*)_{ij})$ and $\sigma_\infty(((A \oplus L_xC)^* L_xCA^*)_{ij}) = \sigma_\infty((A^*)_{ij})$, which leads to $\sigma_\infty(x_i) = \sigma_\infty(\hat{x}_i)\ \forall i \in [1, n]$. ■

Proposition 4: If matrix $A^*\overline{B}$ is in $\mathsf{Im}\Psi_{CA^*\overline{B}}$, matrix L_x is such that $\hat{x} = x$.

Proof: First, let us recall that

$$\begin{array}{lcl} A^*\overline{B} \in \mathsf{Im}\Psi_{CA^*\overline{B}} & \Leftrightarrow & \exists z \text{ s.t. } A^*\overline{B} = zCA^*\overline{B} \\ & \Leftrightarrow & ((A^*\overline{B}) \phi (CA^*\overline{B}))(CA^*\overline{B}) = A^*\overline{B}. \end{array}$$

If $\exists z$ s.t. $A^*\overline{B} = zCA^*\overline{B}$ then

$$\begin{array}{lcl} L_xCA^*\overline{B} & = & ((A^*\overline{B}) \phi (CA^*\overline{B}))CA^*\overline{B} \\ & = & ((zCA^*\overline{B}) \phi (CA^*\overline{B}))CA^*\overline{B} \\ & = & zCA^*\overline{B} = A^*\overline{B} \text{ (see eq. (6),} \end{array}$$

by recalling that $\overline{B} = \begin{pmatrix} B & R \end{pmatrix}$, this equality can be written

$$\begin{pmatrix} L_xCA^*B & L_xCA^*R \end{pmatrix} = \begin{pmatrix} A^*B & A^*R \end{pmatrix}.$$

Therefore $(A \oplus L_xC)^* L_xCA^*R = A^*(L_xCA^*)^* L_xCA^*R = A^*(L_xCA^*)^+ R = A^*(L_xCA^*R \oplus (L_xCA^*)^2 R \oplus (L_xCA^*)^3 R \oplus ...)$ (see equation (23) and a^+ definition). Since $L_xCA^*R = A^*R$, the following equality is satified $(L_xCA^*)^2 R = L_xCA^*A^*R = L_xCA^*R = A^*R$ and more generally $(L_xCA^*)^i R = A^*R$, therefore L_x ensures equality $(A \oplus L_xC)^* L_xCA^*R = A^*(L_xCA^*)^+ R = A^*R$. On the other hand lemma 1 yields the equality $(A \oplus L_xC)^* B = A^*B$, which concludes the proof. ■

Remark 3: This sufficient condition gives an interesting test to know if the number of sensors is sufficient and if they are well localized to allow an exact estimation. Obviously, this condition is fulfilled if matrix C is equal to the identity.

Below, the synthesis of the observer matrices L_x for the TEG of figure 1 is given:

$$L_x = \begin{pmatrix} (4\gamma)^* & 6(4\gamma)^* \\ \gamma^2(4\gamma)^* & 6\gamma^2(4\gamma)^* \\ \varepsilon & (3\gamma)^* \end{pmatrix}$$

Assumptions of proposition 3 being fulfilled, it can easily be checked, by using toolbox Minmaxgd (see [7]), that $\sigma_\infty(x_i) = \sigma_\infty(\hat{x}_i)\ \forall i \in [1, n]$ and that $Cx = C\hat{x}\ \forall (u, w)$ according to corollary 1.

V. Conclusion

This paper[5] has proposed a methodology to design an observer for $(max, +)$ linear systems. The observer matrix is obtained thanks to the residuation theory and is optimal in the sense that it is the greatest which achieves the objective. It allows to compute a state estimation lower than or equal to the real state and ensures that the estimated output is equal to the system output. As a perspective, this state estimation may be used in state feedback control strategies as proposed in [6], [19], and an application to fault detection for manufacturing systems may be envisaged. Furthermore, in order to deal with uncertain systems an extension can be envisaged by considering interval analysis as it is done in [15],[11] and more recently in [8].

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