Modeling and Control for $(\max,+)$-Linear Systems with Set-Based Constraints

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***Abstract*— We consider $(\max,+)$-linear systems (*i.e.,* discrete event systems ruled by conditions of the form: "for all $k \geq l$, occurrence k of event e_2 is at least τ units of time after occurrence $k-l$ of event e_1" with $\tau, l \in \mathbb{N}_0$) with the additional restriction that certain events can only occur at time instants in predefined ultimately periodic sets of integers. Such phenomena frequently occur in transportation networks (*e.g.,* traffic lights). In this paper, we extend transfer function matrix models and model reference control developed for $(\max,+)$-linear systems to the above class of systems, which is more general than $(\max,+)$-linear systems. To illustrate these results, we consider a road traffic example with traffic lights.**

I. INTRODUCTION

A discrete event system is a dynamical system driven by the instantaneous occurrences of events. In a discrete event system, two basic elements are distinguished: the event set and the rule describing the behavior of the system. When this rule is only composed of standard synchronizations (*i.e.,* conditions of the form: "for all $k \geq l$, occurrence k of event e_2 is at least τ units of time after occurrence $k-l$ of event e_1" with $\tau, l \in \mathbb{N}_0$), the considered system is called $(\max,+)$-linear. This terminology is due to the fact that its behavior is described by linear equations in particular algebraic structures such as the $(\max,+)$-algebra. In a $(\max,+)$-linear system, the event set is partitioned into

- input events: these events are the source of standard synchronizations, but not subject to standard synchronizations
- output events: these events are subject to standard synchronizations, but not the source of standard synchronizations
- state events: these events are both subject to and the source of standard synchronizations

Based on this partition, and by possibly introducing further state events, a $(\max,+)$-linear state-space model, similar to the one from standard control theory, is obtained. Therefore, much effort was made during the last decades to adapt key concepts from standard control theory to $(\max,+)$-linear systems. For example, transfer function matrix models have been introduced for $(\max,+)$-linear systems by using formal power series [1]. Furthermore, some standard control approaches such as model reference control [2] or model predictive control [3] have been extended to $(\max,+)$-linear systems.

In this paper, we consider $(\max,+)$-linear systems with the additional restriction that certain events can only occur at time instants in predefined ultimately periodic sets of integers (the set $S \subseteq \mathbb{N}_0$ is said to be ultimately periodic if there exist $n_0 \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that $\forall n \geq n_0$, $n \in S \Rightarrow n+p \in S$). Such constraints are called set-based constraints and such systems are referred to as $(\max,+)$-linear systems with set-based constraints. As $(\max,+)$-linear systems are time-invariant while $(\max,+)$-linear systems with set-based constraints are not necessarily time-invariant (see Ex. 1), $(\max,+)$-linear systems with set-based constraints are strictly more general. In the following, based on an analogy with $(\max,+)$-linear systems, we extend transfer function matrix models and model reference control to $(\max,+)$-linear systems with set-based constraints. Note that, under some assumptions, $(\max,+)$-linear systems with set-based constraints are the dual in the time-domain of the discrete event systems investigated in [4]. Note also that the authors have already defined in [5] a more general class of systems using partial synchronization (*i.e.,* conditions of the form: "event e_2 can only occur *when*, not after, event e_1 occurs"). However, as shown in [6], it is not possible to extend transfer function matrix models and model reference control to this class of systems by using an analogy with $(\max,+)$-linear systems.

This paper is structured as follows. In the next section, necessary mathematical tools are recalled. The dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ is introduced in § III. Then, in § IV, transfer function matrix models in the dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ are developed for $(\max,+)$-linear systems with set-based constraints. Finally, § V focuses on model reference control for such systems. Throughout this paper, the following simple road traffic example with traffic lights is used to illustrate and clarify the presented concepts. Note that standard synchronizations cannot model intersections, as they cannot model choices. Hence, the considered models are not as general as other models for transportation networks [7], [8].

Example 1: This example deals with a road from P_1 to P_3 via P_2. The road is equipped with two traffic lights in P_2 and in P_3. The traffic light in P_2 allows other users such as pedestrians or trains to cross the road, but is not regulating an intersection with another road. Therefore, a vehicle entering the road in P_1 passes by P_2 and leaves the road in P_3. Next, the characteristics of the road are made explicit. The travel time from P_1 to P_2 and from P_2 to P_3 is at least ten units of time. The capacity of each section (*i.e.,* from P_1 to P_2 or from P_2 to P_3) is assumed to be infinite. When the traffic light is

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green, at most one vehicle can pass the traffic light per unit of time. Furthermore, the behavior of the traffic lights is known: each traffic light is green at time instants in the ultimately periodic set of integers $S = \{6k, 6k+1, 6k+2$ with $k \in \mathbb{N}_0\}$. Initially, no vehicles are on the road.

In the following, the system is modeled by a $(\max,+)$-linear systems with set-based constraints. The model is based on the following events:

u	a vehicle arrives on the road (input event)
x_i	a vehicle passes by P_i with $i = 1,2,3$ (state events)
y	a vehicle leaves the road (output event)

The behavior of the system is described by the following constraints:

- for all $k \geq 0$, occurrence k of event x_2 (resp. x_3) is at least ten units of time after occurrence k of event x_1 (resp. x_2)
- for all $k \geq 1$, occurrence k of event x_2 (resp. x_3) is at least one unit of time after occurrence $k-1$ of event x_2 (resp. x_3)
- for all $k \geq 0$, occurrence k of event x_1 (resp. y) is at least zero units of time after occurrence k of event u (resp. x_3)
- event x_2 (resp. x_3) can only occur at time instants in S

The first three items represent standard synchronizations and define a $(\max,+)$-linear system. The last item is composed of set-based constraints. Hence, the proposed model is a $(\max,+)$-linear system with set-based constraints.

A graphical representation of the road traffic model by a timed Petri net is given in Fig. 1. As usual, transitions represent events and are indicated by bars, places are shown as circles and holding times are written next to the corresponding places. Further, the dotted arrows picture the influence of traffic lights.

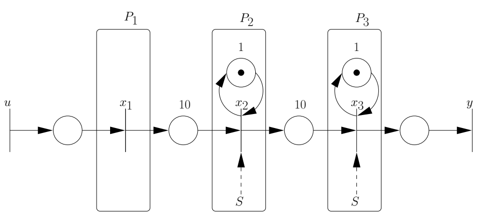

Fig. 1. Timed Petri net representing the considered road traffic example

In the following, we assume that the systems operates under the earliest functioning rule, *i.e.,* all output and state events occur as soon as possible. Consider a single vehicle entering the road at $t = 0$, then it leaves the road at $t = 24$. However, if the vehicle enters the road at $t = 1$, it also leaves the road at $t = 24$. Hence, the considered system is not time-invariant (*i.e.,* delaying the input by one unit of time does not induce an output delayed by one unit of time).

II. MATHEMATICAL PRELIMINARIES

The following is a short summary of basic results from dioid and residuation theory. The reader is invited to consult [1] for more details.

A dioid $\mathscr{D}$ is a set endowed with two internal operations $\oplus$ (addition) and $\otimes$ (multiplication, often denoted by juxtaposition), both associative and both having a neutral element denoted ε and e respectively. Moreover, $\oplus$ is commutative and idempotent ($\forall a \in \mathscr{D}, a \oplus a = a$), $\otimes$ is distributive with respect to $\oplus$, and ε is absorbing for $\otimes$ ($\forall a \in \mathscr{D}, \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). The operation $\oplus$ induces an order relation $\preceq$ on $\mathscr{D}$, defined by $\forall a, b \in \mathscr{D}, a \preceq b \Leftrightarrow a \oplus b = b$.

A dioid $\mathscr{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. A complete dioid $\mathscr{D}$ admits a greatest element denoted $\top$ defined by $\top = \bigoplus_{x \in \mathscr{D}} x$. For a in a complete dioid $\mathscr{D}$, the Kleene star of a, denoted a^* and defined by $a^* = \bigoplus_{k=0}^{+\infty} a^k$ (where $a^0 = e$ and $a^{k+1} = a \otimes a^k$ for $k \in \mathbb{N}_0$), exists and belongs to $\mathscr{D}$.

Example 2 (Dioid $\overline{\mathbb{N}}_{\max}$): A well-known complete dioid is the $(\max,+)$-algebra $\overline{\mathbb{N}}_{\max}$: the set $\mathbb{N}_0 \cup \{-\infty, +\infty\}$ is endowed with max as addition and $+$ as multiplication. Then, ε is equal to $-\infty$, e to 0, and $\top$ to $+\infty$. The associated order relation $\preceq$ is the usual order relation $\leq$.

Let $f : E \to F$ with E and F ordered sets. f is said to be residuated if f is isotone (*i.e.,* order-preserving) and if, for all $y \in F$, the least upper bound of the subset $\{x \in E | f(x) \preceq y\}$ exists and lies in this subset. This element in E is denoted $f^\sharp(y)$. Mapping $f^\sharp$ from F to E is called the residual of f. Over complete dioids, the mapping $L_a : x \mapsto a \otimes x$ (left-product by a), respectively $R_a : x \mapsto x \otimes a$ (right-product by a), is residuated. Its residual is denoted by $L_a^\sharp(x) = a \backslash\!\!\circ\, x$ (left-division by a), resp. $R_a^\sharp(x) = x \circ\!\!/ a$ (right-division by a).

A. Matrix Dioid

By analogy with standard linear algebra, $\oplus$ and $\otimes$ are defined for matrices with entries in a dioid $\mathscr{D}$. For $A, B \in \mathscr{D}^{n \times m}$ and $C \in \mathscr{D}^{m \times p}$,

- $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$
- $(A \otimes C)_{ij} = \bigoplus_{k=1}^{m} A_{ik} C_{kj}$

Endowed with these operations, the set of square matrices with entries in a complete dioid is also a complete dioid. Furthermore, the operations $\backslash\!\!\circ$ and $\circ\!\!/$ are extended to matrices with entries in a complete dioid.

B. Dioid of Formal Power Series

A formal power series in a single variable γ with coefficients in the complete dioid $\mathscr{D}$ is a mapping from $\mathbb{Z}$ to $\mathscr{D}$ equal to ε for $k < 0$. Then, a formal power series s is often written using the following notation:

$$s = \bigoplus_{k \in \mathbb{N}_0} s(k) \gamma^k$$

The set of formal power series is endowed with the operation $\oplus$ and $\otimes$ defined by

$$\forall k \in \mathbb{N}_0, \quad (s_1 \oplus s_2)(k) = s_1(k) \oplus s_2(k)$$
$$(s_1 \otimes s_2)(k) = \bigoplus_{j \in \mathbb{N}_0} s_1(j) \otimes s_2(k-j)$$

Endowed with these operations, the set of formal power series with coefficients in $\mathscr{D}$, denoted $\mathscr{D}[\![\gamma]\!]$, is a complete dioid [1].

1) Dioid of Isotone Formal Power Series: A series s in $\mathscr{D}[\![\gamma]\!]$ is said to be isotone (*i.e.,* order-preserving) if

$$\forall k,l \in \mathbb{Z}, \quad k \geq l \Rightarrow s(k) \succeq s(l)$$

Endowed with the above operations, the set of isotone formal power series with coefficients in $\mathscr{D}$, denoted $\mathscr{D}_\gamma[\![\gamma]\!]$, is a complete dioid [1]. By taking into account the isotony, compact notations for elements in $\mathscr{D}_\gamma[\![\gamma]\!]$ are available as shown in the following example.

Example 3: Let us consider the series s in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ defined by

$$s(k) = \begin{cases} \varepsilon \text{ if } k < 1 \\ 5 \text{ if } k = 1,2 \\ 8 \text{ if } k \geq 3 \end{cases}$$

Using the above notation,

$$s = 5\gamma \oplus 5\gamma^2 \oplus 8\gamma^3 \oplus 8\gamma^4 \oplus 8\gamma^5 \oplus \ldots$$

However, to simplify notation, we often write $s = 5\gamma \oplus 8\gamma^3$, as it is possible to reconstruct s from this simplified notation using isotony.

C. Dioid of Residuated Mappings

Let $\mathscr{D}$ be a complete dioid. The set of residuated mappings over $\mathscr{D}$, denoted $\mathscr{F}_{\mathscr{D}}$, endowed with the operations $\oplus$ and $\otimes$ defined by

$$\forall x \in \mathscr{D}, \quad (f_1 \oplus f_2)(x) = f_1(x) \oplus f_2(x)$$
$$f_1 \otimes f_2 = f_1 \circ f_2$$

is a complete dioid [1].

III. ALGEBRAIC TOOLS

In the following, the theoretical foundation for modeling and control of $(\max,+)$-linear systems with set-based constraints is given. First, the dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ is introduced. Second, some properties of the dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ (*i.e.,* the dioid of isotone formal power series in γ with coefficients in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$) are presented. The following results come mainly from [6].

A. Dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$

The dioid of residuated mappings over $\overline{\mathbb{N}}_{\max}$ is denoted $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$. As recalled in § II, the dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ is complete. In the following, some classes of mappings in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ are introduced.

Definition 1 (Causality): A mapping f in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ is said to be causal if $f = \varepsilon$ or if, for all $x \in \overline{\mathbb{N}}_{\max}$, $f(x) \succeq x$.

Definition 2 (Periodicity): A mapping f in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ is said to be periodic if there exist $X \in \mathbb{N}_0$ and $\omega \in \mathbb{N}$ such that

$$\forall x \succeq X, \quad f(\omega x) = \omega f(x)$$

Next, particular mappings in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ useful for the modeling of $(\max,+)$-linear systems with set-based constraints are introduced.

Example 4 (Mapping δ): The mapping δ in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ is defined by $\delta(x) = 1x$. This mapping is causal and periodic.

Example 5 (α-mappings): Let us consider $S \subseteq \mathbb{N}_0$. The α-mapping associated with S, denoted α_S, is defined by

$$\alpha_S(x) = \bigwedge \{z \succeq x | z \in S \cup \{-\infty, +\infty\}\}$$

where $\bigwedge A$ denotes the greatest lower bound of the set A. The mapping α_S is a causal element of $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$. Furthermore, if S is ultimately periodic, then α_S is periodic.

B. Dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$

The dioid of isotone formal power series in γ with coefficients in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ is denoted $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$. As recalled in § II, the dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ is complete.

Example 6: Let s be a series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ defined by $s = e\gamma \oplus f\gamma^3$ with

$$e(x) = x \text{ and } f(x) = \begin{cases} \varepsilon & \text{if } x = \varepsilon \\ 3\left\lceil \frac{x}{3} \right\rceil & \text{if } x \in \mathbb{N}_0 \\ \top & \text{if } x = \top \end{cases}$$

where the expression $3\left\lceil \frac{x}{3} \right\rceil$ is in the standard algebra.

A graphical representation of series s is drawn in Fig. 2, where the plane at a given k (*i.e.,* the plane $(x, s(k)(x))$) is the 2D-representation of the mapping $s(k)$.

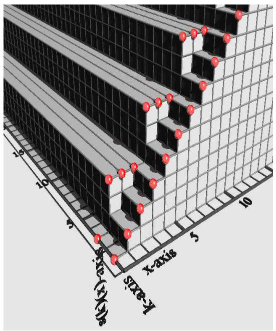

Fig. 2. Series $s = e\gamma \oplus f\gamma^3$

1) Slicing Mapping:

Definition 3 (Slicing mapping ψ): The slicing mapping ψ is a mapping from $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ to the set of mappings from $\overline{\mathbb{N}}_{\max}$ to $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ defined by

$$\forall s \in \mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!], \forall x \in \overline{\mathbb{N}}_{\max}, \quad \psi(s)(x) = \bigoplus_{k \in \mathbb{Z}} s(k)(x)\gamma^k$$

Example 7: The mapping $\psi(s)$ associated with the series s given in Ex. 6 is defined by

$$\psi(s)(x) = \begin{cases} \varepsilon & \text{if } x = \varepsilon \\ x\gamma & \text{if } x = 3^j \text{ with } j \in \mathbb{N}_0 \\ x\gamma \oplus 2x\gamma^3 & \text{if } x = 1 \otimes 3^j \text{ with } j \in \mathbb{N}_0 \\ x\gamma \oplus 1x\gamma^3 & \text{if } x = 2 \otimes 3^j \text{ with } j \in \mathbb{N}_0 \\ \top\gamma & \text{if } x = \top \end{cases}$$

The series $\psi(s)(x)$ provides the planes $(k, s(k)(x))$ for $x \in \overline{\mathbb{N}}_{\max}$ (*i.e.,* corresponding to the 2D-representation of the series $\psi(s)(x)$ in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$): $\psi(s)(x)$ corresponds to the slice of the series s at $x \in \overline{\mathbb{N}}_{\max}$ in Fig. 2.

2) Periodicity: First, the definition of periodicity is recalled for series in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$.

Definition 4 (Periodicity in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$): A series s in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ is said to be periodic if there exist $N, \tau \in \mathbb{N}_0$ and $\nu \in \mathbb{N}$ such that

$$\forall n \geq N, \quad s(n+\nu) = \tau s(n)$$

The throughput of a periodic series s in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$, denoted by $\sigma(s)$, is defined by $\frac{\nu}{\tau}$.

Second, periodicity for series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ is introduced.

Definition 5 (Periodicity): A series s in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ is said to be periodic if there exist $N \in \mathbb{N}$ periodic mappings $f_1,\ldots,f_N$ in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$, $n_1,\ldots,n_N$ in $\mathbb{N}_0$, $\tau_1,\ldots,\tau_N$ in $\mathbb{N}_0$, and ν in $\mathbb{N}$ such that

$$s = \bigoplus_{k=1}^{N} (\delta^{\tau_k}\gamma^{\nu})^* f_k \gamma^{n_k}$$

A matrix with entries in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ is said to be periodic if all its entries are periodic.

A link between periodicity in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ and in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ is provided by the slicing mapping ψ. For a periodic series $s \in \mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$, $\psi(s)(x)$ is a periodic series in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ for all $x \in \overline{\mathbb{N}}_{\max}$. Furthermore, there exists $X \in \mathbb{N}_0$ such that

$$\forall x \in \mathbb{N}_0, \quad x \geq X \Rightarrow \sigma(\psi(s)(x)) = \sigma(\psi(s)(X))$$

Then, the throughput of a series s in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$, denoted $\sigma(s)$, is defined by $\sigma(s) = \sigma(\psi(s)(X))$.

Example 8: The series $s = f_1 \oplus (\delta^2\gamma)^* f_2 \oplus (\delta^3\gamma)^* f_3$ where f_1, f_2, and f_3 are periodic mappings in $\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ defined by

$$f_1(x) = \begin{cases} \varepsilon \text{ if } x = \varepsilon \\ 3 \text{ if } x = 0,1,2 \\ x \text{ if } x \succeq 3 \end{cases}$$
$$f_2(x) = \begin{cases} \varepsilon \text{ if } x \preceq 2 \\ 5 \text{ if } x = 3,4 \\ x \text{ if } x \succeq 5 \end{cases}$$
$$f_3(x) = \begin{cases} \varepsilon \text{ if } x \preceq 3 \\ 7 \otimes 3^j \text{ if } 4 \otimes 3^j \preceq x \prec 7 \otimes 3^j \text{ with } j \in \mathbb{N}_0 \\ \top \text{ if } x = \top \end{cases}$$

is a periodic series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ drawn in Fig. 3. The slicing mapping applied to series s leads to

$$\psi(s)(x) = \begin{cases} \varepsilon \text{ if } x = \varepsilon \\ 3 \text{ if } x = 0,1,2 \\ 5(2\gamma)^* \text{ if } x = 3 \\ 3^j \otimes 7(3\gamma)^* \\ \quad \text{if } 4 \otimes 3^j \preceq x \prec 7 \otimes 3^j \text{ with } j \in \mathbb{N}_0 \\ \top \text{ if } x = \top \end{cases}$$

As expected, $\psi(s)(x)$ is a periodic series in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ for all $x \in \overline{\mathbb{N}}_{\max}$. Further,

$$\sigma(s) = \sigma(\psi(s)(4)) = \frac{1}{3}$$

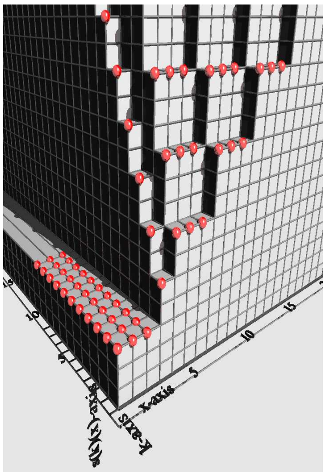

Fig. 3. Series $s = f_1 \oplus (\delta^2\gamma)^* f_2 \oplus (\delta^3\gamma)^* f_3$.

3) Fundamental Theorem: First, some additional classes of series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ are introduced. Second, the fundamental theorem (*i.e.,* the theorem providing the theoretical foundation for the modeling of $(\max,+)$-systems with set-based constraints) is given.

Definition 6 (Causality): A series s in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ is said to be causal if $s(k)$ is causal for all $k \in \mathbb{Z}$. A matrix with entries in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ is said to be causal if all its entries are causal series.

Definition 7 (Realizability): A matrix H in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{m\times p}$ is said to be realizable if there exist $n \in \mathbb{N}$ and N ultimately periodic subsets of $\mathbb{N}_0$ (denoted $S_1,\ldots,S_N$) such that $H = CA^*B$ where

- C is a matrix in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{m\times n}$ with entries in $\{\varepsilon, e\}$
- B is a matrix in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{n\times p}$ with entries in $\{\varepsilon, e\}$
- A is a matrix in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{n\times n}$ with diagonal entries in $\{\varepsilon, e, \delta, \alpha_{S_1}, \ldots, \alpha_{S_N}, \gamma\}$ and non-diagonal entries in $\{\varepsilon, e, \delta, \gamma\}$

Theorem 1 (Fundamental Theorem [6]): Let H be a matrix in $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{m\times p}$. Then,

$$H \text{ is causal and periodic} \Leftrightarrow H \text{ is realizable}$$

4) Operations $\backslash\!\!\circ$ and $\circ\!\!/$: The following properties of the operations $\backslash\!\!\circ$ and $\circ\!\!/$ in the complete dioid $\mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]$ are shown in [6].

Proposition 1 (Left-division): Let $A \in \mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{m\times n}$ and $B \in \mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{m\times p}$ be causal periodic matrices. Then, matrix $B \backslash\!\!\circ A$ is periodic.

Proposition 2 (Right-division): Let $A \in \mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{n\times p}$ and $B \in \mathscr{F}_{\overline{\mathbb{N}}_{\max,\gamma}}[\![\gamma]\!]^{m\times p}$ be causal periodic matrices such that, for any entry s of A or B, $\sigma(s) = \sigma(\psi(s)(e))$. Then, matrix $A \circ\!\!/ B$ is periodic.

IV. OPERATORIAL REPRESENTATION

Operatorial representation is a method to obtain transfer function matrix models for $(\max,+)$-linear systems [1]. In the following, we extend this approach to $(\max,+)$-linear systems with set-based constraints.

Let us first introduce daters and operators. A dater d is an isotone (*i.e.,* order-preserving) mapping from $\mathbb{Z}$ to $\overline{\mathbb{N}}_{\max}$ such that $d(k) = \varepsilon$ for $k < 0$. To capture the behavior of a discrete event system, we associate with an event e a dater, also denoted e, describing the timing pattern of event e (*i.e.,* for $k \in \mathbb{N}_0$, $e(k)$ is the time of occurrence k of event e).

Obviously, a dater is a series in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ and, conversely, a series in $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]$ is a dater.

An operator is a residuated mapping over daters. As recalled in § II, the set of operators, denoted $\mathscr{O}$, is a complete dioid. Furthermore, matrices with entries in $\mathscr{O}$ define mappings between vectors of daters: matrix O in $\mathscr{O}^{m\times p}$ corresponds to the mapping from $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]^p$ to $\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]^m$ with

$$\forall u \in \overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!]^p, \quad (O(u))_i = \bigoplus_{k=1}^{p} O_{ik}(u_k)$$

The previous definitions allow us to formally define operatorial representations for discrete event systems.

Definition 8 (Operatorial representation): Let $\mathscr{S}$ be a discrete event system subject to standard synchronization, such that its event set is partitioned into n state events, denoted $x_1,\ldots,x_n$, m input events, denoted $u_1,\ldots,u_m$, and p output events, denoted $y_1,\ldots,y_p$. The system $\mathscr{S}$ admits an operatorial representation if there exist $A\in\mathscr{O}^{n\times n}$, $B\in\mathscr{O}^{n\times m}$, and $C\in\mathscr{O}^{p\times n}$ such that its behavior is given by the least solution (x,y) of

$$\begin{cases} x \succeq A(x)\oplus B(u) \\ y\succeq C(x)\end{cases}$$

An interesting feature provided by operatorial representation is transfer function matrix model. The input-output mapping is captured by the relation $y=H(u)$ where the matrix H, called transfer function matrix, is equal to CA^*B.

In the following, we restrict ourselves to a particular class of operators represented by series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$. Let $f\gamma^l$ with $f\in\mathscr{F}_{\overline{\mathbb{N}}_{\max}}$ and $l\in\mathbb{N}_0$ be a series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$. The operator $f\gamma^l$ is defined by

$$\forall d\in\overline{\mathbb{N}}_{\max,\gamma}[\![\gamma]\!], \forall k\in\mathbb{Z}, \quad \left(f\gamma^l\right)(d)(k) = f(d(k-l)) \quad (1)$$

Then, the operations $\oplus$ and $\otimes$ over series in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ coincide with the operations $\oplus$ and $\otimes$ over operators.

A. Operatorial Representation for $(\max,+)$-linear Systems

The standard synchronizations "for all $k\geq l_1$, occurrence k of event e_2 is at least τ_1 units of time after occurrence $k-l_1$ of event $e_{1,1}$" and "for all $k\geq l_2$, occurrence k of event e_2 is at least τ_2 units of time after occurrence $k-l_2$ of event $e_{1,2}$" are both expressed by the following inequality in the standard algebra

$$\forall k\in\mathbb{Z}, \quad e_2(k)\geq \max\left(\tau_1+e_{1,1}(k-l_1),\tau_2+e_{1,2}(k-l_2)\right)$$

In $\overline{\mathbb{N}}_{\max}$, this corresponds to

$$\forall k\in\mathbb{Z}, \quad e_2(k)\succeq \tau_1 e_{1,1}(k-l_1)\oplus\tau_2 e_{1,2}(k-l_2)$$

This leads directly to an inequality over daters using the operators γ and δ:

$$e_2\succeq\left(\delta^{\tau_1}\gamma^{l_1}\right)(e_{1,1})\oplus\left(\delta^{\tau_2}\gamma^{l_2}\right)(e_{1,2})$$

Note that, if events $e_{1,1}$ and $e_{1,2}$ are the same event, the above relation is rewritten as

$$e_2\succeq\left(\delta^{\tau_1}\gamma^{l_1}\oplus\delta^{\tau_2}\gamma^{l_2}\right)(e_{1,1})$$

As a $(\max,+)$-linear system is only ruled by standard synchronization, a $(\max,+)$-linear system admits an operatorial representation. Further, the entries of the matrices A, B and C in the operatorial representation are obtained by combining elements in $\{\varepsilon,e,\delta,\gamma,\oplus,\otimes\}$. A convenient dioid to deal with operatorial representations for $(\max,+)$-linear systems is the dioid $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (see [1]). However, it is not possible to model $(\max,+)$-linear systems with set-based constraints in the dioid $\mathscr{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Hence, this justifies the need for a more general dioid such as $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$.

B. Operatorial Representation for $(\max,+)$-linear Systems with Set-Based Constraints

The modeling of standard synchronization has been addressed in § IV-A. It remains to develop an operatorial representation for set-based constraints (*i.e.,* conditions of the form: "event x can only occur at time instants in S"). In the following, we only consider set-based constraints on state events. This assumption is not restrictive: a set-based constraint on an input or output event comes down to a set-based constraint on a state event after extending the state set.

Let us consider the set-based constraint "event x can only occur at time instants in S". As $a\in S\Leftrightarrow\alpha_S(a)=a$, this set-based constraint is rewritten as $\alpha_S(x(k))=x(k)$ for all $k\in\mathbb{Z}$. Using the operatorial interpretation of $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ given in (1), this is equivalent to $x=\alpha_S(x)$. As $\alpha_S\succeq\mathsf{Id}$, this comes down to $x\succeq\alpha_S(x)$.

Hence, set-based constraints and standard synchronizations are both modeled by relations of the form $e_2\succeq o(e_1)$ where o is an operator in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$. Therefore, the discussion in § IV-A allows us to derive an operatorial representation for $(\max,+)$-linear systems with set-based constraints. Furthermore, this modeling approach leads to a system-theoretic interpretation of Th. 1:

- the transfer function matrix of a $(\max,+)$-linear system with set-based constraints is a causal and periodic matrix with entries in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$
- a causal and periodic matrix with entries in $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$ is the transfer function matrix of a $(\max,+)$-linear system with set-based constraints

Example 9: An operatorial representation for the $(\max,+)$-linear system with set-based constraints introduced in Ex. 1 is

$$\begin{cases} x\succeq\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \delta^{10} & \alpha_S\oplus\delta\gamma & \varepsilon\\ \varepsilon & \delta^{10} & \alpha_S\oplus\delta\gamma\end{pmatrix}(x)\oplus\begin{pmatrix}e\\ \varepsilon\\ \varepsilon\end{pmatrix}(u)\\ y\succeq\begin{pmatrix}\varepsilon & \varepsilon & e\end{pmatrix}(x)\end{cases}$$

The transfer function matrix H is

$$H=f_1\oplus f_2\gamma\oplus\left(\delta^6\gamma^3\right)^*\left(g_1\gamma^2\oplus g_2\gamma^3\oplus g_3\gamma^4\right)$$

where, for example,

$$f_1(x)=\begin{cases} x \text{ if } x\in\{\varepsilon,+\infty\}\\ 24\otimes 6^k \text{ if } 6^k\preceq x\prec 5\otimes 6^k \text{ with } k\in\mathbb{N}_0\\ 30\otimes 6^k \text{ if } x=5\otimes 6^k \text{ with } k\in\mathbb{N}_0\end{cases}$$

V. MODEL REFERENCE CONTROL

Model reference control is a method to obtain prefilters and feedbacks for $(\max,+)$-linear systems [2]. In the following, we extend this approach to $(\max,+)$-linear systems with set-based constraints.

Model reference control consists in modifying (*e.g.,* by adding a prefilter or a feedback) the dynamics of the plant, specified by its transfer function matrix $H \in \mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]^{p\times m}$, to match a model reference, specified by a matrix $G \in \mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]^{p\times m}$. In the following, we consider output feedback.

Let us consider an output feedback F. The control input is then given by

$$u = F\left(y\right) \oplus v$$

where v denotes the external input. Note that, to avoid non-causal behaviors (*i.e.,* the feedback needs at time t information which is available at time $t+1$ or later), feedback F has to be causal. The control structure is pictured in Fig. 4.

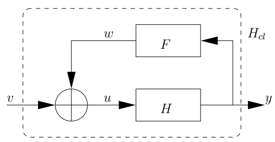

Fig. 4. Control with output feedback

The closed-loop transfer function matrix of the system, denoted H_{cl} in Fig. 4, is equal to $(HF)^* H$ (see [2]). The aim of the considered control approach is to enforce the just-in-time policy (*i.e.,* delay input events as much as possible) while ensuring that the controlled system is not slower than the model reference. Hence, the control problem corresponds to finding the greatest causal solution F of

$$(HF)^* H \preceq G$$

This problem may not admit a solution. However, in the particular case $G = H$ (*i.e.,* the model reference is the transfer function matrix of the plant, which implies that the input-output relation may not be slower than the uncontrolled input), the output feedback is

$$F = H \backslash\!\!\!\circ\, H \circ\!\!\!/\, H$$

A complete proof is given in [6]. To ensure the realizability of feedback F, $H \backslash\!\!\!\circ\, H \circ\!\!\!/\, H$ must be periodic (see Th. 1). According to Def. 2, this holds if, for all entries h of H, $\sigma\left(h\right) = \sigma\left(\psi\left(h\right)\left(e\right)\right)$. Furthermore, as $(HF)^* \succeq e$, $(HF)^* H \succeq H$. Hence, as $G = H$, $(HF)^* H = H$: the selected output feedback F does not affect the input-output behavior of the system.

Example 10: Output feedback for the $(\max,+)$-linear system with set-based constraints introduced in Ex. 1 is computed. Note that model reference control is suitable for this application: we cannot specify the timing pattern of the input (*i.e.,* vehicles entering the road in P_1), but we can delay the input (*e.g.,* by adding an additional traffic light in P_1). The expected advantage of this control approach is to reduce congestion between P_1 and P_3 without decreasing the input-output performance. The output feedback F is given by

$$F = H \backslash\!\!\!\circ\, H \circ\!\!\!/\, H = \left(\delta^6\gamma^3\right)^* \left(f_1\gamma^{12} \oplus f_2\gamma^{13} \oplus f_3\gamma^{14}\right)$$

where, for example,

$$f_1\left(x\right) = \begin{cases} x \text{ if } x \in \{\varepsilon, +\infty\} \\ 26 \text{ if } e \preceq x \prec 21 \\ 26 \otimes 6^k \text{ if } 21 \otimes 6^k \preceq x \prec 24 \otimes 6^k \text{ with } k \in \mathbb{N}_0 \\ 27 \otimes 6^k \text{ if } x = 25 \otimes 6^k \text{ with } k \in \mathbb{N}_0 \\ 28 \otimes 6^k \text{ if } x = 26 \otimes 6^k \text{ with } k \in \mathbb{N}_0 \end{cases}$$

According to the notation in Fig. 4, a realization of feedback F using a $(\max,+)$-linear system with set-based constraints is

$$\begin{cases} z \succeq \begin{pmatrix} \alpha_{S_1} & \varepsilon & \varepsilon & \varepsilon \\ e & \alpha_{S_2} & \varepsilon & \varepsilon \\ \varepsilon & e & \alpha_{S_3} & \varepsilon \\ \delta^2\gamma^{12} & \delta^3\gamma^{13} & \delta^4\gamma^{14} & \delta^6\gamma^3 \end{pmatrix} (z) \oplus \begin{pmatrix} e \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} (y) \\ w \succeq \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix} (z) \end{cases}$$

where z denotes the vector of state events of feedback F and, for example, $S_1 = \{24 + 6k, 25 + 6k, 26 + 6k \text{ with } k \in \mathbb{N}_0\}$.

By applying feedback F, we ensure that at most 12 vehicles are on the road (instead of possibly an infinite number for the uncontrolled system) without decreasing the input-output performance of the road.

VI. CONCLUSION

In this paper, we extend operatorial representation and model reference control to $(\max,+)$-linear systems with set-based constraints by introducing a new dioid, namely $\mathscr{F}_{\overline{\mathbb{N}}_{\max},\gamma}[\![\gamma]\!]$. As future work, we intend to investigate the effect of on-line changes in set-based constraints on transfer function matrix models.

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