Modeling and Control of Weight-Balanced Timed Event Graphs in Dioids

Bertrand Cottenceau, Laurent Hardouin, and Jean-Louis Boimond

***Abstract*—The class of timed event graphs (TEGs) has widely been studied thanks to an approach known as the theory of max-plus linear systems. In particular, the modeling of TEGs via formal power series in a dioid called $\mathcal{M}_{\rm in}^{\rm ax}[\![\gamma,\delta]\!]$ has led to input–output representations on which some model matching control problems have been solved. Our work attempts to extend the class of systems for which a similar control synthesis is possible. To this end, a subclass of timed Petri nets that we call weight-balanced timed event graphs (WBTEGs) will be first defined. They can model synchronization and delays (WBTEGs contain TEGs) and can also describe dynamic phenomena such as batching and event duplications (unbatching). Their behavior is described by rational compositions (sum, product and Kleene star) of four elementary operators γ^n, δ^t, μ_m, and β_b on a dioid of formal power series denoted $\mathcal{E}^*[\![\delta]\!]$. The main feature is that the transfer series of WBTEGs have a property of ultimate periodicity (such as rational series in $\mathcal{M}_{\rm in}^{\rm ax}[\![\gamma,\delta]\!]$). Finally, the existing results on control synthesis for max-plus linear systems find a natural application in this framework.**

***Index Terms*—Controller synthesis, dioids, discrete-event systems, formal power series, residuation, weighted timed event graphs (WTEGs).**

I. Introduction

SINCE the beginning of the eighties, it has been known that the class of timed event graphs (TEGs) can be studied thanks to linear models in specific algebraic structures called dioids (or idempotent semirings) [1], [5], [9], [14], [19]. Among different representations, a description of TEGs by the means of operators is possible. By denoting Σ the semimodule of counter functions[1], one can describe their behavior by combining two shift operators (see [1, Ch. 5], [5]) denoted, respectively, $\gamma : \Sigma \to \Sigma$ and $\delta : \Sigma \to \Sigma$

$$(\gamma x)(t) = x(t) + 1;\ (\delta x)(t) = x(t-1). \tag{1}$$

The input–output behavior of a TEG is then described by a transfer matrix the entries of which are elements of the rational closure of the set $\{\varepsilon, e, \gamma, \delta\}$, where ε (resp. e) is the null (resp. neutral) operator. In other words, the input–output behavior of any TEG can be written with a finite expression involving these operators. Moreover, it is shown in [1], [10], and [9] that a rational expression can be turned into a canonical form which is ultimately periodic. The algebraic structure for the calculus of transfer series for TEGs is a dioid called $\mathcal{M}_{\rm in}^{\rm ax}[\![\gamma,\delta]\!]$ introduced in [5]. It is a set of formal series in two variables γ and δ corresponding, respectively, to the event-shift operator and to the time-shift operator given in (1). This modeling has made it possible to elaborate software tools to compute the transfer matrix of any TEG in $\mathcal{M}_{\rm in}^{\rm ax}[\![\gamma,\delta]\!]$ [7], [9]. These tools also contribute to the performance evaluation of discrete event systems since the ultimate periodicity of a TEG corresponds to its production rate (number of events by time unit).

Moreover, an input–output model is well suited to address some model matching control problems such as the ones studied in [6], [15], [17], and [13]. These control strategies have clearly been elaborated by analogy with the classical control theory, i.e., controllers are computed so that the closed-loop system matches a given reference model. The role of the controller is to filter the system input in order to achieve some given performance. When applied on a manufacturing production system, the controller obtained with that approach leads to improve the internal flows of products and to reduce the internal stocks.

The objective of the work presented here is to study the class of weighted timed event graphs (WTEGs) [18], i.e., TEGs the arcs of which are valued by positive integers. In comparison with TEGs, the modeling power is greatly increased since in addition to synchronisations and delays, WTEGs can describe batch constitution (several successive input events are necessary to release one output event) and duplication (one input event releases several output events). These phenomena are usual in manufacturing systems (batch/unbatch) and cannot be accurately modeled with ordinary TEGs.

In the literature of discrete event systems, the analysis of WTEGs is discussed for instance in [2], [18], and also under an equivalent graphical model called Synchronous Data Flow graphs (SDF) in [16], [21]. In these works, WTEGs are considered as modeling tools both for manufacturing systems and for computation in the field of concurrent applications. The main concerns are the throughput computation of a given system [8] and the possibility of elaborating a periodic schedule [2], [18]. Due to the importance of the synchronisation phenomena in these systems, several papers based on the max-plus theory are available. In [8], [11], the throughput of a SDF is computed thanks to a max-plus model. In [4] and [12], a class of *fluid* TEGs with multipliers (TEGMs) is modeled by formal series in a specific dioid. The TEGMs can be seen as a continuous approxi-

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[1]A counter function $x : \mathbb{Z} \to \mathbb{Z}, t \mapsto x(t)$ gives the cumulative number of occurrences of the event labeled x at date t. Such a function plays the role of signal.

mation of the (discrete) WTEGs considered here. Nevertheless, as mentioned in the conclusion of [4], the fluid behavior can be arbitrarily far from the discrete one. Therefore, the (discrete) WTEGs deserve a specific study since they cannot be analysed with the tools introduced in [4].

The main motivation of our work is to tackle the controller synthesis for WTEGs. More specifically, we aim to solve some model matching control problems (such as [6], [13], [17]) for the class of WTEGs. Let us note that this adaptation of existing results is not so immediate since all these works rely on an input–output representation (transfer function) which was not available yet for WTEGs. In that context, our main contribution is to provide a description of the input–output behavior of WTEGs by formal power series the variable of which can be assimilate to operators. In addition to operators γ and δ defined in (1), two additional ones denoted β_b and μ_m are introduced to describe a batch operation (division operator) and a duplication phenomenon (multiplier operator). The main result of our paper is to show that for a subclass of WTEGs that we call weight-balanced timed event graphs (WBTEGs), the input–output behavior is necessarily described by a rational expression (with operators γ^n, δ^t, μ_m, and β_b) that can be turned into an ultimately periodic form. As for TEGs, periodic phenomena are still a prevailing aspect of the behavior of WBTEGs. The algebraic structure used to obtain these results is a new dioid called $\mathcal{E}^*[\![\delta]\!]$ that encompasses dioid $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$.

The paper is organized as follows. In Section II, the subclass of weight-balanced timed event graphs is first defined. Then, the modeling of WBTEGs thanks to the additive operators γ^n, δ^t, μ_m, and β_b is presented. A new dioid of formal power series denoted $\mathcal{E}^*[\![\delta]\!]$ is introduced in Section III. The formal series have a graphical representation that is also given in that section. The results concerning the ultimate periodicity of WBTEGs' transfer series are stated in Section IV. This part lies on technical proofs adapted from the work on the rational calculus in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ introduced in [9], [10]. Finally, the question of control synthesis for WBTEGs is addressed in Section V after some reminders on the residuation theory.

II. Weight-Balanced Timed Event Graphs (WBTEGs)

Definitions

Weighted Event Graphs (WEGs) [18] constitute a subclass of generalized Petri Nets given by a set of places $P = \{p_1, \ldots, p_m\}$ and a set of transitions $T = \{t_1, \ldots, t_n\}$ (see [20] for a survey on Petri nets). An event graph cannot describe concurrency phenomena, then every place $p_k \in P$ is defined between one input transition t_i and one output transition t_o. The arcs $t_i \rightarrow p_k$ and $p_k \rightarrow t_o$ are oriented and valued[2] by strictly positive integers denoted respectively $w_i(p_k)$ and $w_o(p_k)$. A transition without input (resp. output) place is called a source or input (resp. sink or output) transition. An initial marking (a set of initial tokens depicted with black dots) denoted $M_0(p_k)$ is associated to each place $p_k \in P$. A given transition t_j is said *enabled* as soon as each input place p_l contains at least $w_o(p_l)$ tokens. A transition can be fired only

[2]From a graphical point of view, the valuations are depicted directly on the arcs.

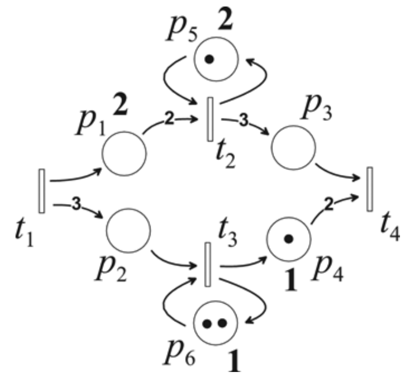

Fig. 1. Weight-balanced timed event graph.

if it is enabled. At each firing of a transition, $w_o(p_l)$ tokens are removed from each input place p_l, and $w_i(p_k)$ tokens are added to each output place p_k.

1) Example 1: For the WEG depicted on Fig. 1, t_1 (resp. t_4) is an input (resp. output) transition. The initial marking is given by $M_0(p_4) = 1$, $M_0(p_5) = 1$ and $M_0(p_6) = 2$. All arcs are assumed to be 1-valued except when mentioned, for instance $w_i(p_3) = 3$ and $w_o(p_1) = 2$. Transition t_4 is enabled when place p_4 has two tokens and place p_3 has one token. The firing of transition t_2 adds three tokens to place p_3.

Definition 1 (Gain of a Path): The gain of an elementary (oriented) path $t_i \rightarrow p_k \rightarrow t_o$ is defined as $\Gamma(t_i, p_k, t_o) \triangleq w_i(p_k)/w_o(p_k) \in \mathbb{Q}$. For a general path π passing through several places, the gain corresponds to the product of elementary paths, i.e., $\Gamma(\pi) = \prod_{p_j \in \pi} w_i(p_j)/w_o(p_j)$.

Definition 2 (Neutral and Weight-Balanced Event Graph): A WEG is said *neutral* if all its circuits have a gain of 1. A WEG is said *weight-balanced* if $\forall t_i, t_j \in T$, all the paths from t_i to t_j have the same gain (gains are balanced for parallel paths). Therefore, a weight-balanced event graph is necessarily neutral.

A holding time denoted $\Delta(p_k) \in \mathbb{N}$ can be associated to each place $p_k \in P$ of a WEG. Each token entering in a place p_k has to wait $\Delta(p_k)$ time units before contributing to enable the output transition. A WEG with holding times is called a weighted timed event graph (WTEG). Hereafter, we will only consider weight-balanced timed event graphs (in short WBTEGs).

Definition 3 (Earliest Functioning): The *earliest functioning* of a WTEG consists in firing transitions as soon as they are enabled, except for input transitions that are fired in accordance with input trajectories.

2) Example 2: For the WTEG depicted on Fig. 1, holding times are attached to places: $\Delta(p_1) = 2$, $\Delta(p_6) = 1$, $\Delta(p_4) = 1$, and $\Delta(p_5) = 2$. This is a Weight-Balanced TEG since it is neutral and all the parallel paths from t_1 to t_4 are balanced (they have the same gain equal to $3/2$). For instance, $\Gamma(t_1, p_1, t_2) = 1/2$ and $\Gamma(t_1, p_2, t_3) = 3$. The input–output gain of $3/2$ represents the fact that the average functioning of the system produces three output events for each two input events.

A. Algebra of Additive Operators

A dioid [1, Ch. 4] (or idempotent semiring) is an algebraic structure with two inner operations, a sum and a product. The sum is commutative, associative and idempotent ($a \oplus a = a$) and the product is associative and distributive over the sum. The neutral elements of these operations are usually denoted ε for the sum, and e for the product. Since the sum is idempotent, a natural order can be associated to a dioid:

$$a \succeq b \iff a = a \oplus b. \tag{2}$$

When the sum of any finite or infinite subset of a dioid is defined, and the product distributes over infinite sums, the dioid is said complete. A complete dioid is a partially ordered set (poset) with a complete lattice structure: the infimum operator is defined as $a \wedge b = \bigoplus\{x | x \oplus a = a \text{ and } x \oplus b = b\}$. On a dioid, since the order is partial, two elements a and b may be incomparable: $a \npreceq b$ and $b \npreceq a$, which is denoted by $a \| b$.

For WBTEGs modeling, a counter function $x_i : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ is associated to each transition t_i: $x_i(\tau)$ gives the cumulative number of events t_i at date τ. A counter function is naturally non-decreasing: $\tau_a \leq \tau_b \Rightarrow x(\tau_a) \leq x(\tau_b)$. The set of counter functions denoted Σ has a semimodule structure for the internal operation $\oplus = \min$ and for the scalar operation defined by $\lambda.x(t) = x(t) + \lambda$. An operator is a map $\mathcal{H} : \Sigma \to \Sigma$ which is said *linear* if $\forall x, y \in \Sigma$, a) $\mathcal{H}(x \oplus y) = \mathcal{H}(x) \oplus \mathcal{H}(y)$ and b) $\mathcal{H}(\lambda.x) = \lambda.\mathcal{H}(x)$. An operator is said *additive* if only a) is satisfied.

Definition 4 (Dioid $\mathcal{O}$ of Additive Operators [19]): The set of additive operators on Σ, with the operations defined below, is a non commutative complete dioid denoted $\mathcal{O}$: $x \in \Sigma$, $\forall \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{O}$

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \triangleq \forall x, (\mathcal{H}_1 \oplus \mathcal{H}_2)(x) = \min(\mathcal{H}_1(x), \mathcal{H}_2(x))$$
$$\mathcal{H}_1 \circ \mathcal{H}_2 \triangleq \forall x, (\mathcal{H}_1 \circ \mathcal{H}_2)(x) = \mathcal{H}_1(\mathcal{H}_2(x)).$$

The null operator (neutral for $\oplus$ and absorbing for $\circ$) is denoted $\varepsilon : \forall x \in \Sigma, (\varepsilon x)(t) = +\infty$ and the unit operator (neutral for $\circ$) is denoted $e : \forall x \in \Sigma, (ex)(t) = x(t)$.

For the sequel, we will simply denote $\mathcal{H}x$ (instead of $\mathcal{H}(x)$) the image of the counter $x \in \Sigma$ by the additive operator $\mathcal{H} \in \mathcal{O}$, and we will also often omit symbol $\circ$ for the product of $\mathcal{O}$, $\mathcal{H}_1\mathcal{H}_2 = \mathcal{H}_1 \circ \mathcal{H}_2$. Two additive operators $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{O}$ are equal if $\forall x \in \Sigma, \mathcal{H}_1 x = \mathcal{H}_2 x$.

Definition 5 (Elementary Operators in WBTEGs): The dynamical phenomena arising in WBTEGs can be described thanks to the next additive operators in $\mathcal{O}$: let $x \in \Sigma$ be a counter

$$\tau \in \mathbb{Z}, \delta^\tau : \forall x,\ (\delta^\tau x)(t) = x(t - \tau),$$
$$\nu \in \mathbb{Z}, \gamma^\nu : \forall x,\ (\gamma^\nu x)(t) = x(t) + \nu,$$
$$b \in \mathbb{N}^*, \beta_b : \forall x,\ (\beta_b x)(t) = \lfloor x(t)/b \rfloor,$$
$$m \in \mathbb{N}^*, \mu_m : \forall x,\ (\mu_m x)(t) = x(t) \times m$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to $a \in \mathbb{Q}$.

We can remark that the unit operator e has various expressions: $e = \gamma^0 = \delta^0 = \mu_1 = \beta_1$.

Proposition 1: The next formal identities can be stated

$$\gamma^n \gamma^{n'} = \gamma^{n+n'};\ \delta^t \delta^{t'} = \delta^{t+t'} \tag{3}$$
$$\gamma^n \oplus \gamma^{n'} = \gamma^{\min(n,n')};\ \delta^t \oplus \delta^{t'} = \delta^{\max(t,t')} \tag{4}$$
$$\gamma^1 \delta^1 = \delta^1 \gamma^1;\ \mu_m \delta^1 = \delta^1 \mu_m;\ \beta_b \delta^1 = \delta^1 \beta_b \tag{5}$$
$$\mu_m \gamma^n = \gamma^{m \times n} \mu_m;\ \gamma^n \beta_b = \beta_b \gamma^{n \times b}. \tag{6}$$

Proof: For all counter $x \in \Sigma$ we have (3): $\forall t, (x(t) + n') + n = x(t) + (n' + n)$ and $x(\tau - t - t') = x(\tau - (t + t'))$. Equation (4): $\forall t, \min(x(t) + n, x(t) + n') = x(t) + \min(n, n')$. Since $\forall t, x(t) \geq x(t - 1)$, then $\min(x(\tau - t), x(\tau - t')) = x(\tau - \max(t, t'))$. Equation (5): immediate equation (6): $m \times (x(t) + n) = m \times x(t) + m \times n$ and $\lfloor x(t)/b \rfloor + n = \lfloor (x(t) + n \times b)/b \rfloor$. ■

Definition 6 (Kleene Star): The Kleene star of an operator is defined by: $\forall \mathcal{H} \in \mathcal{O}, \mathcal{H}^* = \bigoplus_{i \in \mathbb{N}} \mathcal{H}^i = e \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \ldots$ with $\mathcal{H}^n = \mathcal{H} \circ \ldots \circ \mathcal{H}$ (n times).

Theorem 1: On a complete dioid $\mathcal{D}$, the implicit equation $x = ax \oplus b$ has $x = a^*b$ as least solution.

Proof: see [1, Th. 4.75] ■

Theorem 2: For all operator $\mathcal{H} \in \mathcal{O}$, the next equalities are satisfied: $\mathcal{H} = \mathcal{H}(\delta^{-1})^* = (\delta^{-1})^*\mathcal{H} = (\gamma^1)^*\mathcal{H} = \mathcal{H}(\gamma^1)^*$.

Proof: Since a counter function x is monotone, then $\forall t, x(t + 1) \geq x(t) \iff \delta^{-1}x \preceq x$. For the same reason, $\forall t, x(t) + 1 \geq x(t) \iff \gamma x \preceq x$. Therefore, $\forall x \in \Sigma, \forall \mathcal{H} \in \mathcal{O}, \mathcal{H}(\gamma^1)^* x = \mathcal{H}x = (\gamma^1)^*\mathcal{H}x = \mathcal{H}(\delta^{-1})^* x = (\delta^{-1})^*\mathcal{H}x$. ■

Theorem 3: On a complete dioid $\mathcal{D}$, with $a, b \in \mathcal{D}$, $k \in \mathbb{N}^*$, one has

$$(a \oplus b)^* = a^*(ba^*)^* \tag{7}$$
$$(ab^*)^* = e \oplus a(a \oplus b)^* \tag{8}$$
$$a^* = (a^k)^*(e \oplus a \oplus \ldots \oplus a^{k-1}) \tag{9}$$
$$\text{if } \mathcal{D} \text{ is commutative } (a \oplus b)^* = a^* b^*. \tag{10}$$

Proof: These results can be found in [9, Prop. 4.1.6] and in [1, Sec. 4.8] ■

Definition 7 (Redundancy): Let us consider an expression $w = o_1 \oplus o_2 \oplus \ldots \oplus o_n$, with $o_i \in \mathcal{O}$. A term o_j is said redundant for w if $w = \bigoplus o_i = \bigoplus\{o_i | o_i \neq o_j\}$. In other words, removing o_j does not change the expression w.

1) Example 3: In the following expression $\gamma^1\delta^2 \oplus \gamma^3\delta^1 \oplus \gamma^4\delta^3$, operator $\gamma^3\delta^1$ is redundant. Indeed, by applying (4), $\gamma^1\delta^2 = \gamma^1\delta^2 \oplus \gamma^3\delta^2 \oplus \gamma^3\delta^1 \iff \gamma^3\delta^1 \preceq \gamma^1\delta^2$.

B. Modeling of WBTEGs

The WBTEGs are analyzed here with the earliest functioning rule (see Def. 3). We can model a path of a WBTEG by a product of operators in $\mathcal{O}$, the synchronization of parallel paths by a sum $\oplus$, and the circuits by the Kleene star of operators. Each elementary path $t_i \to p_k \to t_j$ of a WBTEG, where $M_0(p_k)$ is the initial marking of place p_k and $\tau = \Delta(p_k)$ its holding time, can be described by the relation

$$x_j = \beta_{w_o(p_k)} \gamma^{M_0(p_k)} \mu_{w_i(p_k)} \delta^\tau x_i \tag{11}$$

where x_i (resp. x_j) is the counter function associated to transition t_i (resp. t_j).

1) Example 4 (WBTEG of Fig. 1): For the WBTEG depicted in Fig. 1 and for the earliest functioning, we have

$$x_2(t) = \min(\lfloor \tfrac{x_1(t-2)}{2} \rfloor, x_2(t-2) + 1)$$
$$x_3(t) = \min(x_1(t) \times 3, x_3(t-1) + 2).$$

Therefore, the counter functions are linked by $x_2 = \beta_2\delta^2 x_1 \oplus \gamma^1\delta^2 x_2$ and thanks to Theorem 1, $x_2 = (\gamma^1\delta^2)^*\beta_2\delta^2 x_1$. Similarly, $x_3 = (\gamma^2\delta^1)^*\mu_3 x_1$. Finally, the counter function associated to the output transition is $x_4 = \mu_3 x_2 \oplus \beta_2\gamma^1\delta^1 x_3 = (\mu_3(\gamma^1\delta^2)^*\beta_2\delta^2 \oplus \beta_2\gamma^1\delta^1(\gamma^2\delta^1)^*\mu_3)x_1$. The input–output behavior (or transfer function) of the WBTEG is described by the rational expression $\mu_3(\gamma^1\delta^2)^*\beta_2\delta^2 \oplus \beta_2\gamma^1\delta^1(\gamma^2\delta^1)^*\mu_3$ in $\mathcal{O}$.

Theorem 4 (Transfer Matrix of a WBTEG): The behavior of a WBTEG is described by a matrix the elements of which belong to the rational closure of the set of operators $O_{M,B} = \{\varepsilon, e, \gamma^1, \delta^1, \mu_2, \ldots, \mu_M, \beta_2, \ldots, \beta_B\}$ where $B = \max_i w_o(p_i)$ and $M = \max_i w_i(p_i)$ with $p_i \in P$.

Proof: For each place p_k an operator $\beta_b \gamma^n \mu_m \delta^t$ [see (11)] is associated. Therefore, the different graph compositions (parallel, serial, feedback) are expressed by operations in $\{\oplus, \circ, *\}$. Since a WBTEG is a finite graph, the rationality is straightforward. ■

III. Graphical Representations of Operators

According to (5) in Prop. 1, operator δ^1 can commute with any simple or composed event operator. For instance, $\delta^1\gamma^1\delta^2\mu_3\beta_2\delta^1 = \gamma^1\mu_3\beta_2\delta^4 = \delta^4\gamma^1\mu_3\beta_2$. Hence, in every finite composition (product) of elementary operators in $\{\delta^t, \gamma^n, \mu_m, \beta_b\}$, the time-shift operator can be factorized. That is why operations can be evaluated separately, on the one hand on event operators and on the other hand on time-shift operators.

A. Bi-Dimensional Graphical Representation of E-Operators

1) Event Operators: The set of operators generated by sum and composition of operators in γ^n, μ_m and β_b has a dioid structure.

Definition 8 (Dioid $\mathcal{E}$): We denote by $\mathcal{E}$ (for event) the dioid of operators obtained by sums and compositions of operators in $\{\varepsilon, e, \gamma^n, \mu_m, \beta_b\}$, with $n \in \mathbb{Z}$, and $m, b \in \mathbb{N}^*$. The elements of $\mathcal{E}$ are called E-operators hereafter.

Dioid $\mathcal{E}$ is a complete subdioid of $\mathcal{O}$ (additive operators). Since operation $\circ$ is not commutative on $\mathcal{E}$, checking the equality of two E-operators is not immediate. Nevertheless, the comparison of E-operators is possible thanks to an associate map called *counter-to-counter (C/C) function*. Since an E-operator $w \in \mathcal{E}$ induces modifications only on the event numbering (no time shift), its instantaneous behavior is described by a function denoted $\mathcal{F} : \mathbb{Z} \to \mathbb{Z}$.

Definition 9 (C/C Function $\mathcal{F}$): For a given E-operator w in $\mathcal{E}$, we denote by $\mathcal{F}_w : \mathbb{Z} \to \mathbb{Z}, k_i \mapsto k_o$ the mapping which maps its input counter value k_i to its output counter value k_o. $\mathcal{F}_w$ is obtained by replacing $x(t)$ by k_i in the counter equation $(wx)(t)$ where $x \in \Sigma$.

2) Example 5: For instance $(\beta_2\gamma^1\mu_3 x)(t) = \lfloor (3 \times x(t) + 1)/2 \rfloor$. By replacing $x(t) \in \mathbb{Z}$ by a value $k_i \in \mathbb{Z}$, we obtain $\mathcal{F}_{\beta_2\gamma^1\mu_3}(k_i) = \lfloor (3 \times k_i) + 1)/2 \rfloor$ (see Fig. 2). For operator $\beta_2\gamma^1\mu_3$, if k_i input events have occurred at a given date t, then $k_o = \lfloor (3 \times k_i) + 1)/2 \rfloor$ output events have occurred at date t.

The C/C function $\mathcal{F}_w$ gives an unambiguous representation of E-operator w and leads to a natural bi-dimensional graphical representation in $\mathbb{Z}^2$. On the graphical representation, the axis are labeled by I-count (input count) and O-count (output count).

Due to the non commutativity of product in $\mathcal{O}$, checking the formal identity of operators is not immediate. Nevertheless, when restricted to operators in $\mathcal{E}$, the equality can be checked thanks to the C/C function

$$w_1, w_2 \in \mathcal{E}, w_1 = w_2 \iff \mathcal{F}_{w_1} = \mathcal{F}_{w_2}. \tag{12}$$

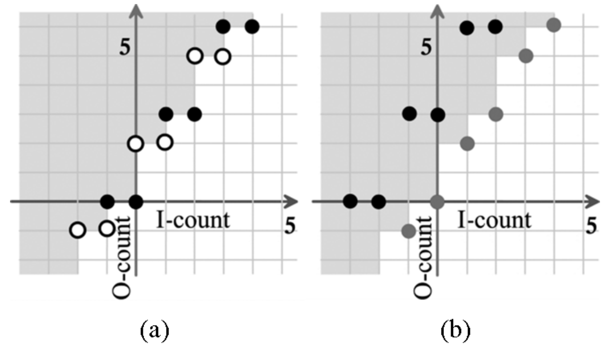

Fig. 2. (a) On the left, $\mathcal{F}_{\mu_3\beta_2\gamma^1}$ (black dots) and $\mathcal{F}\gamma^2\mu_3\beta_2$ (white dots). (b) On the right, $\mathcal{F}_{\beta_2\gamma^1\mu_3}$ (gray dots) and $\mathcal{F}_{\gamma^3\mu_3\beta_2\gamma^1}$ (black dots).

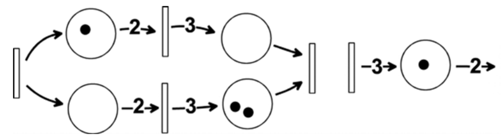

Fig. 3. Input–output equivalence: $\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2 = \beta_2\gamma^1\mu_3$.

Moreover, we have an isomorphism between $\mathcal{E}$ and the set of C/C functions

$$\mathcal{F}_{w_1 \oplus w_2} = \min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2}) \text{ and } \mathcal{F}_{w_1 \circ w_2} = \mathcal{F}_{w_1} \circ \mathcal{F}_{w_2}. \tag{13}$$

In other words, the calculus on C/C functions is an alternative to the formal calculus on $\mathcal{E}$.

3) Example 6: Thanks to (12) and (13), we can check the equality $\mu_3\beta_2\gamma^1 \oplus \gamma^2\mu_3\beta_2 = \beta_2\gamma^1\mu_3$, even if Prop. 1 does not give all the formal equalities necessary to obtain that result. On the right-hand side of Fig. 2, $\mathcal{F}_{\beta_2\gamma^1\mu_3}$ is depicted with gray dots, and on the left-hand side, $\mathcal{F}_{\mu_3\beta_2\gamma^1}$ is depicted with black dots and $\mathcal{F}_{\gamma^2\mu_3\beta_2}$ with white dots. Obviously, $\mathcal{F}_{\beta_2\gamma^1\mu_3} = \min(\mathcal{F}_{\mu_3\beta_2\gamma^1}, \mathcal{F}_{\gamma^2\mu_3\beta_2})$. When translated into a WBTEG model, the previous identity means that the two WBTEGs depicted in Fig. 3 are equivalent from an input–output point of view: the same input sequence will produce the same output sequence.

4) Graphical Considerations: The relation on E-operators can be viewed from a geometrical point of view.

Partial Order on $\mathcal{E}$: thanks to (2), the comparison in $\mathcal{E}$ is interpreted as follows:

$$\begin{aligned} w_1 \preceq w_2 &\iff w_1 \oplus w_2 = w_2 \\ &\iff min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2}) = \mathcal{F}_{w_2} \\ &\iff \mathcal{F}_{w_1} \geq \mathcal{F}_{w_2}. \end{aligned}$$

In Fig. 2, the gray zone corresponds to the domain of E-operators less than w according to the order (2). For instance, we can see on the right-hand side of Fig. 2 that $\mathcal{F}_{\gamma^3\mu_3\beta_2\gamma^1} \geq \mathcal{F}_{\beta_2\gamma^1\mu_3}$, which corresponds to $\gamma^3\mu_3\beta_2\gamma^1 \preceq \beta_2\gamma^1\mu_3$. The right side of Fig. 2 shows that operator $\beta_2\gamma^1\mu_3$ dominates[3] operator $\gamma^3\mu_3\beta_2\gamma^1$. From a practical point of view, if these two operators are synchronized, the behavior of operator $\beta_2\gamma^1\mu_3$ is dominant.

[3]The domination designation comes from [5].

Left and Right Product by γ^n: For $w \in \mathcal{E}$, the left and the right product by γ^n are graphically described by shifts in $\mathbb{Z}^2$, say

$$\mathcal{F}_{\gamma^n w} \iff \mathcal{F}_w \text{ shifted by } n \text{ units to the top in } \mathbb{Z}^2(\uparrow)$$
$$\mathcal{F}_{w\gamma^n} \iff \mathcal{F}_w \text{ shifted by } n \text{ units to the left in } \mathbb{Z}^2(\leftarrow).$$

B. Periodic E-Operators

Definition 10 (Periodic E-Operator): An E-operator $w \in \mathcal{E}$ is said (n, n')-periodic, or simply periodic, if $\mathcal{F}_w$ satisfies $\forall k_i \in \mathbb{Z}, \mathcal{F}_w(k_i + n) = \mathcal{F}_w(k_i) + n'$.

Clearly, γ^n, μ_m, β_b are periodic E-operators since

$$\mathcal{F}_{\gamma^n}(0) = n, \mathcal{F}_{\gamma^n}(k_i + 1) = \mathcal{F}_{\gamma^n}(k_i) + 1$$
$$\mathcal{F}_{\mu_m}(0) = 0, \mathcal{F}_{\mu_m}(k_i + 1) = \mathcal{F}_{\mu_m}(k_i) + m$$
$$\mathcal{F}_{\beta_b}(0 \le k_i < b) = 0, \mathcal{F}_{\beta_b}(k_i + b) = \mathcal{F}_{\beta_b}(k_i) + 1.$$

γ^n is $(1,1)$-periodic, μ_m is $(1, m)$-periodic and β_b is $(b, 1)$-periodic. The set of periodic E-operators is denoted $\mathcal{E}_{per}$.

Definition 11 (Gain of $w \in \mathcal{E}_{per}$): Let $w \in \mathcal{E}_{per}$ be a (k, k')-periodic E-operator. The gain of w is defined as $\Gamma(w) = k'/k$. It is the average slope of $\mathcal{F}_w$. An E-operator is said conservative if $\Gamma(w) = 1$.

Proposition 2: Let $w_1, w_2 \in \mathcal{E}_{per}$ be two periodic E-operators. We have

$$w_1 w_2 \in \mathcal{E}_{per} \text{ and } \Gamma(w_1 w_2) = \Gamma(w_1) \times \Gamma(w_2)$$
$$\text{if } \Gamma(w_1) = \Gamma(w_2) \text{ then } w_1 \oplus w_2 \in \mathcal{E}_{per}$$
$$\text{if } \Gamma(w_1) = \Gamma(w_2) \text{ then } w_1 \wedge w_2 \in \mathcal{E}_{per}.$$

Proof: Since w_1 and w_2 are periodic then $\mathcal{F}_{w_1}(k_i + k_1) = \mathcal{F}_{w_1}(k_i) + k'_1$ and $\mathcal{F}_{w_2}(k_i + k_2) = \mathcal{F}_{w_2}(k_i) + k'_2$. Hence, $\mathcal{F}_{w_2}(k_i + k_1 k_2) = \mathcal{F}_{w_2}(k_i) + k_1 k'_2$ and $\mathcal{F}_{w_1}(\mathcal{F}_{w_2}(k_i + k_1 k_2)) = \mathcal{F}_{w_1}(\mathcal{F}_{w_2}(k_i) + k_1 k'_2) = \mathcal{F}_{w_1}(\mathcal{F}_{w_2}(k_i)) + k'_1 k'_2 = \mathcal{F}_{w_1 w_2}(k_i) + k'_1 k'_2$. Therefore, operator $w_1 w_2$ is a periodic operator the gain of which is $(k'_1 k'_2)/(k_1 k_2)$. For the sum of operators with the same gain, both operators can be written with the same periodicity: $\mathcal{F}_{w_1}(k_i + k_1 k_2) = \mathcal{F}_{w_1}(k_i) + k'_1 k_2$ and $\mathcal{F}_{w_2}(k_i + k_1 k_2) = \mathcal{F}_{w_2}(k_i) + k'_2 k_1$ with $k'_1 k_2 = k_1 k'_2$ since $\Gamma(w_1) = \Gamma(w_2)$. Hence, $\min(\mathcal{F}_{w_1}, \mathcal{F}_{w_2})$ is also periodic. By symmetry, the max ($\wedge$) of two periodic E-operators with the same gain is also periodic. ■

Proposition 3: The E-operators arising in WBTEGs are periodic.

Proof: Due to the definition of WBTEGs (see Def. 2), the parallel paths have the same gain. Thanks to Prop. 2, the periodicity of E-operators is therefore kept by the serial and the parallel compositions of WBTEGs. ■

A periodic E-operator $w \in \mathcal{E}_{per}$ can be handled by the means of a finite representation, which will make a finite data description adapted to a software implementation possible. A (k, k')-periodic E-operator can be represented by a pair $(k, k') \in \mathbb{N}^{*2}$ describing the gain $\Gamma(w) = k'/k$ and the values of $\mathcal{F}_w(k_i)$ for one period $k_i \in \{0, \ldots, k-1\}$. The canonical form of a periodic E-operator is strongly linked to the possibility of describing a periodic C/C function with a canonical form. First, it is clear that a (k, k')-periodic C/C function $\mathcal{F}_w$ is also (nk, nk')-periodic, for $n \in \mathbb{N}^*$. Conversely, a (k, k')-periodic C/C function $\mathcal{F}_w$ may have a shorter periodicity if there exists $n' \in \mathbb{N}$ such that $\forall k_i, \mathcal{F}_w(k_i + k/n') = k'/n'$. All the periodicities of a given E-operator are therefore totally ordered.

Definition 12 (Least Periodicity): The periodicity of a periodic operator is defined as the least pair (k, k') such that $\forall k_i, \mathcal{F}_w(k_i + k) = k'$.

Remark 1: For a given representation of a (k, k')-periodic operator $w \in \mathcal{E}_{per}$, finding the least periodicity amounts to finding the greatest integer n such that w is also $(k/n, k'/n)$-periodic.

The canonical decomposition of an operator $w \in \mathcal{E}_{per}$ is based on a sum of specific E-operators defined hereafter.

Definition 13 (∇ Operator): Let us denote by $\nabla_{m|b}$ and ∇_n the next composed E-operators $\nabla_{m|b} \triangleq \mu_m \beta_b$ and $\nabla_n \triangleq \mu_n \beta_n$.

Let us note that $\nabla_{m|b}$ is (b, m)-periodic and that it is graphically represented by a staircase C/C function: $\mathcal{F}_{\nabla_{m|b}}(k_i) = m\lfloor k_i/b \rfloor$, say $\mathcal{F}_{\nabla_{m|b}}(0 \le k_i \le b-1) = 0$, and $\mathcal{F}_{\nabla_{m|b}}(k_i + b) = \mathcal{F}_{\nabla_{m|b}}(k_i) + m$. For instance in Fig. 2, operator $\mathcal{F}_{\gamma^3 \nabla_{3|2} \gamma^1}$, which is $(2, 3)$-periodic, is depicted on the right-hand side with black dots.

Proposition 4 (Canonical Form of $w \in \mathcal{E}_{per}$): A periodic E-operator w has a canonical form which is a finite sum given by $w = \bigoplus_{i=1}^{N} \gamma^{n_i} \nabla_{m|b} \gamma^{n'_i}$ without redundant terms and such that $0 \le n'_i < b$.

Proof: First, $\mathcal{F}_w$ is written with its least periodicity (see Def. 12). This form is a canonical (b, m)-periodic form of $\mathcal{F}_w$. Then, $\mathcal{F}_w$ can be expressed as the minimum of b functions $\mathcal{F}_w = min(S_0, \ldots, S_{b-1})$ where function S_i is a (b, m)-periodic function given by $S_i(0 \le j \le i) = \mathcal{F}_w(i)$ and $S_i(i < j \le b-1) = \mathcal{F}_w(i) + m$. Each function S_i is a staircase (b, m)-periodic function which is also the C/C function of an operator $\gamma^n \nabla_{m|b} \gamma^{n'}$, more specifically $S_i = \mathcal{F}_{\gamma^{\mathcal{F}_w(i)} \nabla_{m|b} \gamma^{b-1-i}}$. Finally, the canonical form of w is obtained by summing $\bigoplus_{i=0}^{b-1} \gamma^{\mathcal{F}_w(i)} \nabla_{m|b} \gamma^{b-1-i}$ and by removing all the redundant E-operators from that expression, if any. At the end, w is expressed by a sum of incomparable E-operators which is unique since the considered representation of $\mathcal{F}_w$ is canonical. ■

1) Example 7: Let us consider operator $\beta_2 \gamma^1 \mu_3$ the C/C function of which is $\mathcal{F}_{\beta_2 \gamma^1 \mu_3}(k_i) = \lfloor (3k_i + 1)/2 \rfloor$. This $(2, 3)$-periodic function is such that $\mathcal{F}_{\beta_2 \gamma^1 \mu_3}(0) = 0$ and $\mathcal{F}_{\beta_2 \gamma^1 \mu_3}(1) = 2$. Therefore, the canonical form of $\beta_2 \gamma^1 \mu_3$ is $\gamma^0 \nabla_{3|2} \gamma^1 \oplus \gamma^2 \nabla_{3|2} \gamma^0$, or simply $\nabla_{3|2} \gamma^1 \oplus \gamma^2 \nabla_{3|2}$ since γ^0 is the unit operator (see Fig. 2). As a second example, let us consider the E-operator $\beta_4 \mu_3$. We have $\mathcal{F}_{\beta_4 \mu_3}(k_i) = \lfloor 3k_i/4 \rfloor$ which is $(4, 3)$-periodic. Then, by expressing the values of the C/C function for one period we obtain: $\mathcal{F}_{\beta_4 \mu_3}(0 \le k_i \le 1) = 0$, $\mathcal{F}_{\beta_4 \mu_3}(2) = 1$ and $\mathcal{F}_{\beta_4 \mu_3}(3) = 2$. Hence, we have $\beta_4 \mu_3 = \gamma^0 \nabla_{3|4} \gamma^3 \oplus \gamma^0 \nabla_{3|4} \gamma^2 \oplus \gamma^1 \nabla_{3|4} \gamma^1 \oplus \gamma^2 \nabla_{3|4} \gamma^0$. In this expression $\gamma^0 \nabla_{3|4} \gamma^3$ is redundant since $\gamma^3 \preceq \gamma^2 \Rightarrow \gamma^0 \nabla_{3|4} \gamma^3 \preceq \gamma^0 \nabla_{3|4} \gamma^2$. Finally, the canonical form of $\beta_3 \mu_4$ is $\nabla_{3|4} \gamma^2 \oplus \gamma^1 \nabla_{3|4} \gamma^1 \oplus \gamma^2 \nabla_{3|4}$.

C. Dioid $\mathcal{E}^*[\![\delta]\!]$

The previous subsection shows that E-operators generated by WBTEGs are periodic and have a canonical form. Since

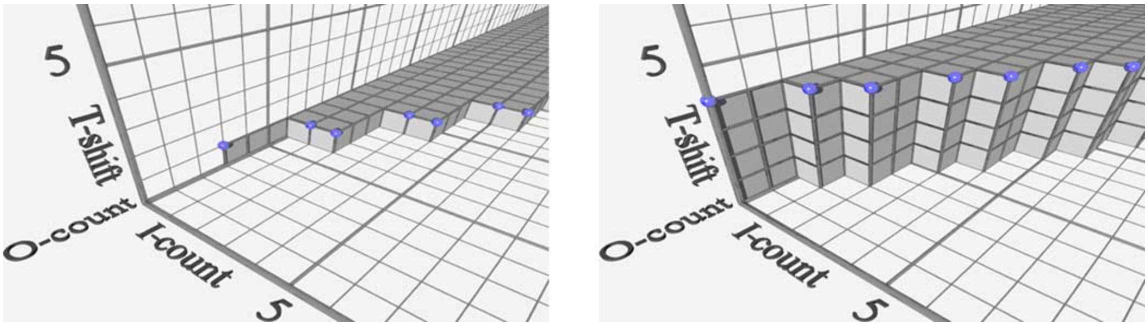

Fig. 4. Graphical representation of $\gamma^3\mu_3\beta_2\gamma^1\delta^1$ (on the left) and $\beta_2\gamma^1\mu_3\delta^4$ (on the right).

time-shift operator δ^τ can commute with all the E-operators (see Prop. 1), then the operators generated by WBTEGs can be described by the means of formal power series in one variable δ denoted $\bigoplus_i w_i\delta^{t_i}$, where coefficients w_i are taken in $\mathcal{E}_{per}$ and the exponents are in $\mathbb{Z}$.

Theorem 5 ([1]): The set of formal power series in variable z with exponents in $\mathbb{Z}$ and coefficients in a complete dioid $\mathcal{D}$ is a complete dioid denoted $\mathcal{D}[\![z]\!]$.

Theorem 6 ([1]): The quotient of a dioid $(\mathcal{D},\oplus,\otimes)$ by an equivalence relation $\mathcal{R}$ compatible with $\oplus$ and $\otimes$ is a dioid denoted $\mathcal{D}_{/\mathcal{R}}$.

Definition 14 (Dioid $\mathcal{E}^*[\![\delta]\!]$): We denote by $\mathcal{E}[\![\delta]\!]$ the complete dioid of formal power series in one variable δ with exponents in $\mathbb{Z}$ and coefficients in the non commutative complete dioid $\mathcal{E}$. We denote by $\mathcal{E}^*[\![\delta]\!]$ the quotient $\mathcal{E}[\![\delta]\!]_{/\equiv^*}$ where the equivalence relation considered is defined as: $s_a, s_b \in \mathcal{E}[\![\delta]\!]$

$$s_a \equiv^* s_b \iff s_a(\delta^{-1})^* = s_b(\delta^{-1})^*. \tag{14}$$

A series $s \in \mathcal{E}^*[\![\delta]\!]$ is written $s = \bigoplus_{t\in\mathbb{Z}} s(t)\delta^t$ with $s(t) \in \mathcal{E}$. For two series $s_1, s_2 \in \mathcal{E}^*[\![\delta]\!]$:

$$s_1 \oplus s_2 = \bigoplus_{t\in\mathbb{Z}} (s_1(t) \oplus s_2(t))\,\delta^t$$

$$s_1 \otimes s_2 = \bigoplus_{t\in\mathbb{Z}} \left(\bigoplus_{\tau+\tau'=t} s_1(\tau) \circ s_2(\tau') \right) \delta^t.$$

Adding the quotient structure by (14) allows us to assimilate the variable δ in dioid $\mathcal{E}^*[\![\delta]\!]$ to the time-shift operator $\delta^1 : \Sigma \to \Sigma, x(t) \mapsto x(t-1)$ in dioid $\mathcal{O}$. Therefore, all the identities given in Prop. 1 hold in $\mathcal{E}^*[\![\delta]\!]$. Dioid $\mathcal{E}^*[\![\delta]\!]$ is the algebraic structure the best adapted to handle operators given in Def. 5.

The series of $\mathcal{E}^*[\![\delta]\!]$ have a graphical representation which consists in describing each monomial $s(t)\delta^t$ in $\mathbb{Z}^3$, where coefficient $s(t) \in \mathcal{E}$ is described by its C/C function in a $\mathrm{I\text{-}count} \times \mathrm{O\text{-}count}$ plane which is located at value t along the T-shift axis. To improve the readability, the three-dimensional representation has been truncated to the positive values.

Thanks to equalities in Prop. 1, let us note that for $w_1, w_2 \in \mathcal{E}_{per}$ and $t_1, t_2 \in \mathbb{Z}$, we have

$$w_1\delta^{t_1} \preceq w_2\delta^{t_2} \iff w_1 \preceq w_2 \text{ and } t_1 \le t_2. \tag{15}$$

Therefore, a monomial $a = w_a\delta^{t_a}$ in $\mathcal{E}^*_{per}[\![\delta]\!]$ dominates all the monomials $w\delta^t$ such that $w \preceq w_a$ and $t \le t_a$. For each monomial, the subspace of $\mathbb{Z}^3$ which corresponds to the dominated operators is depicted as a gray shadow.

1) Example 8: In order to give a graphical interpretation of the partial order in $\mathcal{E}^*[\![\delta]\!]$, the representation of $\gamma^3\mu_3\beta_2\gamma^1\delta^1$ and $\beta_2\gamma^1\mu_3\delta^4$ have been juxtaposed in Fig. 4. Each monomial is depicted as well as the gray shadow corresponding to the domain of dominated monomials. Since $\gamma^3\mu_3\beta_2\gamma^1 \preceq \beta_2\gamma^1\mu_3$ (see Fig. 2) and $\delta^1 \preceq \delta^4$, then $\gamma^3\mu_3\beta_2\gamma^1\delta^1 \preceq \beta_2\gamma^1\mu_3\delta^4$. From a graphical point of view, if the representations of $\gamma^3\mu_3\beta_2\gamma^1\delta^1$ and $\beta_2\gamma^1\mu_3\delta^4$ are merged into the same picture, the shadow of $\beta_2\gamma^1\mu_3\delta^4$ clearly hides the representation of $\gamma^3\mu_3\beta_2\gamma^1\delta^1$.

Remark 2 (Simplifications): By representing each monomial of a series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ with its gray shadow in $\mathbb{Z}^3$, the graphical representation naturally hides the redundant monomials of the series. It is the geometrical interpretation of the simplifications that can be done in $\mathcal{E}^*[\![\delta]\!]$. Let us note that the gray shadow of monomials in $\mathcal{E}^*[\![\delta]\!]$ is the natural extension in $\mathbb{Z}^3$ of the south-east cones used to describe monomials in $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[\![\gamma,\delta]\!]$ (see [5]).

D. Polynomials in $\mathcal{E}^*_{per}[\![\delta]\!]$

Due to the specific structure of WBTEGs, we do not consider the whole set of series of $\mathcal{E}^*[\![\delta]\!]$ but only the series the coefficients of which are periodic E-operators. This subset is denoted $\mathcal{E}^*_{per}[\![\delta]\!]$.

Definition 15 (Balanced Series in $\mathcal{E}^*_{per}[\![\delta]\!]$): A series $s = \bigoplus s(t)\delta^t \in \mathcal{E}^*_{per}[\![\delta]\!]$ is said balanced if all its coefficients $s(t) \in \mathcal{E}_{per}$ have the same gain. The gain of s is denoted $\Gamma(s)$ and corresponds to the gain of all its coefficients. A balanced series is said *conservative* if $\Gamma(s) = 1$.

The series that can be described by finite sums $\bigoplus_{i=1}^T s(t_i)\delta^{t_i}$ are called polynomials.

Proposition 5 (Canonical Form of Balanced Polynomials): Let $p = \bigoplus_{i=1}^N w_i\delta^{t_i}$ be a balanced polynomial of $\mathcal{E}^*_{per}[\![\delta]\!]$. The canonical form of p is the unique expression $p = \bigoplus_{j=1}^{N'} w'_j\delta^{t'_j}$ such that w'_j are in the canonical form of Prop. 4 and coefficients and exponents are strictly ordered, that is to say

$$1 \le j \le N'-1, t'_j < t'_{j+1} \text{ and } w'_j \succ w'_{j+1}. \tag{16}$$

Proof: This form is obtained by sorting monomials according to the exponents of δ. Then, each coefficient is modified as follows $w'_j = \bigoplus_{i\ge j}^N w_i$. Finally, if $t_i < t_{i+1}$ and $w'_i = w'_{i+1}$, monomial $w'_i\delta^{t_i}$ is redundant and can be removed. In the final

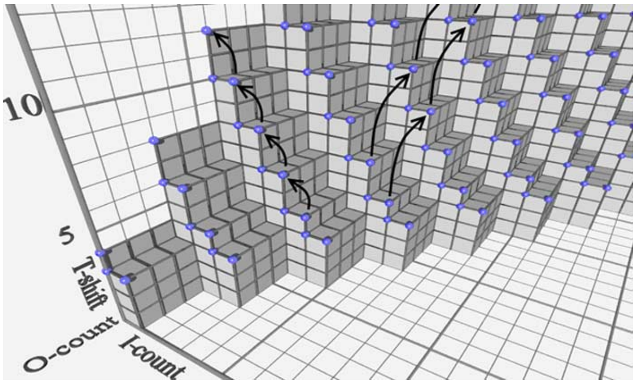

Fig. 5. Ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$.

form $p = \bigoplus_{j=1}^{N'} w'_j \delta^{t'_j}$ (with $N' \leq N$), coefficients and exponents are strictly ordered. In other words, the remaining monomials are incomparable (no further simplification is possible). ■

1) Example 9: Let us consider polynomial $p = \gamma^1\nabla_3\gamma^1\delta^2 \oplus \gamma^2\delta^3 \oplus \gamma^2\nabla_3\delta^4 \oplus \nabla_3\gamma^2\delta^6 \oplus \gamma^2\nabla_3\gamma^1\delta^7$. First, the monomials are sorted according to the exponent of δ. Next, the coefficient of δ^2 is replaced by $\gamma^1\nabla_3\gamma^1 \oplus \gamma^2 \oplus \gamma^2\nabla_3 \oplus \nabla_3\gamma^2 \oplus \gamma^2\nabla_3\gamma^1 = \gamma^0$. Then, the coefficient of δ^3 is replaced by the sum $\gamma^2 \oplus \gamma^2\nabla_3 \oplus \nabla_3\gamma^2 \oplus \gamma^2\nabla_3\gamma^1 = \nabla_3\gamma^2 \oplus \gamma^2\nabla_3$, and so on. At this step, $p = \gamma^0\delta^2 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3)\delta^3 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3)\delta^4 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3\gamma^1)\delta^6 \oplus \gamma^2\nabla_3\gamma^1\delta^7$. Monomial $(\nabla_3\gamma^2 \oplus \gamma^2\nabla_3)\delta^3$ is redundant since the coefficients of δ^3 and δ^4 are the same. Once it is removed, the form obtained is such that the coefficients and the exponents are strictly ordered, say $p = \delta^2 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3)\delta^4 \oplus (\nabla_3\gamma^2 \oplus \gamma^2\nabla_3\gamma^1)\delta^6 \oplus \gamma^2\nabla_3\gamma^1\delta^7$.

IV. WBTEGs ARE DESCRIBED BY ULTIMATELY PERIODIC SERIES OF $\mathcal{E}^*_{per}[\![\delta]\!]$

In this section, we will show that the behavior of a WBTEG is described by ultimately periodic and balanced series of $\mathcal{E}^*_{per}[\![\delta]\!]$. This result has to be compared to the well known result for ordinary Timed Event Graphs [5, Th. 21]: the entries of the transfer matrix of a TEG are ultimately periodic series of $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$. For TEGs, operations (and algorithms) on ultimately periodic series of $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ have already been investigated in [10] and [9]. The developments given here are clearly in the same spirit.

Since we consider WBTEGs, only balanced series of $\mathcal{E}^*_{per}[\![\delta]\!]$ are considered hereafter.

Definition 16 (Ultimately Periodic Series of $\mathcal{E}^*_{per}[\![\delta]\!]$): A balanced series $s \in \mathcal{E}^*_{per}[\![\delta]\!]$ is said ultimately periodic if it can be written as $s = p \oplus q(\gamma^\nu\delta^\tau)^*$, where $\nu, \tau \in \mathbb{N}$, p and q are balanced polynomials $p = \bigoplus_{i=1}^{n} w_i\delta^{t_i}$, $q = \bigoplus_{j=1}^{N} W_j\delta^{T_j}$ with $\forall i, j, w_i, W_j \in \mathcal{E}_{per}$.

The property of periodicity has a natural graphical interpretation. In the graphical representation of s, the monomials of q are depicted as a group of C/C functions that are periodically shifted by τ units to the increasing values along the T-shift axis and by ν units towards the decreasing values along the I-count axis.

1) Example 10: Fig. 5 gives the graphical description of $s = \nabla_{3|2}\delta^3 \oplus \nabla_{3|2}\gamma^1\delta^4(\gamma^1\delta^2)^*$.

Remark 3: A periodic form is not unique. For instance, $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ and $s = p \oplus q \oplus q\gamma^\nu\delta^\tau(\gamma^\nu\delta^\tau)^*$ are two different ultimately periodic forms of the same series.

Remark 4: Balanced polynomials in $\mathcal{E}^*_{per}[\![\delta]\!]$ can always be considered as ultimately periodic series since $(\gamma^1\delta^0)^* = e$.

Proposition 6 (Left and Right Periodicity): An ultimately right-periodic series $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ in $\mathcal{E}^*_{per}[\![\delta]\!]$ has also an ultimately left-periodic form $s = p \oplus (\gamma^{\nu'}\delta^{\tau'})^* q'$ where q' is a balanced polynomial. The left (resp. right) asymptotic slope is defined as $\sigma_l(s) = \tau'/\nu'$ (resp. $\sigma_r(s) = \tau/\nu$), and the next equality is satisfied $\Gamma(s) = \sigma_r(s)/\sigma_l(s)$.

Proof: Let $\Gamma(s) = k'/k$ be the gain of s. The coefficients of polynomial $q = \bigoplus w_j\delta^{t_j}$ are given by $w_j = \bigoplus_i \gamma^{n_{ij}}\nabla_{m_j|b_j}\gamma^{n'_{ij}}$ with $\forall j, m_j/b_j = k'/k$. Let us remark that thanks to (6), $\nabla_{m_j|b_j}\gamma^{b_j} = \mu_{m_j}\beta_{b_j}\gamma^{b_j} = \mu_{m_j}\gamma^1\beta_{b_j} = \gamma^{m_j}\mu_{m_j}\beta_{b_j} = \gamma^{m_j}\nabla_{m_j|b_j}$. More generally, for $\alpha \in \mathbb{N}$, $\nabla_{m_j|b_j}\gamma^{\alpha b_j} = \gamma^{\alpha m_j}\nabla_{m_j|b_j}$. Therefore, if we take $B = lcm(b_j)$ and $M = Bk'/k$, then $\forall i, j, \gamma^{n_{ij}}\nabla_{m_j|b_j}\gamma^{n'_{ij}}\gamma^B = \gamma^M\gamma^{n_{ij}}\nabla_{m_j|b_j}\gamma^{n'_{ij}}$, and consequently $\forall i, w_i\gamma^B = \gamma^M w_i$. By applying (9)

$$\begin{aligned} q(\gamma^\nu\delta^\tau)^* &= q(\gamma^{B\nu}\delta^{B\tau})^*(e \oplus \ldots \oplus \gamma^{(B-1)\nu}\delta^{(B-1)\tau}) \\ &= (\gamma^{M\nu}\delta^{B\tau})^* q(e \oplus \ldots \oplus \gamma^{(B-1)\nu}\delta^{(B-1)\tau}) \\ &= (\gamma^{M\nu}\delta^{B\tau})^* q'. \end{aligned}$$

Finally, $\sigma_r(s) = \tau/\nu$ and $\sigma_l(s) = (B\tau)/(M\nu)$ and $\sigma_r(s)/\sigma_l(s) = \Gamma(s) = k'/k$. ■

2) Example 11: For the series depicted on Fig. 5, a right-periodic form and a left-periodic form are given by

$$\begin{aligned} s &= \nabla_{3|2}\delta^3 \oplus \nabla_{3|2}\gamma^1\delta^4(\gamma^1\delta^2)^* \\ &= \nabla_{3|2}\delta^3 \oplus (\gamma^3\delta^4)^*(\nabla_{3|2}\gamma^1\delta^4 \oplus \gamma^3\nabla_{3|2}\delta^6). \end{aligned}$$

The way the coefficients are periodically shifted is illustrated by a set of arrows depicted on the picture. For this series, the slopes are, respectively, $\sigma_l(s) = 4/3$ and $\sigma_r(s) = 2$.

The main result concerning the class of WBTEGs is that the serial, the parallel and the feedback composition keep the ultimate periodicity property. To obtain this result, one has to analyze how the sum, the product and the Kleene star operations behave on ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$.

For the next propositions, series s_1 and s_2 are periodic series defined as $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $s_2 = p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$ with $p_1 = \bigoplus_{i=1}^{n_1} w_{1i}\delta^{t_{1i}}$, $q_1 = \bigoplus_{j=1}^{N_1} W_{1j}\delta^{T_{1j}}$, $p_2 = \bigoplus_{k=1}^{n_2} w_{2k}\delta^{t_{2k}}$, $q_2 = \bigoplus_{l=1}^{N_2} W_{2l}\delta^{T_{2l}}$, and $w_{1i}, w_{2k}, W_{1j}, W_{2l} \in \mathcal{E}_{per}$.

Proposition 7: Let s_1 and s_2 be two ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$. If $\Gamma(s_1) = \Gamma(s_2)$ then $s_1 \oplus s_2$ is an ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$ such that

$$\begin{aligned} \sigma_r(s_1 \oplus s_2) &= \max(\sigma_r(s_1), \sigma_r(s_2)) \\ \sigma_l(s_1 \oplus s_2) &= \max(\sigma_l(s_1), \sigma_l(s_2)). \end{aligned}$$

Proof: If $(\tau_1/\nu_1) = (\tau_2/\nu_2)$, by taking $N = lcm(\nu_1, \nu_2) = k_1\nu_1 = k_2\nu_2$ and $T = k_1\tau_1 = k_2\tau_2$, then by applying (9) we can express $(\gamma^{\nu_1}\delta^{\tau_1})^* = (e \oplus \gamma^{\nu_1}\delta^{\tau_1} \oplus \ldots \oplus \gamma^{(k_1-1)\nu_1}\delta^{(k_1-1)\tau_1})(\gamma^{k_1\nu_1}\delta^{k_1\tau_1})^* g$ and $(\gamma^{\nu_2}\delta^{\tau_2})^* =$

$(e\oplus\gamma^{\nu_2}\delta^{\tau_2}\oplus\ldots\oplus\gamma^{(k_2-1)\nu_2}\delta^{(k_2-1)\tau_2})(\gamma^{k_2\nu_2}\delta^{k_2\tau_2})^*$. Therefore, $q_1(\gamma^{\nu_1}\delta^{\tau_1})^* = q_1'(\gamma^N\delta^T)^*$ and $q_2(\gamma^{\nu_2}\delta^{\tau_2})^* = q_2'(\gamma^N\delta^T)^*$. Therefore, $s_1 \oplus s_2 = p_1 \oplus p_2 \oplus (q_1' \oplus q_2')(\gamma^N\delta^T)^*$ is also ultimately periodic. If $(\tau_1/\nu_1) > (\tau_2/\nu_2)$, according to Lemma 4 (see the Appendix) we obtain that $\forall j,l$ s.t. $1 \le j \le N_1, 1 \le l \le N_2$, then $W_{1j}\delta^{T_{1j}}(\gamma^{\nu_1}\delta^{\tau_1})^*$ is ultimately greater than $W_{2l}\delta^{T_{2l}}(\gamma^{\nu_2}\delta^{\tau_2})^*$. In other words, $q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ is ultimately greater than $q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$. Finally, $s_1 \oplus s_2$ is ultimately periodic with the periodicity of s_1. ■

Proposition 8: Let s_1 and s_2 be two ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$. If $\Gamma(s_1) = \Gamma(s_2)$ then $s_1 \wedge s_2$ is an ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$.

Proof: (Sketch of proof) The proof is similar. Two cases have to be considered. If $(\tau_1/\nu_1) = (\tau_2/\nu_2)$ then we can write the infimum as $s_1 \wedge s_2 = (p_1 \oplus q_1'(\gamma^N\delta^T)^*) \wedge (p_2 \oplus q_2'(\gamma^N\delta^T)^*)= (p_1 \wedge p_2) \oplus (p_1 \wedge q_2'(\gamma^N\delta^T)^*) \oplus (p_2 \wedge q_1'(\gamma^N\delta^T)^*)\oplus(q_1'(\gamma^N\delta^T)^* \wedge q_2'(\gamma^N\delta^T)^*)$. The first three terms are polynomials and the last one is an ultimately periodic series the slope of which is T/N. If $(\tau_1/\nu_1) > (\tau_2/\nu_2)$, we previously obtained that $q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ is ultimately greater than $q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$. In other words, $s_1 \wedge s_2$ has the ultimate periodicity of s_2. ■

Proposition 9: Let $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $s_2 = p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$ be two ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$. Then $s_1 \otimes s_2$ is an ultimately periodic series such that $\Gamma(s_1 \otimes s_2) = \Gamma(s_1) \times \Gamma(s_2)$

$$\sigma_r(s_1 \otimes s_2) = max(\sigma_r(s_2), \Gamma(s_2) \times \sigma_r(s_1))$$
$$\sigma_l(s_1 \otimes s_2) = max(\sigma_l(s_1), \sigma_l(s_2)/\Gamma(s_1)).$$

Proof: Thanks to Prop. 6, we can write s_1 and s_2, respectively, with a right-periodic form and a left-periodic form such as $s_1 \otimes s_2 = (p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*) \otimes (p_2 \oplus (\gamma^{\nu_2'}\delta^{\tau_2'})^* q_2')= p_1p_2 \oplus p_1(\gamma^{\nu_2'}\delta^{\tau_2'})^* q_2' \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^* p_2 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*(\gamma^{\nu_2'}\delta^{\tau_2'})^* q_2'$. The first term p_1p_2 is a polynomial. The second and the third term are explicitly equal to $p_1(\gamma^{\nu_2'}\delta^{\tau_2'})^* q_2' = (\bigoplus w_{1i}\delta^{t_{1i}})(\gamma^{\nu_2'}\delta^{\tau_2'})^*(\bigoplus W_{2l}'\delta^{T_{2l}'})$ and $q_1(\gamma^{\nu_1}\delta^{\tau_1})^* p_2 = (\bigoplus W_{1j}\delta^{T_{1j}})(\gamma^{\nu_1}\delta^{\tau_1})^*(\bigoplus w_{2k}\delta^{t_{2k}})$. These terms are two finite sums of periodic series with the same gain. Due to Prop. 7, these terms are periodic too. In the last term, $(\gamma^{\nu_1}\delta^{\tau_1})^*(\gamma^{\nu_2'}\delta^{\tau_2'})^*$ is an ultimately periodic series in $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$ (Th. 8 in the Appendix), and therefore in $\mathcal{E}^*[\![\delta]\!]$ too. The term $q_1(\gamma^{\nu_1}\delta^{\tau_1})^*(\gamma^{\nu_2'}\delta^{\tau_2'})^* q_2'$ is consequently a finite sum of ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. The product $s1 \otimes s_2$ is finally a finite sum of ultimately series with the same gain which is periodic by applying Prop. 7 again. ■

Let us now focus on the behavior of circuits in WBTEGs. They are algebraically described by Kleene star operations on conservative series of $\mathcal{E}^*_{per}[\![\delta]\!]$.

Proposition 10: Let $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ be a conservative ($\Gamma(s) = 1$) ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. Then s^* is a conservative ultimately periodic series.

Proof: Thanks to Lemma 7 in the Appendix, we can write polynomials p and q with a non canonical form $p = \bigoplus_i \gamma^{n_i}\nabla_M\gamma^{n_i'}\delta^{t_i}$ and $q = \bigoplus_j \gamma^{N_j}\nabla_M\gamma^{N_j'}\delta^{T_j}$. By taking $r = \gamma^{M\nu}\delta^{M\tau}$, then monomial r commutes with p and q, *i.e.*, $pr = rp$ and $qr = rq$. Thanks to (9), series s can be written $s = p \oplus q(e \oplus \gamma^\nu\delta^\tau \oplus \ldots \oplus \gamma^{(M-1)\nu}\delta^{(M-1)\tau})r^* = p \oplus q'r^*$. Moreover, r also commutes with q', i.e., $q'r = rq'$. By applying (7), one has $s^* = (p \oplus q'r^*)^* = p^*(q'r^*p^*)^*$. Since $rp = pr$ and thanks to (10), then $r^*p^* = (r \oplus p)^*$. Finally, by using (8), we can write $s^* = p^*(q'(r \oplus p)^*)^* = p^*(e \oplus q'(q' \oplus r \oplus p)^*)$. In that expression, $(q' \oplus r \oplus p)$ and p are two conservative polynomials. Thanks to Prop. 15 shown in the Appendix, $(q' \oplus r \oplus p)^*$ and p^* are two conservative ultimately periodic series. Finally, it can be inferred that $(e \oplus q'(q' \oplus r \oplus p)^*)$ is a conservative ultimately periodic series, and thanks to Prop. 9, s^* is periodic as well. ■

Proposition 11 (Transfer of a WBTEG): The transfer matrix of a weight-balanced timed event graph is composed of ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$.

Proof: We recall first that all the elementary operators $\gamma^n, \delta^t, \mu_m$ and β_b can be considered as ultimately periodic series (see remark 4). Then, due to the specific structure of WBTEGs, the modeling by series in $\mathcal{E}^*_{per}[\![\delta]\!]$ is such that:

- the sum ($\oplus$) of series in $\mathcal{E}^*_{per}[\![\delta]\!]$ is necessarily done on series with the same gain (balanced property) and the periodicity is kept by the balanced synchronization (see Prop. 7);
- the product of ultimately periodic series is done when the serial composition of systems arises, and the product keeps the periodicity property (see Prop. 9);
- the Kleene star is done only on conservative ultimately periodic series since the loops of a WBTEG are neutral. Thanks to Prop. 10, the Kleene star of conservative ultimately periodic series keeps the periodicity property.

■

Remark 5 (Canonical Form): The property of ultimate periodicity of WBTEGs does not depend on a specific periodic form. However, by comparison to existing results for periodic series in $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$, we think that a canonical periodic form exists also for ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. Such a canonical form would be useful for a software tool dedicated to periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$ (such as [7] for $\mathcal{M}^{ax}_{in}[\![\gamma,\delta]\!]$). By transposing the results given in [10] and [9] and by writing a series $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ with $p = \bigoplus_{i=1}^n w_i\delta^{t_i}$, $q = \bigoplus_{j=1}^N W_j\delta^{T_j}$, the canonical form of s would be the unique one such that p and q are in a canonical form ($\forall i,j\ w_i \succ w_{i+1}$, $t_i < t_{i+1}$, $W_j \succ W_{j+1}, T_j < T_{j+1}$), $w_n \succ W_1$ and $W_N \succ W_1\gamma^\nu$ (such a form is called a proper form in [10]), and finally the number n of monomials in p is as small as possible, and the periodic pattern q is also as short as possible (pair (ν,τ) is as small as possible).

V. Control of WBTEGs

The input–output model obtained in the previous section for WBTEGs allows us to consider some model matching control problems such as the ones studied in [6], [12], [13], [15], and [17]. We only need to express the residuation of the product in $\mathcal{E}^*_{per}[\![\delta]\!]$. The first step consists in expressing the residuation of the product in $\mathcal{E}_{per}$.

A. Residuation in $\mathcal{E}_{per}$

On a complete dioid, the product is not invertible, but the theory of residuation developped in [3], and applied to idempotent semirings in [1], can be used to find optimal solutions to some inequalities. On a complete dioid, mappings $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ are residuated. It means that $\forall b$, $L_a(x) \preceq b$ and $R_a(x) \preceq b$ have maximal solutions, that are, respectively, denoted $L_a^\sharp(b) = a \diagdown\!\!\!\circ\, b = \bigoplus\{x|ax \preceq b\}$ and $R_a^\sharp(b) = b \phi a = \bigoplus\{x|xa \preceq b\}$. Mappings $L_a^\sharp$ and $R_a^\sharp$ are said residual mappings of L_a and R_a. When the dioid product is commutative, then $L_a^\sharp = R_a^\sharp$.

Theorem 7 ([1], [3]): On a complete dioid $\mathcal{D}$,

$$abx \preceq c \iff x \preceq b\backslash a\backslash c = (ab)\backslash c \tag{17}$$
$$xba \preceq c \iff x \preceq c\phi a \phi b = c\phi(ba) \tag{18}$$
$$(a \oplus b)x \preceq c \iff x \preceq a\backslash c \wedge b\backslash c \tag{19}$$
$$x(a \oplus b) \preceq c \iff x \preceq c\phi a \wedge c\phi b. \tag{20}$$

The dioid of E-operators denoted $\mathcal{E}$ is complete. It is then possible to define the residual mappings of L_a and R_a on $\mathcal{E}$. More precisely, concerning the elementary operators of $\mathcal{E}$, the following results can be obtained.

Proposition 12: Let $w \in \mathcal{E}$ be an E-operator, then

$$\gamma^n\backslash w = \gamma^{-n}w \qquad w\phi\gamma^n = w\gamma^{-n} \tag{21}$$
$$\mu_m\backslash w = \beta_m\gamma^{m-1}w \qquad w\phi\mu_m = w\beta_m \tag{22}$$
$$\beta_b\backslash w = \mu_b w \qquad w\phi\beta_b = w\gamma^{b-1}\mu_b. \tag{23}$$

Proof: Since operator γ^n is invertible ($\gamma^n\gamma^{-n} = \gamma^{-n}\gamma^n = e$), then we obtain (21). For (22), the right product by μ_m is invertible since $\beta_m\mu_m = e$. For the left product, by definition of the residual mapping: $\mu_m\backslash w = \bigoplus\{v \in \mathcal{E}|\mu_m v \preceq w\}$. Since $w_1 \preceq w_2 \iff \mathcal{F}_{w_1} \geq \mathcal{F}_{w_2}$, then we have $\mu_m\backslash w = \bigoplus\{v \in \mathcal{E}|\mathcal{F}_{\mu_m v} \geq \mathcal{F}_w\} = \bigoplus\{v \in \mathcal{E}|m\mathcal{F}_v \geq \mathcal{F}_w\} = \bigoplus\{v \in \mathcal{E}|\mathcal{F}_v \geq \mathcal{F}_w/m\}$. Finally, $\mathcal{F}_{\mu_m\backslash w}$ must satisfy $\forall k \in \mathbb{Z},\ \mathcal{F}_{\mu_m\backslash w}(k) \geq \mathcal{F}_w(k)/m$, which is equivalent to $\forall k \in \mathbb{Z}, \mathcal{F}_{\mu_m\backslash w}(k) = \lceil \mathcal{F}_w(k)/m \rceil = \lfloor (\mathcal{F}_w(k) + m - 1)/m \rfloor$ where $\lceil x \rceil$ denotes the least integer greater than or equal to x. Translated into operators, we have $\mu_m\backslash w = \beta_m\gamma^{m-1}w$.

For (23), we know that the left product by β_b is invertible. For the right product, we have

$$\begin{aligned} w\phi\beta_b &= \bigoplus\{v \in \mathcal{E}|\mathcal{F}_{v\beta_b} \geq \mathcal{F}_w\} \\ &= \bigoplus\{v \in \mathcal{E}|\forall k \in \mathbb{Z}, \mathcal{F}_v(\lfloor k/b \rfloor) \geq \mathcal{F}_w(k)\}. \end{aligned}$$

Therefore, the C/C function of $w\phi\beta_b$ has to satisfy the following constraints:

$$\begin{gathered} 0 \leq k \leq b-1, \mathcal{F}_{w\phi\beta_b}(0) \geq \mathcal{F}_w(k) \\ b \leq k \leq 2b-1, \mathcal{F}_{w\phi\beta_b}(1) \geq \mathcal{F}_w(k) \\ \cdots \\ nb \leq k \leq (n+1)b-1, \mathcal{F}_{w\phi\beta_b}(n) \geq \mathcal{F}_w(k). \end{gathered}$$

Since $\mathcal{F}_w$ is a non decreasing function, $\mathcal{F}_{w\phi\beta_b}$ must satisfy $\mathcal{F}_{w\phi\beta_b}(n) = \mathcal{F}_w((n+1)b-1)$ say $\mathcal{F}_{w\phi\beta_b}(k) = \mathcal{F}_w(bk + (b-1))$, which amounts to $w\phi\beta_b = w\gamma^{b-1}\mu_b$. ■

1) Example 12: Let us develop the computation of $(\gamma^1\mu_2)\backslash(\gamma^2\beta_3\mu_4) \in \mathcal{E}$. By applying results from Prop. 12 and from Prop. 1, we obtain

$$\begin{aligned} (\gamma^1\mu_2)\backslash(\gamma^2\beta_3\mu_4) &= \mu_2\backslash(\gamma^1\backslash(\gamma^2\beta_3\mu_4)) \\ &= \mu_2\backslash(\gamma^{-1}(\gamma^2\beta_3\mu_4)) \\ &= \beta_2\gamma^1(\gamma^1\beta_3\mu_4) = \beta_2\gamma^2\beta_3\mu_4 \\ &= \gamma^1\beta_2\beta_3\mu_4 = \gamma^1\beta_6\mu_4 = \gamma^1\beta_3\mu_2. \end{aligned}$$

Let us note that the canonical form of $\gamma^1\beta_3\mu_2$ is $\gamma^1\nabla_{2|3}\gamma^1 \oplus \gamma^2\nabla_{2|3}$. Since residuation is not an exact inversion, we can remark here that the canonical form of $(\gamma^1\mu_2)[(\gamma^1\mu_2)\backslash(\gamma^2\beta_3\mu_4)] = \gamma^3\nabla_{4|3}\gamma^1 \oplus \gamma^5\nabla_{4|3}$ is different from $(\gamma^2\beta_3\mu_4) = \gamma^2\nabla_{4|3}\gamma^2 \oplus \gamma^3\nabla_{4|3}\gamma^1 \oplus \gamma^4\nabla_{4|3}$.

Proposition 13: Let us consider $w_1, w_2 \in \mathcal{E}_{per}$. Then $w_2\backslash w_1$ and $w_1\phi w_2$ are periodic E-operators such that $\Gamma(w_2\backslash w_1) = \Gamma(w_1)/\Gamma(w_2)$ and $\Gamma(w_1\phi w_2) = \Gamma(w_1)/\Gamma(w_2)$.

Proof: Thanks to Th. 7 and Prop. 12, and since we can write periodic E-operators as finite sums, $w_1 = \bigoplus_i \gamma^{n_i}\nabla_{m|b}\gamma^{n'_i}$ and $w_2 = \bigoplus_j \gamma^{n_j}\nabla_{M|B}\gamma^{n'_j}$, then

$$\begin{aligned} w_2\backslash w_1 &= [\bigoplus_j \gamma^{n_j}\nabla_{M|B}\gamma^{n'_j}]\backslash[\bigoplus_i \gamma^{n_i}\nabla_{m|b}\gamma^{n'_i}] \\ &= \bigwedge_j \left([\gamma^{n_j}\nabla_{M|B}\gamma^{n'_j}]\backslash[\bigoplus_i \gamma^{n_i}\nabla_{m|b}\gamma^{n'_i}] \right) \\ &= \bigwedge_j \left(\bigoplus_i \gamma^{-n'_j}\mu_B\beta_M\gamma^{M-1}\gamma^{-n_j}\gamma^{n_i}\nabla_{m|b}\gamma^{n'_i} \right). \end{aligned}$$

It is then a finite infimum of periodic E-operators, that is also a periodic E-operator thanks to Prop. 2. ■

B. Residuation in $\mathcal{E}^*_{per}[\![\delta]\!]$

Let us note that $w_a\delta^{t_a}\backslash w_b\delta^{t_b} = w_a\backslash w_b\delta^{t_b - t_a}$. Thanks to (19) and (20), we can express the residuation of the product of balanced polynomials. Let $p_1 = \bigoplus_i w_{1_i}\delta^{t_{1_i}}$ and $p_2 = \bigoplus_j w_{2_j}\delta^{t_{2_j}}$ be two balanced polynomials in $\mathcal{E}^*_{per}[\![\delta]\!]$. Then, we can write $p_2\backslash p_1$ and $p_1\phi p_2$ as

$$\begin{aligned} p_2\backslash p_1 &= (\bigoplus_j w_{2_j}\delta^{t_{2_j}})\backslash[\bigoplus_i w_{1_i}\delta^{t_{1_i}}] \\ &= \bigwedge_j \left((w_{2_j}\delta^{t_{2_j}})\backslash[\bigoplus_i w_{1_i}\delta^{t_{1_i}}] \right) \\ &= \bigwedge_j [\bigoplus_i (w_{2_j}\backslash w_{1_i})\delta^{t_{1_i} - t_{2_j}}] \end{aligned}$$

and

$$p_1\phi p_2 = \bigwedge_j [\bigoplus_i (w_{1_i}\phi w_{2_j})\delta^{t_{1_i} - t_{2_j}}].$$

The computation of operations $\backslash$ and ϕ on balanced polynomials is based on the residuation of coefficients in $\mathcal{E}_{per}$ (see the previous subsection), and it is then equivalent to the infimum operation on a finite set of polynomials in $\mathcal{E}^*_{per}[\![\delta]\!]$ which is a balanced polynomial.

When we extend the computation of operations $\backslash$ and ϕ to ultimately periodic series of $\mathcal{E}^*_{per}[\![\delta]\!]$, we can show that the residuation of two periodic series can be computed with a fixed-point iteration method.

Lemma 1 ([1]): The greatest fixed-point of $\Pi_l(x) = a\backslash x \wedge b$ (resp. $\Pi_r(x) = x\phi a \wedge b$) is $a^*\backslash b$ (resp. $b\phi a^*$).

Proposition 14: Let $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $s_2 = p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$ be two periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. If the mapping $(\gamma^{\nu_2}\delta^{\tau_2})\backslash x \wedge (q_2\backslash s_1)$ has a fixed point, then $s_2\backslash s_1$ is a periodic series.

Proof: We can write $s_2 \circ\!\!\!\!\backslash s_1 = [p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^*] \circ\!\!\!\!\backslash s_1 = p_2 \circ\!\!\!\!\backslash s_1 \wedge (\gamma^{\nu_2}\delta^{\tau_2})^* \circ\!\!\!\!\backslash (q_2 \circ\!\!\!\!\backslash s_1)$ (according to (19)). Thanks to Lemma 1, if $(\gamma^{\nu_2}\delta^{\tau_2}) \circ\!\!\!\!\backslash x \wedge (q_2 \circ\!\!\!\!\backslash s_1)$ has a fixed point, then $s_2 \circ\!\!\!\!\backslash s_1$ is shown to be expressed as the infimum ($\wedge$) of a finite set of periodic series with the same slope. Thanks to Prop. 8, the result is also periodic. ■

Remark 6: Prop. 14 gives a practical way to compute the residuation of two ultimately periodic series in $\mathcal{E}^*_{per}[\![\delta]\!]$. However, it should be kept in mind that the fixed-point iteration method does not necessarily terminate in a finite number of steps. The convergence depends on the ultimate slopes of series. This aspect would need more developments to be clarified.

C. Example of Output Feedback Synthesis

This modeling is applied in this section to obtain an output feedback control for the WBTEG of Fig. 1. First, we state the transfer relation of Fig. 1 in an ultimately periodic form. In Ex. 4 we obtained $x_4 = \left(\mu_3(\gamma^1\delta^2)^*\beta_2\delta^2 \oplus \beta_2\gamma^1\delta^1(\gamma^2\delta^1)^*\mu_3\right) x_1$, i.e., $x_4 = Hx_1$. The transfer function H is expressed as a sum of two periodic series. The first term of H can be written in a left-periodic or in a right-periodic form, $\mu_3(\gamma^1\delta^2)^*\beta_2\delta^2 = (\gamma^3\delta^2)^*\nabla_{3|2}\delta^2 = \nabla_{3|2}\delta^2(\gamma^2\delta^2)^*$. The second one can also be written as $\beta_2\gamma^1\delta^1(\gamma^2\delta^1)^*\mu_3 = (\gamma^1\delta^1)^*(\nabla_{3|2}\gamma^1 \oplus \gamma^2\nabla_{3|2})\delta^1 = [(\nabla_{3|2}\gamma^1 \oplus \gamma^2\nabla_{3|2})\delta^1 \oplus (\gamma^1\nabla_{3|2}\gamma^1 \oplus \gamma^3\nabla_{3|2})\delta^2 \oplus (\gamma^2\nabla_{3|2}\gamma^1 \oplus \gamma^4\nabla_{3|2})\delta^3](\gamma^2\delta^3)^*$. According to Prop. 7, the sum of these series is ultimately periodic with $\sigma_r(H) = max(1/1, 3/2)$ and $\sigma_l(H) = max(2/3, 1/1)$ (the slope of the second term). The gain of series H is clearly the gain of all paths from t_1 to t_4, $\Gamma(H) = 3/2$. A left and a right periodic form of H are given below (where coefficients are described in their canonical form in $\mathcal{E}_{per}$):

$$H = p \oplus q(\gamma^2\delta^3)^* = p \oplus (\gamma^1\delta^1)^* q'$$

with $p = \nabla_{3|2}\delta^2 \oplus (\gamma^2\nabla_{3|2}\gamma^1 \oplus \gamma^3\nabla_{3|2})\delta^3 \oplus \gamma^3\nabla_{3|2}\delta^4 \oplus (\gamma^4\nabla_{3|2}\gamma^1 \oplus \gamma^6\nabla_{3|2})\delta^5 \oplus (\gamma^5\nabla_{3|2}\gamma^1 \oplus \gamma^6\nabla_{3|2})\delta^6$, $q = [(\gamma^6\nabla_{3|2}\gamma^1 \oplus \gamma^8\nabla_{3|2})\delta^7 \oplus (\gamma^7\nabla_{3|2}\gamma^1 \oplus \gamma^9\nabla_{3|2})\delta^8 \oplus (\gamma^8\nabla_{3|2}\gamma^1 \oplus \gamma^{10}\nabla_{3|2})\delta^9]$ and $q' = [(\gamma^6\nabla_{3|2}\gamma^1 \oplus \gamma^8\nabla_{3|2})\delta^7]$.

Thanks to results obtained in [6], we can compute the greatest neutral output feedback for the WBTEG described by the transfer matrix H. From a practical point of view, it is the slowest controller that we can add between the output and the input so that the closed-loop system has the same behavior as the system alone. The benefit from this controller is to reduce the internal stocks as much as possible while keeping the system throughput. By knowing H, this controller is expressed by (see [6]) $\hat{F} = H \circ\!\!\!\!\backslash H \circ\!\!\!\!/ H$. Series $H \circ\!\!\!\!\backslash H$ is computed first with the fixed-point iteration given in Prop. 14 which terminates in a finite number of steps. Then, the same method is applied again to obtain the result of $H \circ\!\!\!\!\backslash H \circ\!\!\!\!/ H$. For the WBTEG of Fig. 1, the computation gives

$$\begin{aligned}\hat{F} &= (\gamma^3\nabla_{2|3}\gamma^1 \oplus \gamma^4\nabla_{2|3})\delta^0 \oplus \gamma^4\nabla_{2|3}\delta^2 \\ &\quad \oplus (\gamma^2\delta^3)^*[\gamma^6\nabla_{2|3}\delta^4] \\ &= (\gamma^3\nabla_{2|3}\gamma^1 \oplus \gamma^4\nabla_{2|3})\delta^0 \oplus \gamma^4\nabla_{2|3}\delta^2 \\ &\quad \oplus [\gamma^6\nabla_{2|3}\delta^4](\gamma^3\delta^3)^*.\end{aligned}$$

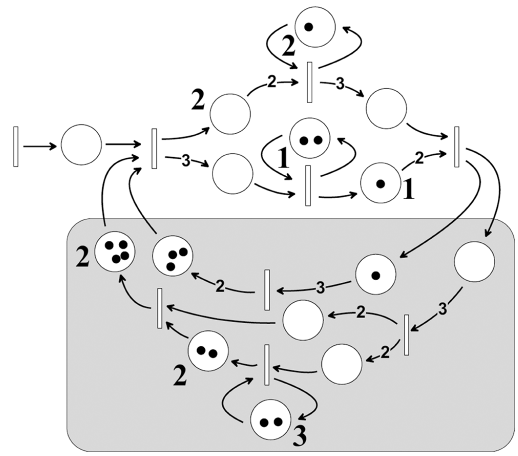

Fig. 6. Greatest neutral output feedback.

The controller is described by an ultimately periodic series the slopes of which are $\sigma_r(\hat{F}) = 3/3$ and $\sigma_l(\hat{F}) = 3/2$. We naturally obtain that $\Gamma(\hat{F}) = 2/3$ is equal to $1/\Gamma(H)$: the additional circuit due to the feedback loop is neutral, and therefore the closed-loop system is still a WBTEG. Controller $\hat{F}$ can be described by a WBTEG which is depicted in Fig. 6. The gray zone corresponds to the realization of controller $\hat{F}$. Let us note that the closed-loop system becomes bounded since it is a strongly connected WBTEGs.

VI. Conclusion

This work presents a modeling approach for the class of WBTEGs in a dioid of additive operators. Four elementary operators denoted $\gamma^n, \delta^t, \mu_m$ and β_b are necessary to describe the dynamical phenomena modeled by a WBTEG. The input–output behavior of WBTEGs can be embedded into rational formal power series in a non commutative dioid denoted $\mathcal{E}^*[\![\delta]\!]$. More specifically, we show that the transfer series of WBTEGs expressed in $\mathcal{E}^*[\![\delta]\!]$ have an ultimate periodicity property. This input–output representation is well suited to address some model matching control problems already tackled in literature for TEGs. As an example, the computation of a neutral output feedback controller for a WBTEG is given in this paper. The main contribution of this work is to show that the study of WBTEGs can be done with algebraic tools similar to the ones presented in [1] for the analysis and the control of TEGs.

Appendix

A. Rational Calculus in $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[\![\gamma, \delta]\!]$

Definition 17 ($\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[\![\gamma, \delta]\!]$): Dioid $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[\![\gamma, \delta]\!]$ is the set of formal power series in two commutative variables γ and δ with boolean coefficients quotiented by the equivalence $s_1 \equiv s_2 \iff \gamma^*(\delta^{-1})^* s_1 = \gamma^*(\delta^{-1})^* s_2$.

Theorem 8 (Operations on Periodic Series): Let $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $s_2 = p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^*$ be two periodic series of $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[\![\gamma, \delta]\!]$, where $\nu_1, \nu_2, \tau_1, \tau_2 \in \mathbb{N}$, p_1, p_2, q_1 and q_2 are polynomials in $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[\![\gamma, \delta]\!]$. The asymptotic slope of s_1 (resp.

s_2) is denoted $\sigma(s_1) = \tau_1/\nu_1$ (resp. $\sigma(s_2) = \tau_2/\nu_2$). Let $s_1 \neq \varepsilon$ and $s_2 \neq \varepsilon$, then

- $s_1 \oplus s_2$ is a periodic series such that $\sigma(s_1 \oplus s_2) = max(\sigma(s_1), \sigma(s_2))$;
- $s_1 \otimes s_2$ is a periodic series such that $\sigma(s_1 \otimes s_2) = max(\sigma(s_1), \sigma(s_2))$;
- $(s_1)^*$ is a periodic series

Proof: Proofs are detailed in [9] and in a more concise way in [10]. ■

B. Intermediate Results in $\mathcal{E}^*_{per}[\![\delta]\!]$

First, we recall a result given in [5, Lem 6], the proof of which is detailed in [9, Lem. 4.1.4]. This result is stated in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ and is still valid in $\mathcal{E}^*_{per}[\![\delta]\!]$ since $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ is a subdioid.

Lemma 2 (Domination Lemma): Let $m_1 = \gamma^{n_1}\delta^{t_1}(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $m_2 = \gamma^{n_2}\delta^{t_2}(\gamma^{\nu_2}\delta^{\tau_2})^*$ be two simple periodic series in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ such that $\tau_1/\nu_1 > \tau_2/\nu_2$. There exists an integer K such that

$$\gamma^{n_2}\delta^{t_2}\gamma^{K\nu_2}\delta^{K\tau_2}(\gamma^{\nu_2}\delta^{\tau_2})^* \preceq m_1.$$

Remark 7: The previous lemma means that in the expression $m_1 \oplus m_2$, monomials $\gamma^{n_2}\delta^{t_2}\gamma^{K\nu_2}\delta^{K\tau_2} \oplus \gamma^{n_2}\delta^{t_2}\gamma^{(K+1)\nu_2}\delta^{(K+1)\tau_2} \oplus \ldots$ are redundant. Asymptotically, series m_1 dominates series m_2.

Lemma 3: Let $w_1 = \bigoplus_i \gamma^{n_{1i}}\nabla_{m_1|b_1}\gamma^{n'_{1i}}$ and $w_2 = \bigoplus_j \gamma^{n_{2j}}\nabla_{m_2|b_2}\gamma^{n'_{2j}}$ be two periodic E-operators in $\mathcal{E}_{per}$ such that $\Gamma(w_1) = \Gamma(w_2)$. There exists a positive integer K such that $w_2\gamma^K \preceq w_1$.

Proof: First we show that for all i, j there exists an integer K_{ji} such that $\gamma^{n_{2j}}\nabla_{m_2|b_2}\gamma^{n'_{2j}}\gamma^{K_{ji}} \preceq \gamma^{n_{1i}}\nabla_{m_1|b_1}\gamma^{n'_{1i}}$. For all counter value $k \in \mathbb{Z}$, K_{ji} must satisfy

$$\left\lfloor \frac{k + n'_{2j} + K_{ji}}{b_2} \right\rfloor m_2 + n_{2j} \geq \left\lfloor \frac{k + n'_{1i}}{b_1} \right\rfloor m_1 + n_{1i}. \quad (24)$$

Since for all $x \in \mathbb{R}, x \geq \lfloor x \rfloor \geq x - 1$, if K_{ji} is chosen such that $\forall k \in \mathbb{Z}, ((k+n'_{2j}+K_{ji})/b_2-1)m_2+n_{2j} \geq ((k+n'_{1i})/b_1)m_1+n_{1i}$, then (24) is satisfied too. By assumption, $m_2/b_2 = m_1/b_1$; therefore, the previous inequality does not depend on k. It suffices to take $K_{ji} = n'_{1i} - n'_{2j} + b_2 + \lceil \frac{b_2}{m_2}(n_{1i} - n_{2j}) \rceil$. Finally, by taking $K = \max(0, \max_{i,j} K_{ji})$, we have $w_2\gamma^K \preceq w_1$. ■

The next Lemma is an extension of Lemma 2 in $\mathcal{E}^*_{per}[\![\delta]\!]$.

Lemma 4: Let us consider $m_1 = w_1\delta^{t_1}(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $m_2 = w_2\delta^{t_2}(\gamma^{\nu_2}\delta^{\tau_2})^*$ with $w_1, w_2 \in \mathcal{E}_{per}$ such that $\Gamma(w_1) = \Gamma(w_2)$. If $\tau_1/\nu_1 > \tau_2/\nu_2$, there exists a positive integer K such that

$$w_2\delta^{t_2}\gamma^{K\nu_2}\delta^{K\tau_2}(\gamma^{\nu_2}\delta^{\tau_2})^* \preceq m_1.$$

Proof: Thanks to Lemma 3 we can find an integer N s.t. $w_2\gamma^N \preceq w_1$. By applying Lemma 2, since $\tau_1/\nu_1 > \tau_2/\nu_2$, then $\delta^{t_1}(\gamma^{\nu_1}\delta^{\tau_1})^*$ is asymptotically greater than $\gamma^{-N}\delta^{t_2}(\gamma^{\nu_2}\delta^{\tau_2})^*$. Therefore, since $w_2\gamma^N \preceq w_1$ then $w_1\delta^{t_1}(\gamma^{\nu_1}\delta^{\tau_1})^*$ is asymptotically greater than $w_2\delta^{t_2}(\gamma^{\nu_2}\delta^{\tau_2})^*$. ■

Lemma 5: In $\mathcal{E}_{per}$, operator ∇_m can be written in a non canonical form as

$$\nabla_m = \bigoplus_{j=0}^{j=n-1} \gamma^{jm}\nabla_{nm}\gamma^{(n-1-j)m}.$$

Proof: The C/C function of ∇_m is clearly (m, m)-periodic. By considering $\mathcal{F}_{\nabla_m}$ as a (nm, nm)-periodic function, we obtain this non canonical form. ■

1) Example 13: For instance, we can write ∇_2 in a non canonical form as $\nabla_2 = \nabla_6\gamma^4 \oplus \gamma^2\nabla_6\gamma^2 \oplus \gamma^4\nabla_6$.

Lemma 6: In $\mathcal{E}_{per}$, we have $\nabla_M\gamma^K\nabla_M = \gamma^{\lfloor K \rfloor_M}\nabla_M = \nabla_M\gamma^{\lfloor K \rfloor_M}$ where $\lfloor K \rfloor_M \triangleq M\lfloor K/M \rfloor$ is the greatest integer in $M\mathbb{Z}$ less than or equal to K.

Proof: $\forall x \in \Sigma$, we have $(\nabla_M x)(t) = M\lfloor x(t)/M \rfloor = \lfloor x(t) \rfloor_M$. Since $K = \lfloor K \rfloor_M + K \bmod M$, we have $(\nabla_M\gamma^K\nabla_M x)(t) = \lfloor \lfloor x(t) \rfloor_M + K \rfloor_M = \lfloor \lfloor x(t) \rfloor_M + M\lfloor K/M \rfloor + (K \bmod M) \rfloor_M = \lfloor \lfloor x(t) \rfloor_M \rfloor_M + \lfloor K \rfloor_M = \lfloor x(t) \rfloor_M + \lfloor K \rfloor_M$. ■

Lemma 7: Let $p = \bigoplus_i w_i\delta^{t_i} = \bigoplus_i(\bigoplus_j \gamma^{n_{ij}}\nabla_{m_i}\gamma^{n'_{ij}})\delta^{t_i}$ be a conservative ($\Gamma(p) = 1$) balanced polynomial of $\mathcal{E}^*_{per}[\![\delta]\!]$. Polynomial p can be written in a non canonical form as $p = \bigoplus_k \gamma^{n_k}\nabla_M\gamma^{n'_k}\delta^{t_k}$ where $M = lcm(m_i)$.

Proof: By applying Lemma 5, each coefficient w_i of p can be developed as a sum of $\gamma^n\nabla_M\gamma^{n'}$ operators with $M = lcm(m_i)$. ■

2) Example 14: Let $p = \gamma^2\nabla_3\gamma^2\delta^2 \oplus \gamma^3\nabla_2\gamma^1\delta^3$. Thanks to Lemma 5, we can write p as a sum of $\gamma^n\nabla_{lcm(2,3)}\gamma^{n'}$ operators: $p = \gamma^2\nabla_6\gamma^5\delta^2 \oplus \gamma^5\nabla_6\gamma^2\delta^2 \oplus \gamma^3\nabla_6\gamma^5\delta^3 \oplus \gamma^5\nabla_6\gamma^3\delta^3 \oplus \gamma^7\nabla_6\gamma^1\delta^3$.

Thanks to Lemma 7, a conservative polynomial p can always be written as $p = \bigoplus_{i=1}^N w_i\delta^{t_i} = \bigoplus_{i=1}^N \gamma^{n_i}\nabla_M\gamma^{n'_i}\delta^{t_i}$ (see Ex. 14), i.e., all terms depend on the same operator ∇_M. Thanks to this form, the expression of the Kleene star of p can now be studied. In the expression of p^*, the products of L elements in $\{w_1\delta^{t_1}, .., w_N\delta^{t_N}\}$ is central. First, we introduce the following notation with $L \geq 2$:

$$M^L_{IJ} \triangleq \bigoplus_{i_x \in \{1,..,N\}} \left\{ \bigotimes_{x=1}^{L} w_{i_x}\delta^{t_{i_x}} | i_1 = I, i_L = J \right\}.$$

Operator M^L_{IJ} is obtained by summing all the products of exactly L operators from the set $\{w_1\delta^{t_1}, .., w_N\delta^{t_N}\}$ and such that the left factor is $w_I\delta^{t_I}$ and the right factor is $w_J\delta^{t_J}$, with $I, J \in \{1, .., N\}$. Thanks to the notation above, p^* can be expressed as follows:

$$p^* = e \oplus p \oplus \bigoplus_{i=1}^{N}\bigoplus_{j=1}^{N}\bigoplus_{l=2}^{\infty} M^l_{ij}. \quad (25)$$

First, let us focus on product $\bigotimes_{x=1}^{L} w_{i_x}\delta^{t_{i_x}}$ such that $i_x \in \{1, .., N\}$:

$$\begin{aligned}\bigotimes_{x=1}^{x=L} w_{i_x}\delta^{t_{i_x}} &= w_{i_1}w_{i_2}\ldots w_{i_L}\delta^{(t_{i_1}+t_{i_2}+\ldots+t_{i_L})}\\ &= \gamma^{n_{i_1}}\nabla_M\gamma^{n'_{i_1}}\ldots\gamma^{n_{i_L}}\nabla_M\gamma^{n'_{i_L}}\delta^{(t_{i_1}+\ldots+t_{i_L})}.\end{aligned}$$

By applying Lemma 6, one can simplify as follows:

$$\bigotimes_{x=1}^{L} w_{i_x}\delta^{t_{i_x}} = \gamma^{\kappa}\delta^{\tau}\gamma^{n_{i_1}}\nabla_M\gamma^{n'_{i_L}}$$

with

$$\begin{aligned}\kappa &= \lfloor n'_{i_1} + n_{i_2} \rfloor_M + \ldots + \lfloor n'_{i_{L-1}} + n_{i_L} \rfloor_M\\ &= \sum_{j=1}^{j=L-1} \lfloor n'_{i_j} + n_{i_{j+1}} \rfloor_M,\\ \tau &= \sum_{j=1}^{j=L} t_{i_j}.\end{aligned}$$

Finally, such an operator can be written

$$\bigotimes_{x=1}^{L} w_{i_x}\delta^{t_{i_x}} = \left[\bigotimes_{j=1}^{L-1}\gamma^{\lfloor n'_{i_j}+n_{i_{j+1}}\rfloor_M}\delta^{t_{i_j}}\right]\gamma^{n_{i_1}}\nabla_M\gamma^{n'_{i_L}}\delta^{t_{i_L}}.$$

It can be inferred that expression $\bigotimes_{j=1}^{j=L} w_{i_j}\delta^{t_{i_j}}$ such that $i_1 = I$ and $i_L = J$ can be written $\gamma^{\kappa}\delta^{\tau}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}$.

Lemma 8: Let $L \geq 2$ be a finite integer, then we have

$$M_{IJ}^L = (\varphi^{L-1})_{IJ}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}$$

where φ is a square matrix of $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]^{N\times N}$ defined by

$$\forall a,b \in \{1,..,N\}, \varphi_{ab} = \gamma^{\lfloor n'_a+n_b\rfloor_M}\delta^{t_a}.$$

Proof: According to this definition of φ, for the exponent $L-1$ of matrix φ we have

$$(\varphi^{L-1})_{IJ} = \bigoplus_{i_x\in\{1,..,N\}}\{\bigotimes_{x=1}^{L-1}\gamma^{\lfloor n'_{i_x}+n_{i_{x+1}}\rfloor_M}\delta^{t_{i_x}} | i_1 = I, i_L = J\}.$$

■

Lemma 9: The infinite sum $\bigoplus_{L=2}^{+\infty} M_{IJ}^L$ is described by the following equivalent expression:

$$\bigoplus_{L=2}^{+\infty} M_{IJ}^L = (\varphi^+)_{IJ}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}$$

with $\varphi^+ = \bigoplus_{n\geq 1}\varphi^n = \varphi\varphi^*$.

Proof: By using Lemma 8 for operators M_{IJ}^L where $L \geq 2$, we obtain

$$\begin{aligned}\bigoplus\nolimits_{L\geq 2} M_{IJ}^L &= \bigoplus\nolimits_{L\geq 2}\left[(\varphi^{L-1})_{IJ}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}\right]\\ &= (\bigoplus\nolimits_{n\geq 1}\varphi^n)_{IJ}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}\\ &= (\varphi^+)_{IJ}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}.\end{aligned}$$

■

Proposition 15: Let p be a conservative polynomial defined as $p = \bigoplus_{i=1}^{i=N} w_i\delta^{t_i}$ with $\forall i \in \{1,..,N\}, w_i = \gamma^{n_i}\nabla_M\gamma^{n'_i}$. The Kleene star of p is a conservative and ultimately periodic series in $\mathcal{E}_{per}^*[\![\delta]\!]$ defined by

$$p^* = e \oplus \left(\bigoplus_{I=1}^{N}\bigoplus_{J=1}^{N}\left((\varphi^*)_{IJ}\gamma^{n_I}\nabla_M\gamma^{n'_J}\delta^{t_J}\right)\right). \quad (26)$$

Proof: Thanks to Lemma 9 and (25), then

$$\begin{aligned}p^* &= e \oplus \bigoplus\nolimits_{k=1}^{N}\gamma^{n_k}\nabla_M\gamma^{n'_k}\delta^{t_k}\\ &\quad\oplus\left(\bigoplus\nolimits_{i=1}^{i=N}\bigoplus\nolimits_{j=1}^{j=N}\left((\varphi^+)_{ij}\gamma^{n_i}\nabla_M\gamma^{n'_j}\delta^{t_j}\right)\right)\\ &= e \oplus\left(\bigoplus\nolimits_{i=1}^{i=N}\bigoplus\nolimits_{j=1}^{j=N}\left((\varphi^*)_{ij}\gamma^{n_i}\nabla_M\gamma^{n'_j}\delta^{t_j}\right)\right).\end{aligned}$$

The Kleene star of matrix φ, since it is a matrix of $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$, is known to be composed of periodic series of $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$ (thanks to Th. 8, rational series $\iff$ periodic series in $\mathcal{M}_{\text{in}}^{\text{ax}}[\![\gamma,\delta]\!]$). Then, for all i,j, series $(\varphi^*)_{ij}\gamma^{n_i}\nabla_M\gamma^{n'_j}\delta^{t_j}$ is ultimately periodic in $\mathcal{E}_{per}^*[\![\delta]\!]$. Therefore, the Kleene star p^* is obtained by summing N^2 ultimately periodic series of $\mathcal{E}_{per}^*[\![\delta]\!]$. Thanks to proposition 7, the result is periodic too. ■

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