

Stochastic filtering scheme of implicit forms of Uncertain Max-plus linear systems

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Abstract—This work aims to improve the stochastic filtering algorithm with bounded disturbances, proposed in [1]. This filter is efficient for max-plus linear (MPL) systems in explicit form, i.e., the Timed Event Graph (TEG) described by this system is initially with one token on each place. Nevertheless, it needs strong assumptions in order to be accurate for systems in implicit form, i.e., the corresponding TEG is initially with some token-free places which implies that some entries of the system state vector are dependent on each other. In this paper, we consider a more general method without these assumptions. It is based on an iterative procedure that widely increases the accuracy of the estimation.

Index Terms—Discrete event systems, Timed event graphs, Max-plus linear systems, Stochastic state estimation, Interval contractors

I. INTRODUCTION

Discrete Event Dynamic Systems (DEDS) are systems whose dynamics are event-driven, i.e., the state evolution depends entirely on the occurrence of asynchronous discrete events over time. Manufacturing systems, telecommunication networks, transportation networks, are example of DEDS [2]. To describe the behaviour of these systems the ordinary or partial differential equations are not suitable, hence more relevant theoretical setting are considered, among them the following can be cited: languages and automata, Markov chain and Petri nets, the reader is invited to consult [3] for an overview.

The DEDS involving only *delay and synchronization phenomena*, i.e., the starting of a task waits for a previous set of tasks to be completed, is worth of interest. These systems can be graphically depicted by Timed Event Graphs (TEGs) which are a subclass of timed Petri nets where each place has one and only one upstream transition and one and only one downstream transition.

The max-plus algebra setting, which is an idempotent semiring, is suitable to describe the behaviour of TEGs thanks to linear state equations which are very analogous to those found in classical linear system theory, i.e., the behaviour of a max-plus linear system (MPL) can be depicted thanks to matrices defined in this algebra.

These linear state equations are useful to deal with control problems addressed similarly to the classical control theory,

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among the problems solved we can cite the optimal control [4, 5], the model-predictive control [6, 7], the robust controller design [8, 9]. These models are also useful to deal with state estimation [10, 11, 12, 13] which is a fundamental problem to address applications such as fault detection and diagnosis [14] and state feedback control [15].

The observer approach, proposed by [10, 16], leads to an estimation of the state as close as possible, from below, to the real state. This approach is efficient for the deterministic systems and to synthesize observer-based controller [17]. In actual dynamical systems, sensor signals are to some extent corrupted with noise, and the state transition of the actual process is to some extent disturbed by exogenous events. If stochastic properties of these *noise sources* are available, state estimation may be performed more efficiently by considering this knowledge. Then, a more sophisticated and general framework for the state estimation is proposed in [1], it is based on the stochastic filtering design.

The Stochastic Max-Plus Linear (SMPL) systems are defined as MPL systems, where the entries of the system matrices are defined by their probability densities. In this work, we are interested in the Uncertain Max-Plus Linear (uMPL) systems, a subclass of SMPL systems, with entries that are bounded random variables with support in a real interval and are also assumed statistically independent of each other. Recently, [1] have proposed a stochastic filtering algorithm for these systems. The algorithm proposed by the authors is decomposed in two steps: a prediction phase based on the calculation of the mathematical expectation $\mathbb{E}[\text{observation}|\text{state}]$ and an update phase using available measurements and the prediction into account in order to compute a state estimate, this latest procedure is based on a constraint satisfaction problem inspired by [18].

This proposed filtering method assumes that the stochastic entries of the system matrices are independent. Unfortunately, in most applications, the entries of system matrices are a summation of processing times, in which various processing times appear in multiple entries. Therefore, the entries of the matrices are not independent and the procedure of [1] is not directly adapted. Indeed, it is based on an *offline* Monte Carlo simulation, which is robust only if the system works in its periodic behaviour. To overcome this problem, we propose a new *online* strategy that is efficient even if the system works in its transient behaviour. Hence, we do not need to perform the imprecise *offline* Monte Carlo simulation, since an *adaptive tuning* occurs in the loop.

This paper is organized as follows. In section II we present some background on MPL systems, interval analysis and

stochastic filtering for uMPL systems. Section III presents the stochastic filtering strategy of implicit forms of uMPL systems as an extension of Algorithm 3 in [1]. In Section IV, simulations are presented to illustrate the efficiency of the method proposed in this paper by considering the results from [13, 1].

II. MATHEMATICAL BACKGROUND

A. Algebraic framework

In this section, we recall some notions we shall use in the following [2, 19].

Define $\varepsilon = -\infty$, $e = 0$ and $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ endowed with two internal operations: *sum* (\oplus) and *product* (\otimes) are defined as $x \oplus y = \max\{x, y\}$ and $x \otimes y = x + y$ for any $x, y \in \mathbb{R}_{\max}$. Particularly, $\top = \bigoplus_{x \in \mathbb{R}_{\max}} x$ is the greatest element of \mathbb{R}_{\max} (\top is called top element of \mathbb{R}_{\max}). The \oplus and \otimes operations can be extended to matrices as follows. If $A, B \in \mathbb{R}_{\max}^{n \times p}$ and $C \in \mathbb{R}_{\max}^{p \times q}$, then $[A \oplus B]_{ij} = a_{ij} \oplus b_{ij}$ and $[A \otimes C]_{ij} = \bigoplus_{k=1}^p a_{ik} \otimes c_{kj}$. As in classical algebra, the operator \otimes will usually be omitted in expressions for the sake of readability.

The implicit equation $x = ax \oplus b$ admits $x = a^*b = (\bigoplus_{k \in \mathbb{N}} a^k)b$, where $a^k = a \otimes a^{k-1}$ and $a^0 = e$, as the smallest solution. All these results admit a natural extension to the matrix case (see [2] for more details concerning the Kleene star operator).

Interval Arithmetic

Interval arithmetic is presented in [20]. An interval in \mathbb{R}_{\max} is defined as $[x] = [\underline{x}, \bar{x}] = \{x \in \mathbb{R}_{\max} : \underline{x} \leq x \leq \bar{x}\}$.

The max-plus operations can be, therefore, extended to intervals as follows [21, 22, 23, 8]:

$$[x] \oplus [y] = \{x \oplus y : x \in [x], y \in [y]\} = [\underline{x} \oplus \underline{y}, \bar{x} \oplus \bar{y}] \quad (1)$$

$$[x] \otimes [y] = \{x \otimes y : x \in [x], y \in [y]\} = [\underline{x} \otimes \underline{y}, \bar{x} \otimes \bar{y}] \quad (2)$$

The \oplus and \otimes are extended to interval matrices as follows: If $[A]$, $[B]$ and $[C]$ are, respectively, $(n \times p)$, $(n \times p)$ and $(p \times q)$ -dimensional interval matrices, then $([A] \oplus [B])_{ij} = [a_{ij} \oplus b_{ij}]$ and $([A] \otimes [C])_{ij} = \bigoplus_{k=1}^p ([a_{ik}] \otimes [c_{kj}])$.

If we consider the max-plus equation $[z] = [C] \otimes x$, with, $[z]$ an q -dimensional interval vector, $[C]$ an $(q \times n)$ -dimensional interval matrix and x an n -dimensional vector, then we can write the i -th component of $[z]$ as follows:

$$[z]_i = \left[\bigoplus_{j=1}^n \underline{c}_{ij} \otimes x_j, \bigoplus_{j=1}^n \bar{c}_{ij} \otimes x_j \right], \text{ for all } i \in \{1, \dots, q\}. \quad (3)$$

B. MPL Systems

The TEGs class can be modelled by max-plus state equations, in which the state is modified exclusively by the occurrence of events (see for example [3, 2, 19]). To establish the equations describing the behaviour of a TEG with n internal transitions, p input transitions (transitions without upstream place), and q output transitions (transitions without downstream place), each internal transition is labelled \tilde{x}_i with $i \in \{1, \dots, n\}$, then $\tilde{x}_i(k)$ represents the date of the k -th firing of this transition, i.e., the occurrence date of the event labelled \tilde{x}_i . In the same way, each input transition is labelled \tilde{u}_i with $i \in \{1, \dots, p\}$ and each output transition is labelled \tilde{z}_i with

$i \in \{1, \dots, q\}$. This labelling leads to the following implicit max-plus linear equation:

$$\tilde{\mathbf{x}}(k) = \bigoplus_{l=0}^M \tilde{A}_l \tilde{\mathbf{x}}(k-l) \oplus \bigoplus_{l=0}^N \tilde{B}_l \tilde{\mathbf{u}}(k-l), \quad (4a)$$

$$\tilde{\mathbf{z}}(k) = \tilde{C} \tilde{\mathbf{x}}(k), \quad (4b)$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1(k), \dots, \tilde{x}_n(k))^T \in \mathbb{R}_{\max}^n$ is the state vector and each entry of matrices $\tilde{A}_l \in \mathbb{R}_{\max}^{n \times n}$ represents the minimal holding time in the corresponding places, i.e., \tilde{a}_l^{ij} represents the minimal holding time of the place linking the transition \tilde{x}_j to \tilde{x}_i and being initially with l tokens¹. The input (or control) vector $\tilde{\mathbf{u}} \in \mathbb{R}_{\max}^p$ represents the input transitions, and $\tilde{B}_l \in \mathbb{R}_{\max}^{n \times p}$ the influence of the input on the state variables, i.e., \tilde{b}_l^{ij} represents the minimal holding time of the place linking the transition \tilde{u}_j to \tilde{x}_i and being initially with l tokens. In the sequel, each place is assumed to be with initially one or zero token, this assumption is done without loss of generality since it is sufficient to add extra places in the corresponding TEG, i.e., to increase n in a suitable way. Eq.(4b) is the observation equation where $\tilde{\mathbf{z}} \in \mathbb{R}_{\max}^q$ represents the output transition (or measurement) and $\tilde{C} \in \mathbb{R}_{\max}^{q \times n}$ depicts the holding time of the corresponding places, which are assumed to be without tokens (extra places can be added).

Thus, the following equations within a standard and implicit form are considered:

$$\mathbf{x}(k) = A_0 \mathbf{x}(k) \oplus A_1 \mathbf{x}(k-1) \oplus B_0 \mathbf{u}(k), \quad (5a)$$

$$\mathbf{z}(k) = C \mathbf{x}(k). \quad (5b)$$

Furthermore, by removing the implicit form, the equation above can be written as:

$$\mathbf{x}(k) = A_0^* A_1 \mathbf{x}(k-1) \oplus A_0^* B_0 \mathbf{u}(k), \quad (6a)$$

$$\mathbf{z}(k) = C \mathbf{x}(k), \quad (6b)$$

where $A_0^* = \bigoplus_{k \in \mathbb{N}} A_0^k$ is with finite entries² if the corresponding graph of this matrix is without circuit, i.e., the corresponding TEG is live³. This assumption means that no transition is frozen, which is, in practice, a non-restrictive assumption.

C. Uncertain Max-Plus Linear Systems

The matrix entries of MPL systems depict mainly transportation or processing times, then in order to take uncertainties on these process duration into account, they have to be considered as random values. In this paper, it is assumed that at each event k the system matrices entries can take an arbitrary value within a real interval. Hence, it is possible to model Uncertain Max-Plus Linear (uMPL) systems [24, 25] as follows:

$$\mathbf{x}(k) = A_0(k) \mathbf{x}(k) \oplus A_1(k) \mathbf{x}(k-1) \oplus B_0(k) \mathbf{u}(k), \quad (7a)$$

$$\mathbf{z}(k) = C(k) \mathbf{x}(k), \quad (7b)$$

¹The number of tokens l in the initial marking is interpreted as backward shift operators in the event domain (formally, \tilde{x}_i with l tokens is shifted l -times, i.e., $\tilde{x}_i(k-l)$).

²The series $\bigoplus_{k \in \mathbb{N}} A^k = E \oplus A \oplus A^2 \oplus \dots$, where E is the identity matrix in max-plus algebra (i.e., a square matrix with the same dimension of A with e on the main diagonal and ε elsewhere), converges to a finite matrix A^* iff $\lambda(A) \leq e$ (its greatest eigenvalue). As a consequence $A^* = E \oplus A \oplus A^2 \oplus \dots \oplus A^{n-1}$ and $\lambda(A^*) = e$.

³A TEG is said to be live if each transition can be fired infinitely often.

where $A_0(k) \in [\underline{A}_0, \overline{A}_0]$, $A_1(k) \in [\underline{A}_1, \overline{A}_1]$, $B_0(k) \in [\underline{B}_0, \overline{B}_0]$, and $C(k) \in [\underline{C}, \overline{C}]$ are matrices of independent random variables with finite support and whose entries are mutually independent⁴. For instance, matrices \underline{A}_0 and \overline{A}_0 are the lower and upper bounds of $[A_0]$, respectively, with the inputs such that $a_0^{ij}(k) \in [\underline{a}_0^{ij}, \overline{a}_0^{ij}]$.

Remark 1: Thanks to the Kleene star matrix A_0^* , Eq.(5a) is equivalent to Eq.(6a) in the deterministic case. However, in the uncertain case it is not possible to apply the same transformation to Eq.(7a), indeed even if the inputs of $A_0(k)$, $A_1(k)$ and $B_0(k)$ are assumed independent, the entries of $A_0^*(k)A_1(k)$ and of $A_0^*(k)B_0(k)$ could be coupled since the entries of these resulting matrices are composed of sums and maximization, which results in multiple occurrences of the same holding times in multiple entries of the resulting matrices [2, 6]. This means that the assumption of independence of the entries does not hold any more.

D. On State Estimation of explicit forms

Bayesian methods provide a rigorous general framework for dynamic state estimation problems. Before providing the main contribution, we shall introduce the stochastic filtering background necessary to properly deal with the class of autonomous systems⁵ whose dynamics are abstractly and purely based on the system of discrete dynamic equations:

$$\begin{cases} \mathbf{x}(k) = f_k(\mathbf{x}(k-1)), \\ \mathbf{z}(k) = g_k(\mathbf{x}(k)), \end{cases} \quad (8)$$

with $\mathbf{x} \in \mathbb{R}^n$ the state vector, $\mathbf{z} \in \mathbb{R}^q$ the observation at time k , $f_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the nonlinear discrete state transition function between k and $k-1$ event-numbers and finally $g_k : \mathbb{R}^n \rightarrow \mathbb{R}^q$ the observation function.

Classical Bayesian formulation

The stochastic filtering scheme that will be carried out in this work is conceptually very similar to those found in discrete time dynamic systems [26] and can be briefly formulated as follows.

From a given set of measurements $\mathcal{Z}(k) = \{\mathbf{z}(1), \dots, \mathbf{z}(k)\}$ we will search for an estimate for the state vector since the sequence $\mathcal{X}(k) = \{\mathbf{x}(0), \dots, \mathbf{x}(k)\}$ is not directly measured. Classically, the estimate is the mathematical expectation of the state vector conditioned to the set of observations $\mathcal{Z}(k)$, i.e., is computed⁶ by $\hat{\mathbf{x}}(k) = \mathbb{E}[\mathbf{x}(k)|\mathcal{Z}(k)]$. As presented in [26, Section 6.6], this estimate can be computed in a recursive way. The probability density function (p.d.f.) $p(\mathbf{x}(k-1), \mathcal{Z}(k-1))$ is assumed to be known and

⁴This assumption of statistical independence between the system matrices inputs means that the minimum task duration or transportation time are independent of each other. This assumption is reasonable for practical problems, e.g., in the field of transport systems, a failure of one train does not affect the potential efficiency of the others, even if they are blocked due to precedence constraint.

⁵Any system described by Eq.(6a) can be transformed into an augmented autonomous model $\mathbf{x}(k) = \mathcal{M}\mathbf{x}'(k-1)$ by considering $\mathcal{M} = (AB)$ and $\mathbf{x}'(k-1) = (\mathbf{x}^T(k-1) \mathbf{u}^T(k))^T$ of appropriate dimensions, i.e., the control input is part of the augmented state vector \mathbf{x}' , and does not require further interest.

⁶The calculation per row of $\mathbb{E}[\mathbf{x}(k)|\mathcal{Z}(k)]$ is given by the first moment of a real-valued variable $x_i(k)$ and defined as $\mathbb{E}[x_i(k)|\mathcal{Z}(k)] = \int_{-\infty}^{+\infty} tp(t|\mathcal{Z}(k))dt$, where $p(x_i(k)|\mathcal{Z}(k))$ is the probability density function of $x_i(k)$ given $\mathcal{Z}(k)$.

$p(\mathbf{x}(k)|\mathcal{Z}(k))$ is obtained from it. As it will be clarified in the sequel, this calculation is two-fold. The first phase, known as *prediction*, is summarized by the following Chapman-Kolmogorov equation⁷:

$$\begin{aligned} p(\mathbf{x}(k)|\mathcal{Z}(k-1)) &= \\ &= \int p(\mathbf{x}(k)|\mathbf{x}(k-1))p(\mathbf{x}(k-1)|\mathcal{Z}(k-1))d\mathbf{x}(k-1). \end{aligned} \quad (9)$$

The latter phase is referred to as *correction* and uses the well known Bayes formula (with the fact of the use of the property of conditional independence of measurements) to finally calculate $p(\mathbf{x}(k)|\mathcal{Z}(k))$ from $p(\mathbf{x}(k)|\mathcal{Z}(k-1))$.

$$\begin{aligned} p(\mathbf{x}(k)|\mathcal{Z}(k)) &= p(\mathbf{x}(k)|\mathbf{z}(k), \mathcal{Z}(k-1)), \\ &= \frac{p(\mathbf{z}(k)|\mathbf{x}(k), \mathcal{Z}(k-1))p(\mathbf{x}(k)|\mathcal{Z}(k-1))}{p(\mathbf{z}(k)|\mathcal{Z}(k-1))}, \\ &= \frac{p(\mathbf{z}(k)|\mathbf{x}(k))p(\mathbf{x}(k)|\mathcal{Z}(k-1))}{\int p(\mathbf{z}(k)|\xi)p(\xi|\mathcal{Z}(k-1))d\xi}. \end{aligned} \quad (10)$$

It should be noted that the above equations depend essentially on $p(\mathbf{x}(k)|\mathbf{x}(k-1))$ and $p(\mathbf{z}(k)|\mathbf{x}(k))$ (measurement likelihood), which are related to the state transition function f_k , the observation function g_k of Eq.(8) and on the prior distribution $p(\mathbf{x}(k)|\mathcal{Z}(k-1))$ obtained from Eq.(9).

An analytical form of this calculation is very difficult because of the nonlinearity of the max-plus equations in the regular algebra, even numerically the computations are not trivial. Therefore, the computation of the integrals in Eq.(9) and in Eq.(10) is intractable in most cases.

Alternative Bayesian formulation

In order to overcome these non-feasible computations, an alternative filtering problem was proposed in [1, Section IV] and can be stated as follows. After each iteration k , we consider the unknown state trajectory $\mathcal{X}(k) = \{\mathbf{x}(0), \dots, \mathbf{x}(k)\}$ and then we define an estimate $\hat{\mathbf{x}}(k)$ of $\mathbf{x}(k)$ as

$$\hat{\mathbf{x}}(k) = \mathbb{E}[\mathbf{x}(k)|\hat{\mathbf{x}}(k-1)], \quad (11)$$

supposing that $\hat{\mathbf{x}}(0)$ is known. Then, from a given sequence of measurement $\mathcal{Z}^\sharp(k) = \{\mathbf{z}^\sharp(1), \dots, \mathbf{z}^\sharp(k)\}$, we will look for the state $\hat{\mathbf{x}}(k)$ such that it leads to

$$\mathbf{z}^\sharp(k) = \mathbb{E}[\mathbf{z}(k)|\hat{\mathbf{x}}(k)]. \quad (12)$$

A sequence $\hat{\mathcal{X}}(k) = \{\hat{\mathbf{x}}(0), \dots, \hat{\mathbf{x}}(k)\}$, ideally tracking $\mathcal{X}(k)$, is obtained if it satisfies the above equations. This sequence can be seen as the classical maximum likelihood estimator in the sense that it is based on the likelihood function $p(\mathbf{z}(k)|\mathbf{x}(k))$. The main difference is that instead of considering the maximum of this function, w.r.t. $\mathbf{x}(k)$, the estimate $\hat{\mathbf{x}}(k)$ (see Eq.(12)) chooses the value of $\mathbf{x}(k)$ such that $\mathbf{z}^\sharp(k)$ is equal to $\mathbb{E}[\mathbf{z}(k)|\hat{\mathbf{x}}(k)]$.

It is important to note that the strategy summarized by Eq.(11) and Eq.(12) effectively takes into account the prior data⁸, since $\hat{\mathbf{x}}(k)$ appears in both equations. Although conceptually useful, these equations are not adequate for the direct

⁷The joint distribution of $\mathbf{x}(k)$ and $\mathbf{x}(k-1)$ given $\mathcal{Z}(k-1)$ can be computed by using the Markov property as follows: $p(\mathbf{x}(k), \mathbf{x}(k-1)|\mathcal{Z}(k-1)) = p(\mathbf{x}(k)|\mathbf{x}(k-1), \mathcal{Z}(k-1))p(\mathbf{x}(k-1)|\mathcal{Z}(k-1)) = p(\mathbf{x}(k)|\mathbf{x}(k-1))p(\mathbf{x}(k-1)|\mathcal{Z}(k-1))$.

⁸According to [26], estimators based on $p(\mathbf{x}(k)|\mathbf{z}(k))$ clearly are Bayesian because they consider the probability density $p(\mathbf{x}(k)|\mathbf{x}(k-1))$ (see Eq.(9) and Eq.(10)). On the other hand, estimators purely based on $p(\mathbf{z}(k)|\mathbf{x}(k))$ are non-Bayesian.

implementation of a filter, because there is no unique solution $\hat{\mathbf{x}}(k)$ guaranteed due to the nature of the max-plus operations.

Alternatively, we can define $\mathbf{x}^\sharp(k)$ as a value that is constrained by $\mathbf{z}^\sharp(k) = \mathbb{E}[\mathbf{z}(k)|\mathbf{x}^\sharp(k)]$ (even though in classical filtering this is not mandatory) and is the closest to $\hat{\mathbf{x}}(k) = \mathbb{E}[\mathbf{x}(k)|\hat{\mathbf{x}}(k-1)]$ (predicted state). To retain this value, we use a suboptimal solver based on interval contraction that will be presented in the sequel.

In short, the approach is also two-fold and can be resumed by the following equations:

Prediction phase

$$\hat{\mathbf{x}}(k|k-1) = \mathbb{E}[\mathbf{x}(k)|\hat{\mathbf{x}}(k-1|k-1)]. \quad (13)$$

Measure-update phase

$$\hat{\mathbf{x}}(k|k) = \arg \min_{\mathbf{x}} \|\mathbf{x} - \hat{\mathbf{x}}(k|k-1)\|_\infty, \quad (14a)$$

$$\text{s.t. } \mathbf{z}^\sharp(k) = \mathbb{E}[\mathbf{z}(k)|\mathbf{x}]. \quad (14b)$$

Remark 2: The filter scheme proposed by Eq.(13) and Eq.(14) falls back on the same proposed by Eq.(11) and Eq.(12) iff $\hat{\mathbf{x}}(k|k-1) = \hat{\mathbf{x}}(k|k)$ holds.

In the sequel, we will introduce the mathematical tools necessary to properly compute Eq.(13) and Eq.(14).

Conditional expectation calculation

Consider the generic max-affine equation

$$z_i = \max_{j=1}^n \{m_{ij}(k) + x_j\}, \quad (15)$$

where $m_{ij}(k) \in [\underline{m}_{ij}, \overline{m}_{ij}]$, with $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, n\}$, and $\mathbf{x} \in \mathbb{R}^n$. In the following, $m_{ij}(k)$ will be considered as independent and uniformly distributed random variables. According to these assumptions the cumulative distribution function (c.d.f.) of $m_{ij}(k)$, denoted by $F_{m_{ij}}(t)$, is given by:

$$F_{m_{ij}}(t) = \begin{cases} 0 & \text{if } t \leq \underline{m}_{ij}, \\ \frac{t - \underline{m}_{ij}}{\overline{m}_{ij} - \underline{m}_{ij}} & \text{if } \underline{m}_{ij} < t \leq \overline{m}_{ij}. \\ 1 & \text{otherwise,} \end{cases} \quad (16)$$

Moreover, $\mathbb{E}[m_{ij}(k)] = \frac{\underline{m}_{ij} + \overline{m}_{ij}}{2}$ (for more details, see [1]).

Max-plus "summation" (or supremum) of independent random variables

Let $v = \max_{j=1}^n \{w_j\}$ be the result of max-plus summation, if w_j are independent random variables then $F_v(t) = P[v \leq t] = P[w_1 \leq t \text{ and } w_2 \leq t \text{ and } w_n \leq t] = \prod_{j=1}^n P[w_j \leq t] = \prod_{j=1}^n F_{w_j}(t)$. As a corollary, if each w_j is shifted by a constant b_j , i.e., $v = \max_{j=1}^n \{w_j + b_j\}$, then $F_v(t) = \prod_{j=1}^n F_{w_j}(t - b_j)$.

C.d.f. of z_i

Summing-up, by analogy, we can calculate the c.d.f. of each z_i , denoted by $\Pi_{z_i}(t)$, as

$$\Pi_{z_i}(t) = \prod_{j=1}^n F_{m_{ij}}(t - x_j), \text{ for all } i \in \{1, \dots, q\}. \quad (17)$$

with the term $F_{m_{ij}}$ related to Eq.(16). The function $\Pi_{z_i}(t)$ is a piece-wise polynomial function of order bounded by n and defined on $[\underline{z}_i = \max_{j=1}^n \{\underline{m}_{ij} + x_j\}, \overline{z}_i = \max_{j=1}^n \{\overline{m}_{ij} + x_j\}]$ according to Eq.(3). It can then be used to compute the conditional expectation as follows:

$$\mathbb{E}[z_i|\mathbf{x}] = \overline{z}_i - \int_{\underline{z}_i}^{\overline{z}_i} \Pi_{z_i}(t) dt, \text{ for all } i \in \{1, \dots, q\}. \quad (18)$$

Remark 3: Since the piece-wise polynomial c.d.f $F_{m_{ij}}(t)$ is given for the uniform law, Eq.(18) is a continuous (possibly piece-wise) and isotonic function of \mathbf{x} (see [1]). By considering other distribution laws (for instance triangular distribution) the computation can become complex (see Eq.(17)). Any other technique to calculate $\mathbb{E}[z_i|\mathbf{x}]$ can be alternatively considered as long as it keeps the properties of continuity and isotony w.r.t. \mathbf{x} . We can mention the results proposed in [27, 13], which are possible alternative methods. In [13], the authors consider an approximation based on the moments of a random variable and is focused on decreasing the computational burden of $\mathbb{E}[\max_{j=1}^n \{w_j\}]$. This method will be used in the numerical simulations for comparison purposes.

Inverse of $\mathbb{E}[\mathbf{z}|\mathbf{x}]$

Consider the matrix form of the same equation $\mathbf{z} = M(k)\mathbf{x}$ written in max-plus algebra that was considered in Eq.(15). This equation will be considered in the observation phase as follows. Consider now the problem: given an observation (or measurement) \mathbf{z}^\sharp , find \mathbf{x}^\sharp such that $\mathbf{z}^\sharp = \mathbb{E}[\mathbf{z}|\mathbf{x}^\sharp]$, i.e., which state \mathbf{x}^\sharp leads to the output \mathbf{z}^\sharp . More formally, we seek to characterize, the set $\chi = \{\mathbf{x}^\sharp \in \mathbb{R}^n \mid \mathbf{z}^\sharp = \mathbb{E}[\mathbf{z}|\mathbf{x}^\sharp]\}$.

Interval contractor

Contractors [18] are powerful tools to solve efficiently the problem of characterization of the set $\chi \subset \mathbb{R}^n$. The operator \mathcal{C}_χ is a contractor for χ if it satisfies $\forall [\mathbf{x}] \in \mathbb{R}^n$,

$$\begin{aligned} \mathcal{C}_\chi([\mathbf{x}]) &\subset [\mathbf{x}] && (\text{contractance}) \text{ and} \\ \mathcal{C}_\chi([\mathbf{x}]) \cap \chi &= [\mathbf{x}] \cap \chi && (\text{completeness}). \end{aligned}$$

A contractor is said to be minimal if $[\mathbf{x}] \cap \chi = \mathcal{C}_\chi([\mathbf{x}])$. In the following, it is assumed that: H1) χ is not empty, H2) $\chi \subset [\mathbf{x}]$, H3) $\mathbb{E}[\mathbf{z}|\underline{\mathbf{x}}] \leq \mathbf{z}^\sharp$, H4) for all $j \in \{1, \dots, n\}$, $\mathbb{E}[\mathbf{z}|\underline{x}_1, \underline{x}_2, \dots, \underline{x}_j, \dots, \underline{x}_{n-1}, \underline{x}_n]^T \geq \mathbf{z}^\sharp$. The Algorithm 1 summarizes the contractor Ω_χ to contract the set χ .

Algorithm 1: Interval contractor

Data: $F(M(k) \in [M])$ (c.d.f. of syst. matrix), $[\mathbf{x}], \mathbf{z}^\sharp$

Result: $\mathcal{I} = \Omega_\chi([\mathbf{x}])$

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1 while  $\underline{\mathbf{x}}' \neq \underline{\mathbf{x}}$  and  $\overline{\mathbf{x}}' \neq \overline{\mathbf{x}}$  do
2    $\underline{\mathbf{x}}' \leftarrow \underline{\mathbf{x}}; \overline{\mathbf{x}}' \leftarrow \overline{\mathbf{x}};$ 
3   foreach  $i \in \{1, \dots, q\}$  do
4     foreach  $j \in \{1, \dots, n\}$  do
5       /* Lower dichotomy */
6        $\mathbf{x} \leftarrow \overline{\mathbf{x}}; x_j \leftarrow \underline{x}_j;$ 
7       if  $\mathbb{E}[z_i|\mathbf{x}] < z_i^\sharp$  then
8          $\underline{\mathbf{x}} \leftarrow \Delta_{ij}^L([\underline{\mathbf{x}}, \overline{\mathbf{x}}]);$ 
9       end
10      /* Upper dichotomy */
11       $\mathbf{x} \leftarrow \underline{\mathbf{x}}; x_j \leftarrow \overline{x}_j;$ 
12      if  $\mathbb{E}[z_i|\mathbf{x}] > z_i^\sharp$  then
13         $\overline{\mathbf{x}} \leftarrow \Delta_{ij}^U([\underline{\mathbf{x}}, \overline{\mathbf{x}}]);$ 
14      end
15    end
16  end
17 return  $[\underline{\mathbf{x}}, \overline{\mathbf{x}}];$ 

```

The computation of Δ_{ij}^L and Δ_{ij}^U is a one-dimensional search that can be efficiently performed by the dichotomy method. Algorithm 2 summarizes this search for each

Algorithm 2: One-dimensional search

Data: $F(M(k) \in [M])$ (c.d.f. of syst. matrix), $[\mathbf{x}]$, (i, j) , z_i^\sharp , type L or U, tol (tolerance)
Result: $\mathbf{w} = \Delta_{ij}^L([\mathbf{x}])$ or $\Delta_{ij}^U([\mathbf{x}])$

```

1  $(\mathbf{y}, \mathbf{w}) \leftarrow \begin{cases} (\bar{\mathbf{x}}, \underline{\mathbf{x}}) & \text{if L;} \\ (\underline{\mathbf{x}}, \bar{\mathbf{x}}) & \text{if U;} \end{cases}$ 
2 while  $|\bar{x}_j - \underline{x}_j| > tol$  do
3    $y_j \leftarrow (\underline{x}_j + \bar{x}_j)/2$ ;
4   if  $\mathbb{E}[z_i | \mathbf{y}] > z_i^\sharp$  then
5      $\bar{x}_j \leftarrow y_j$ ;
6   else
7      $\underline{x}_j \leftarrow y_j$ ;
8   end
9 end
10  $w_j \leftarrow y_j$ ;
11 return  $\mathbf{w}$ ;
```

row/column pair (i, j) . We denote Ω_χ the contractor obtained from the iterated composition of $2qn$ operators Δ , i.e., $\Omega_\chi([\mathbf{x}]) = (\Delta_{11}^L \circ \Delta_{11}^U \circ \dots \circ \Delta_{1n}^L \circ \Delta_{1n}^U \circ \dots \circ \Delta_{q1}^L \circ \Delta_{q1}^U \circ \dots \circ \Delta_{qn}^L \circ \Delta_{qn}^U)([\mathbf{x}])$. Lines 6 and 10 of Algorithm 1 and line 4 of Algorithm 2 refer to Eq.(18) and if they are not satisfied, then the lower-dichotomy is mathematically seen as the identity operator $\Delta_{ij}^L([\mathbf{x}]) = \text{Id}^L([\mathbf{x}]) = \underline{\mathbf{x}}$ and the upper-dichotomy is seen as the identity operator $\Delta_{ij}^U([\mathbf{x}]) = \text{Id}^U([\mathbf{x}]) = \bar{\mathbf{x}}$.

Remark 4: The contractor Ω_χ satisfies the contractance, completeness and monotonic properties (see [1, Lemma 2] for proofs).

Remark 5: $\Omega_\chi([\mathbf{x}])$ converges to a fixed point, i.e., to an interval \mathcal{I} such that $\Omega_\chi(\mathcal{I}) = \mathcal{I}$. Moreover, \mathcal{I} contains χ (see [1, Lemma 3] for proofs).

Deprecation of \mathcal{I} , the suboptimal solver Inv :

In the sequel, the second and last part of the inversion of $\mathbb{E}[z|\mathbf{x}^\sharp]$ is drawn. Generally, the interval $\mathcal{I} = [\underline{\mathbf{x}}^{opt}, \bar{\mathbf{x}}^{opt}]$ is not deprecated⁹ after the contraction procedure of the initial interval $[\mathbf{x}]$. Consider initially the deprecation procedure described below [1, Section IV].

Procedure 1: For an arbitrary $j \in \{1, \dots, n\}$, let $a \in [\underline{x}_j, \bar{x}_j]$. Moreover, let $\chi' = \chi \cap \{x_j = a\}$, with $\{x_j = a\}$ a hyperplane in \mathbb{R}^n , and let $\mathcal{I}' = \mathcal{I} \cap \{x_j = a\}$. Thus, $\chi' \subset \mathcal{I}'$ and, thanks to [1, Lemma 3], χ' is not empty. In general, \mathcal{I}' is not minimal (Remark 5) and the contraction algorithm must be run again to obtain the minimal interval containing χ' .

The procedure above can be iteratively repeated until the minimal interval is reduced to one point (all components are fixed) that necessarily belongs to χ . The remaining question is: which component should be fixed at each step and to which value? To answer this, consider now that one holds a *guess value* (obtained somehow) for a particular \mathbf{x}^\sharp , formally \mathbf{x}^0 . We shall look for a point $\mathbf{x} \in \chi$ that is the closest to this value, i.e., $\mathbf{x}^\sharp = \arg \min_{\mathbf{x} \in \chi} \|\mathbf{x} - \mathbf{x}^0\|_\infty$ (see Eq.(14a)). In general, \mathbf{x}^\sharp is not unique, i.e., multiple solutions yield the same minimum, and an optimal value for this problem cannot be guaranteed. However, following [1, Section IV], a suboptimal heuristic procedure, based on the deprecation method described above, is proposed to solve an alternative optimization problem, stated as follows. As we already know, \mathcal{I} , such that $\chi \subset \mathcal{I}$, is the interval resulting from the Algorithm 1. Then, we consider the

⁹An interval $[x] = [\underline{x}, \bar{x}]$ is said to be deprecated if $\underline{x} = \bar{x}$. The same is applied to interval vectors and matrices.

alternative minimization $\mathbf{x}' = \arg \min_{\mathbf{x} \in \mathcal{I}} \|\mathbf{x} - \mathbf{x}^0\|_\infty$, whose optimal solution¹⁰ is given by the line 7 of Algorithm 3.

Remark 6: The vector \mathbf{x}' not necessarily belongs to χ , but it is useful to determine at each step which component must be deprecated and to which value. Given that, $\chi \subset \mathcal{I}$ then the following statement holds $\min_{\mathbf{x} \in \chi} \|\mathbf{x} - \mathbf{x}^0\|_\infty \geq \min_{\mathbf{x} \in \mathcal{I}} \|\mathbf{x} - \mathbf{x}^0\|_\infty$. The Algorithm 3 summarizes this procedure, and it should be noticed the generation of the initial interval $[\mathbf{x}]$ must satisfy H2, i.e., it must contain at least one solution of the problem characterized by the set χ . A simple rule to guarantee this is to chose $\underline{\mathbf{x}}$ such that $\overline{C}\underline{\mathbf{x}} < \mathbf{z}^\sharp$ and $\bar{\mathbf{x}}$ such that $\underline{C}\bar{\mathbf{x}} > \mathbf{z}^\sharp$ are respected¹¹.

Algorithm 3: Suboptimal solver

Data: $F(M(k) \in [M])$ (c.d.f. of syst. matrix), \mathbf{z}^\sharp , \mathbf{x}^0
Result: $\mathbf{x}^\sharp = \text{Inv}(\mathbf{z}^\sharp, \mathbf{x}^0)$

```

1 generate  $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ ;  $bool \leftarrow true$ ;
2 while  $bool$  do
3    $[\underline{\mathbf{x}}, \bar{\mathbf{x}}] \leftarrow \Omega_\chi([\underline{\mathbf{x}}, \bar{\mathbf{x}}])$ ; // Algorithm 1
4    $bool \leftarrow \underline{\mathbf{x}} \neq \bar{\mathbf{x}}$ ;
5   if  $bool$  then
6     /* Heuristic fixation of the  $j$ -th
7     coordinate of  $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$  based on  $\mathbf{x}^0$ . */
8     foreach  $j \in \{1, \dots, n\}$  do
9        $x'_j \leftarrow \begin{cases} \underline{x}_j & \text{if } x_j^0 \leq \underline{x}_j \\ x_j^0 & \text{if } \underline{x}_j < x_j^0 < \bar{x}_j \\ \bar{x}_j & \text{otherwise.} \end{cases}$ ;
10      //  $\mathbf{x}' = \arg \min_{\mathbf{x} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]} \|\mathbf{x} - \mathbf{x}^0\|_\infty$ 
11     end
12      $j^\sharp \leftarrow \arg \max_{j \in \{1, \dots, n\}} |x'_j - x_j^0|$ ; // Deprecation
13      $\underline{x}_{j^\sharp} \leftarrow x'_{j^\sharp}$ ;  $\bar{x}_{j^\sharp} \leftarrow x'_{j^\sharp}$ ;
14   end
15 end  $bool$ ; // Notice that:  $\underline{\mathbf{x}} = \bar{\mathbf{x}}$ 
```

Stochastic Filtering algorithm

Consider the uMPL system below:

$$\begin{aligned} \mathbf{x}(k) &= A(k)\mathbf{x}(k-1), \\ \mathbf{z}(k) &= C(k)\mathbf{x}(k), \end{aligned} \quad (19)$$

with an n -dimensional vector \mathbf{x} , an q -dimensional vector \mathbf{z} . An $(n \times n)$ -dimensional matrix $A(k) \in [A]$ and an $(q \times n)$ -dimensional matrix $C(k) \in [C]$ such that its entries (for instance, $a_{ij}(k) \in [\underline{a}_{ij}, \bar{a}_{ij}]$) are independent random variables and uniformly distributed according to the c.d.f.: $F(A(k) \in [A])$ and $F(C(k) \in [C])$.

Summing-up, the filtering algorithm, is given by Algorithm 4.

Algorithm 4: Filtering algorithm of explicit forms

Data: $F(A)$, $F(C)$, and $\mathbf{z}^\sharp(k)$
Result: $\hat{\mathbf{x}}(k|k) = \text{Filter}(\mathbf{z}^\sharp(k), \hat{\mathbf{x}}(k-1|k-1))$

```

1  $\hat{\mathbf{x}}(k|k-1) = \mathbb{E}[\mathbf{x}(k) | \hat{\mathbf{x}}(k-1|k-1)]$ ; // Eq.(18)
2  $\hat{\mathbf{x}}(k|k) \leftarrow \text{Inv}(\mathbf{z}^\sharp(k), \hat{\mathbf{x}}(k|k-1))$ ; // Algorithm 3
3 return  $\hat{\mathbf{x}}(k|k)$ 
```

¹⁰As for \mathbf{x}^\sharp , there exist multiple solutions for \mathbf{x}' , but $\min_{\mathbf{x} \in \mathcal{I}} \|\mathbf{x} - \mathbf{x}^0\|_\infty$ is unique.

¹¹H4 implies $\mathbb{E}[z|\bar{\mathbf{x}}] \geq \mathbf{z}^\sharp$ and it must always be respected, otherwise $\bar{\mathbf{x}}$ must be properly modified.

III. ON STATE ESTIMATION OF IMPLICIT FORMS

A. Towards the implementation of a filtering strategy

According to Remark 1, Eq.(7a) can not be directly written in explicit form, however, in practice, the associated TEG is assumed to be live (otherwise some transitions would be frozen), this implies that there is an appropriate permutation of the transitions numbering such that A_0 can be written in strictly lower triangular form ($a_0^{ij} = \varepsilon$ for all $i \leq j$). Thus,

$$\mathbf{x}(k) = A_0(k)\mathbf{x}(k) \oplus \mathbf{y}(k),$$

where $A_0(k)\mathbf{x}(k)$ represents the implicit part and $\mathbf{y}(k) = A_1(k)\mathbf{x}(k-1) \oplus B_0(k)\mathbf{u}(k)$ represents the explicit part. This equation can be component-wise depicted as follows:

$$\begin{aligned} x_1(k) &= y_1(k), \\ x_2(k) &= a_0^{21}(k)x_1(k) \oplus y_2(k), \\ x_3(k) &= a_0^{31}(k)x_1(k) \oplus a_0^{32}(k)x_2(k) \oplus y_3(k), \\ &\vdots \\ x_n(k) &= \left(\bigoplus_{l=1}^{n-1} a_0^{nl}(k)x_l(k)\right) \oplus y_n(k), \end{aligned} \quad (20)$$

where each y_i is either $\max_{j=1}^n \{a_1^{ij}(k) + x_j(k-1)\}$ or $\max_{j=1}^p \{b_0^{ij}(k) + u_j(k)\}$, where $a_1^{ij}(k)$ and $b_0^{ij}(k)$ are assumed to be independent random variables, and $x_j(k-1)$ and $u_j(k)$ are fixed. Hence, the entries of the vector $\mathbf{y}(k)$ are assumed to be independent. Moreover, the first part of the right-hand side of Eq.(20) is $\max_{l=1}^{i-1} \{a_0^{il}(k) + x_l(k)\}$ for all $i \in \{1, \dots, n\}$ and depends on $x_l(k)$ from $l = 1$ up to $i-1$. By assuming that the computation of Eq.(20) is done from $i = 1$ up to n and the terms $x_l(k)$ are fixed and known, we will be able to ensure the independence of the components of the vector $\mathbf{x}(k)$, which is worth of interest in the main contribution of this work¹².

B. Filtering algorithm

The stochastic systems described by Eq.(20) are implicitly component-wise dependent. Hence, to properly calculate the expectation of the i -th component of \mathbf{x} it would be necessary to take into account its joint distribution w.r.t. the other components, which seems to be an intractable problem for most of the cases. From this fact, the stochastic filter scheme that was proposed in the previous section is no longer allowed to be straightforwardly used if we are not interested in conservative results¹³.

In the sequel, we will introduce an alternative way to properly design the component-wise filtering scheme.

Stochastic Filtering algorithm of implicit forms

Consider the uMPL system below:

$$\begin{aligned} \mathbf{x}(k) &= A_0(k)\mathbf{x}(k) \oplus A_1(k)\mathbf{x}(k-1), \\ \mathbf{z}(k) &= C(k)\mathbf{x}(k), \end{aligned} \quad (21)$$

with an n -dimensional vector \mathbf{x} , an q -dimensional vector \mathbf{z} . The $(n \times n)$ -dimensional matrices $A_0(k) \in [A_0]$ and $A_1(k) \in [A_1]$, and an $(q \times n)$ -dimensional matrix $C(k) \in [C]$ such that

¹²We consider that each $x_i(k)$ is recursively known at each subsequent row.

¹³From Remark 1, we can consider a conservative result by taking into account the augmented autonomous and explicit form of Eq.(7a), which is given by: $\mathbf{x}(k) = \mathcal{H}(k)\mathbf{r}(k)$, where $\mathcal{H}(k) \in ([A_0] [A_1] [B_0])$ and $\mathbf{r}(k) = (\mathbf{x}^T(k) \mathbf{x}^T(k-1) \mathbf{u}^T(k))^T$.

their entries (for instance, $a_0^{ij}(k) \in [\underline{a}_0^{ij}, \bar{a}_0^{ij}]$) are independent random variables and uniformly distributed according to the c.d.f.: $F(A_0(k) \in [A_0])$, $F(A_1(k) \in [A_1])$ and $F(C(k) \in [C])$.

Following Eq.(20), A_0 can be given in lower triangular form, and hence we can write

$$x_i(k) = \left(\bigoplus_{j=1}^{i-1} a_0^{ij}(k)x_j(k) \right) \oplus y_i(\mathbf{x}(k-1)), \quad (22)$$

with $y_i(\mathbf{x}(k-1))$ the i -th entry of $\mathbf{y}(\mathbf{x}(k-1)) = A_1(k)\mathbf{x}(k-1)$ (assuming the components of $\mathbf{x}(k-1)$ are obtained from the previous iteration).

By considering Eq.(18) the prediction equation for all $i \in \{1, \dots, n\}$ is given as follows:

$$\hat{x}_i(k|k-1) = \mathbb{E} \left[\left(\bigoplus_{j=1}^{i-1} a_0^{ij}(k)\hat{x}_j(k|k) \right) \oplus y_i(\mathbf{x}(k-1)) \right]. \quad (23)$$

In order to properly obtain $\hat{\mathbf{x}}(k|k)$, one must call n -times the procedure Inv. Clearly, this does not refer to a classical two-fold filter scheme because we update separately the prediction for each i -th component of the state vector.

Summing-up, the filtering algorithm of implicit forms, is given by Algorithm 5.

Algorithm 5: Filtering algorithm of implicit forms

Data: $F(A_0)$, $F(A_1)$, $F(C)$, and $\mathbf{z}^\#(k)$
Result: $\hat{\mathbf{x}}(k|k) = \text{Filter}(\mathbf{z}^\#(k), \hat{\mathbf{x}}(k-1|k-1))$
1 $\mathbf{y} = A_1(k)\hat{\mathbf{x}}(k-1|k-1)$;
2 $\hat{\mathbf{x}}(k|k-1) \leftarrow (\varepsilon, \dots, \varepsilon)^T$; // initialize
3 **foreach** $i \in \{1, \dots, n\}$ **do**
4 $\hat{x}_i(k|k-1) = \mathbb{E}[(\bigoplus_{j=1}^{i-1} a_0^{ij}(k)\hat{x}_j(k|k)) \oplus y_i]$; // Eq.(23)
5 $\hat{\mathbf{x}}(k|k) \leftarrow \text{Inv}(\mathbf{z}^\#(k), \hat{\mathbf{x}}(k|k-1))$; // Algorithm 3
6 **end**
7 **return** $\hat{\mathbf{x}}(k|k)$;

The Algorithm 5 uses n -times the Algorithm 3. This Algorithm 3 returns an estimate $\hat{\mathbf{x}}(k|k)$ for all components of the state vector \mathbf{x} , which is the solution of the constrained minimization problem (see Eq.(14a) and Eq.(14b)). The input $\hat{\mathbf{x}}(k|k-1)$ of the Algorithm 3 is updated at each step i , with $\hat{x}_i(k|k-1)$ which is the corresponding entry of the prediction vector. At this step i , we must note that entries $j \in \{i+1, \dots, n\}$ of $\hat{\mathbf{x}}(k|k-1)$ are still equal to ε , hence these entries j of the estimate $\hat{\mathbf{x}}(k|k)$ given by Algorithm 3 are equal to \underline{x}_j since $x_j^0 \leq \underline{x}_j$ (see line 7).

IV. NUMERICAL SIMULATION

Example: Consider the third-order non-autonomous uMPL system governed by the equation below:

$$\mathbf{x}(k) = A_0(k)\mathbf{x}(k) \oplus A_1(k)\mathbf{x}(k-1) \oplus B\mathbf{u}(k), \quad (24)$$

with $A_0(k) \in \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ [1, 2] & \varepsilon & \varepsilon \\ [6, 10] & [7, 11] & \varepsilon \end{pmatrix}$, $A_1(k) \in \begin{pmatrix} [7, 11] & \varepsilon & [2, 9] \\ \varepsilon & [6, 12] & [4, 8] \\ \varepsilon & \varepsilon & [6, 8] \end{pmatrix}$ and $B = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}$.

The output measurement is defined by

$$\mathbf{z}(k) = C(k)\mathbf{x}(k), \quad (25)$$

with $C(k) \in ([1, 3] [2, 4] \varepsilon)$.

For instance, the prior knowledge of state $\hat{\mathbf{x}}(k-1|k-1) \equiv \mathbf{x}(k-1)$ is initialized with the initial state $\mathbf{x}(0) = (2 \ 5 \ 1)^T$.

Also, it should be mentioned that every calculation of $\hat{x}_i(k|k-1) = \mathbb{E}[x_i(k)|(\mathbf{x}(k-1)^T \mathbf{u}(k)^T)^T]$ takes into account a pseudo-random value $\mathbf{u}(k)$ drawn from the standard uniform distribution on the closed interval vector between $(2, 2)^T$ and $(40, 40)^T$.

A. Filter scenarios

In this section, we compare three different filters strategies for non-autonomous systems in implicit form.

Filter F_1 uses the original filtering method given by Algorithm 5. In order to take the input action into account, the line 1 is replaced by $\mathbf{y} = A_1(k)\hat{\mathbf{x}}(k-1|k-1) \oplus B\mathbf{u}(k)$.

Filter F_2 considers Algorithm 4 with line 1 given by:

$$\hat{x}_i(k|k-1) = \mathbb{E}[x_i(k)|(\hat{\mathbf{x}}^T(k-1|k-1) \mathbf{u}^T(k))^T],$$

with the right-hand side calculated by $\mathbb{E}[\bigoplus_{j=1}^{i-1} a_0^{ij}(k)(\alpha_j \cdot \hat{x}_j(k|k-1)) \oplus \bigoplus_{j=1}^n a_1^{ij}(k)\hat{x}_j(k-1|k-1) \oplus \bigoplus_{j=1}^p b^{ij}u_j(k)]$, for all $i \in \{1, \dots, n\}$ and with \cdot corresponding to the scalar multiplication in regular algebra. It corresponds to the strategy given in [1, Section III] for systems with periodic behaviour¹⁴ with $\alpha_j = 1$ for all $i \in \{1, \dots, n\}$.

Filter F_3 considers an alternative method to deal with the calculation of the expectation, as given in [13]. The authors consider an approximation method for $\mathbb{E}[\max_{i=1}^n \{w_i\}]$, with w_i independent and uniformly distributed random variables such that $w_i \in [\underline{w}_i, \bar{w}_i]$. It is based on lower and upper bounds: $\text{lower}(\mathbb{E}[\max_{i=1}^n \{w_i\}]) \leq \mathbb{E}[\max_{i=1}^n \{w_i\}] \leq \text{upper}(\mathbb{E}[\max_{i=1}^n \{w_i\}])$.

The lower and upper bounds [13, Section 3] of the expectation of max-affine functions are given respectively by

$$\text{lower} \left(\mathbb{E} \left[\max_{i=1}^n \{w_i\} \right] \right) = \max_{i=1}^n \{ \mathbb{E} [w_i] \},$$

$$\text{upper} \left(\mathbb{E} \left[\max_{i=1}^n \{w_i\} \right] \right) = \left(\sum_{i=1}^n \mathbb{E} [|w_i - L|^p] \right)^{\frac{1}{p}} + L.$$

with $L = \min_{i=1}^n \{w_i\}$ (with a careful handling of infinite values).

The upper-approximation can be extended to uMPL systems and will replace all calculations of $\mathbb{E}[\mathbf{z}|\mathbf{x}]$ in the procedure Inv of filter F_1 . This method (with $p = 10$)¹⁵ will be addressed as F_3 in the comparison Table I.

B. Simulation results

Table I shows the obtained results for simulations up to the occurrence of 4000 events. Each position of the table corresponds to root-mean-square-error¹⁶ $\text{RMSE}(x_i(k), \hat{x}_i(k|k))$ with the usage of the corresponding filter.

¹⁴The set of parameters $\{\alpha_1, \dots, \alpha_{n-1}\}$ is computed only once (offline phase) using Monte Carlo techniques. This set is robust under periodic steady-state regime, but if exogenous inputs modify the system behaviour, the parameters should be tuned online. For more details, the reader is invited to refer to [1, Section III].

¹⁵The definition of p -value is properly discussed in [13]. Here the p -value is experimentally tuned by increasing it while the result is improved, indeed for a larger p -value the result deviates from the exact solution.

¹⁶Notation: $\text{RMSE}(\mathbf{a}, \mathbf{b}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\mathbf{a}_i - \mathbf{b}_i)^2}$.

State i	Filter F_1	Filter F_2	Filter F_3
1	2.8883	4.5451	2.9020
2	1.1299	1.7783	1.1418
3	1.7233	16.7846	1.7410

TABLE I
ESTIMATION COMPARISON OF EXAMPLE.

The analysis of Table I indicates that the RMSE between the estimation provided by the F_1 and the true value of the state is almost equal to the RMSE between the estimation provided by the F_3 and the true state. This fact is not surprising since we only changed the way to calculate the mathematical conditional expectation and the interval contraction scheme remains unchanged. However, the RMSE between the estimation provided by the F_2 method and the true value of the state is higher than the two other methods with special attention to $i = 3$ (unobserved state). Indeed, as $x_3(k)$ is not observed and is the noisiest state, then this noise propagates throughout the TEG thanks to $A_1(k)$ and it penalizes the estimate of $x_1(k)$ and $x_2(k)$.

V. CONCLUSION

In this work, we fully explored the capacity of the stochastic filtering algorithm developed in [1] for systems within implicit forms. The optimization is performed such that the state estimated respects the measurement but, in a future work, it would be interesting to consider a criterion ensuring a trade-off between the noise in prediction versus the noise in the measurement as it is performed in the Kalman filtering approach. Another interesting perspective would be to mix the tools developed in [1] and the approximation method of [13, Theorem 8] to circumvent the difficulty of the computation of the joint probabilities involved in the Kleene star operation.

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REFERENCES

- [1] R. S. Mendes, L. Hardouin, and M. Lhommeau, "Stochastic filtering of max-plus linear systems with bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3706–3715, Sep. 2019.
- [2] F. Baccelli, G. Cohen, G. Olsder, and J. Quadrat, *Synchronization and Linearity : An Algebra for Discrete Event Systems*. Wiley and Sons, 1992.
- [3] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*. Kluwer Academic Publishers, 1999.
- [4] G. Cohen, S. Gaubert, and J.-P. Quadrat, "Max-plus algebra and system theory: Where we are and where to go now," *Annual Reviews in Control*, vol. 23, pp. 207 – 219, 1999.
- [5] C. A. Maia, L. Hardouin, R. S. Mendes, and B. Cottenceau, "Optimal Closed-Loop Control for Timed Event

- Graphs in Dioid,” *IEEE-TAC*, vol. 48, no. 12, pp. 2284–2287, 2003.
- [6] T. van den Boom, J. Xu, and B. De Schutter, “Corrections to “Model predictive control for stochastic max-plus linear systems with chance constraints”, [IEEE Trans. on Aut. Control, 64(1): 337–342, 2019],” *IEEE Transactions on Automatic Control*, vol. 65, no. 2, pp. 905–906, Feb. 2020.
- [7] I. Necoara, B. De Schutter, T. van den Boom, and H. Hellendoorn, “Stable model predictive control for constrained max-plus-linear systems,” *Discrete Event Dynamic Systems: Theory and Applications*, vol. 17, no. 3, pp. 329–354, sep 2007.
- [8] M. Lhommeau, L. Hardouin, J.-L. Ferrier, and I. Ouerghi, “Interval analysis in dioid: Application to robust open-loop control for timed event graphs,” in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, 2005, pp. 7744–7749.
- [9] I. Necoara, B. De Schutter, T. van den Boom, and H. Hellendoorn, “Robust control of constrained max-plus-linear systems,” *Int. J. of Robust and Nonlinear Control*, vol. 19, no. 2, pp. 218–242, Jan. 2009.
- [10] L. Hardouin, C. A. Maia, B. Cottenceau, and M. Lhommeau, “Observer design for (max,+) linear systems,” *IEEE Trans. on Automatic Control*, vol. 55 - 2, pp. 538 – 543, 2010.
- [11] M. D. Loreto, S. Gaubert, R. D. Katz, and J. Loiseau, “Duality between invariant spaces for max-plus linear discrete event systems,” *SIAM J. on Control and Optimization*, 2010.
- [12] R. M. F. Cândido, R. Santos-Mendes, L. Hardouin, and C. Maia, “Particle filter for max-plus systems,” *European Control Conference, ECC 2013*, 2013.
- [13] S. Farahani, T. van den Boom, and B. De Schutter, “On optimization of stochastic max-min-plus-scaling systems – An approximation approach,” *Automatica*, vol. 83, pp. 20–27, Sept. 2017.
- [14] C. Paya, E. Le Corrond, Y. Pencolé, and P. Vialletelle, “Observer-based detection of time shift failures in (max,+)-linear systems,” in *The 31st International Workshop on Principles of Diagnosis (DX-2020)*, Nashville, United States, Sept. 2020. [Online]. Available: <https://hal.laas.fr/hal-03023250>
- [15] L. Hardouin, B. Cottenceau, Y. Shang, and J. Raisch, *Control and State Estimation for Max-Plus Linear Systems*. Now Foundations and Trends, 2018.
- [16] V. M. Gonçalves, C. A. Maia, and L. Hardouin, “On max-plus linear dynamical system theory: The observation problem,” *Automatica*, vol. 107, pp. 103–111, 2019. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0005109819302420>
- [17] L. Hardouin, Y. Shang, C. A. Maia, and B. Cottenceau, “Observer-based controllers for max-plus linear systems,” *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2153–2165, 2017.
- [18] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis*. London: Springer-Verlag, 2001.
- [19] B. Heidergott, G. Olsder, and J. van der Woude, *Max Plus at Work: Modeling and Analysis of Synchronized Systems : a Course on Max-Plus Algebra and Its Applications*, ser. Max Plus at work: modeling and analysis of synchronized systems : a course on Max-Plus algebra and its applications. Princeton University Press, 2006, no. v. 13.
- [20] R. E. Moore and F. Bierbaum, *Methods and Applications of Interval Analysis (SIAM Studies in Applied and Numerical Mathematics) (Siam Studies in Applied Mathematics, 2.)*. Soc for Industrial & Applied Math, 1979.
- [21] T. Brunsch, J. Raisch, and L. Hardouin, “Modeling and control of high-throughput screening systems,” *Control Engineering Practice*, vol. 20:1, pp. 14–23, January 2012, doi:10.1016/j.conengprac.2010.12.006.
- [22] L. Hardouin, B. Cottenceau, M. Lhommeau, and E. Le Corrond, “Interval systems over idempotent semiring,” *Linear Algebra and its Applications*, vol. 431, no. 5-7, pp. 855–862, Aug. 2009, doi:10.1016/j.LAA.2009.03.039.
- [23] G. L. Litvinov and A. N. Sobolevskii, “Idempotent interval analysis and optimization problems,” *Reliable Computing*, vol. 7, no. 5, pp. 353–377, 2001.
- [24] R. M. F. Cândido, L. Hardouin, M. Lhommeau, and R. Santos Mendes, “Conditional reachability of uncertain max plus linear systems,” *Automatica*, vol. 94, pp. 426 – 435, 2018. [Online]. Available: <http://www.sciencedirect.com/science/article/pii/S0005109817305721>
- [25] R. M. F. Candido, L. Hardouin, M. Lhommeau, and R. Santos-Mendes, “An algorithm to compute the inverse image of a point with respect to a nondeterministic max plus linear system,” *IEEE Transactions on Automatic Control*, pp. 1–1, 2020.
- [26] A. Jazwinski, *Stochastic Processes and Filtering Theory*, ser. Mathematics in Science and Engineering. Academic Press, 1970, vol. 63.
- [27] T. J. van den Boom and B. De Schutter, “Analytic expressions in stochastic max-plus-linear algebra,” in *53rd IEEE Conference on Decision and Control*, 2014, pp. 1608–1613.

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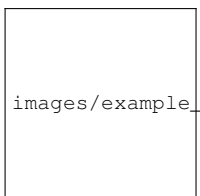
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