SMT-Based and Fixed-Point Approaches for State Estimation in Max-Plus Linear Systems

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Abstract

This work builds on the seminal paper [1] and evaluates an existing method against a new approach for state estimation in Max-Plus Linear systems with bounded uncertainties. Traditional stochastic filtering is inapplicable to this system class, even though the posterior probability density function (PDF) can be computed. Previous research has shown a limited scalability of the *disjunctive* approach using difference-bound matrices. To address this, we investigate an alternative method recently explored in [2, 3], employing Satisfiability Modulo Theory (SMT) techniques, despite their NP-hard nature. The main novelty of this work is the proposal of a *concise* method based on fixed-point iteration in max-plus algebra, which is known to be a pseudo-polynomial time algorithm. To compare both approaches, a representative autonomous system is used in the paper to illustrate the basic computations. The efficiency of both approaches is compared through numerical experiments.

Keywords: Max-Plus Linear Systems, State-Estimation, Satisfiability Modulo Theories, Fixed point algorithm

1 Introduction

Max-plus algebra theory is suitable in analyzing Discrete Event Systems (DES) with delay and synchronization. These phenomena are found in production systems, computing networks, and transportation systems (see [4, 5] for an overview). This theory employs an algebraic structure known as idempotent semiring, enabling the description of these systems as linear models. Thus, Max-Plus Linear (MPL) systems can be defined through recursive state-space equations, where states represent event-times (time instants) within the system, forming a timetable trajectory. Residuation theory [6] further aids in addressing crucial issues in control theory: controllability, observability, stabilization, and feedback synthesis (see [7]).

In problems involving model parameter uncertainties, deterministic considerations are common, disregarding probabilistic aspects. However, in filtering problems affected by random processes influencing model parameters, addressing probabilistic aspects becomes crucial. Stochastic Max-Plus Linear (sMPL) systems handle this by defining MPL systems with matrices containing random variable entries. State-estimation in the Bayesian approach involves computing the *posterior* probability density function (PDF) using available measurements. While the Kalman filter and its extensions are practical for filtering with additive Gaussian noise, they are unsuitable for MPL systems due to their nonlinear discontinuities (see [8, 9] for details). For such systems, we can apply other stochastic filtering strategies as the Sequential Monte-Carlo (SMC) method, also known as Particle Filter but with numerical difficulties related to the generation of the particles (see [10, 11]). This work focuses on systems where uncertain parameters can vary within known intervals, namely uncertain MPL (uMPL) systems, i.e., sMPL systems with bounded random variables.

In this work, we study an *indirect* computation of the support of the *posterior* PDF for uMPL systems. This computation is referred to as set-estimation. In [12], the authors use the works of [13] on difference-bound matrices, in [14] they use max-plus polyhedra [15] and in [16] they use residuation theory [6].

Contribution: we propose the concise approach using a fixed-point algorithm, known to be with pseudo-polynomial complexity, and we compare it with the disjunctive Satisfiability Modulo Theory (SMT) approach of [2, 3], known to be NP-hard, using Z3 solver of [17] to estimate (if it is feasible) the state of uMPL systems. Throughout the paper, a small-sized autonomous system is explored to demonstrate the basic computations of each approach.

The paper is organized as follows: Section 2 recalls the basic notions of MPL systems. Section 3 presents the *indirect* computation of the set of all states that can be reached from a previous state through the transition model and that can lead to the measurement output through the measurement function by using the disjunctive and concise approaches. Section 4 presents the application: proving the feasibility guarantee of set-estimation. Numerical simulations are performed to compare the two approaches. Finally, Section 5 concludes the work and presents some ideas for future works.

2 Preliminaries

2.1 Max-plus algebra

A set $\mathcal{D}$ forms a *dioid* or *idempotent semiring* if it satisfies certain algebraic properties. These properties include the associativity, commutativity and idempotency of the sum $\oplus$, as well as the associativity and left and right distributivity of the product $\otimes$ w.r.t $\oplus$. Dioid $\mathcal{D}$ contains a null element, ε, such that $\forall a \in \mathcal{D}, a \oplus \varepsilon = a$ and an identity element e, such that $\forall a \in \mathcal{D}, a \otimes e = e \otimes a = a$. A partial order relation

$$a \succeq b \iff a = a \oplus b$$

is defined for elements $a, b \in \mathcal{D}$. This order relation makes $\mathcal{D}$ to be a partially ordered set such that each pair of elements a, b admits the lowest upper bound $\sup\{a, b\}$ which coincides with $a \oplus b$. Hence, a dioid is a *sup-semilattice*. Furthermore, the sum and the left and right products preserve this relation, i.e., if $a \succeq b$ then $a \oplus c \succeq b \oplus c$, $a \otimes c \succeq b \otimes c$ and $c \otimes a \succeq c \otimes b$. A dioid $\mathcal{D}$ is complete if it is closed for infinite sums and the left and right distributivity of the product extend to infinite sums. In practice, for $\mathcal{D}$ to be complete, the top element, denoted $\top$, exists and is equal to the sum of all elements of $\mathcal{D}$, i.e., $\top = \bigoplus_{a \in \mathcal{D}} a$, such that $\forall a \in \mathcal{D}, a \oplus \top = \top$. This element respects the absorbing rule, i.e., $\varepsilon \otimes \top = \varepsilon$. For a complete dioid, an inner operation representing the lower bound of the operands, denoted, $\ominus$ automatically exists. The partial order relation can be expressed as

$$a \succeq b \iff a = a \oplus b \iff b = a \ominus b$$

where $a \ominus b = \inf\{a, b\}$ is the greatest lower bound of a, b.

The max-plus algebra, denoted as $\mathbb{R}_{\max}$, is a set that includes $\mathbb{R}$ along with the elements $\varepsilon = -\infty$, $\top = +\infty$ and $e = 0$, i.e., $\mathbb{R} \cup \{-\infty, +\infty\}$, with the two binary operations $a \oplus b \coloneqq \max\{a, b\}$ and $a \otimes b \coloneqq a + b$. This algebra is an example of a complete dioid. This dioid is linearly ordered w.r.t. $\oplus$ and the order $\succeq$ in this set coincides with the usual linear order $\geq$. Furthermore, in this dioid, the operation $a \ominus b$ coincides with $\min\{a, b\}$.

The two binary operations in $\mathbb{R}_{\max}$ are naturally extended to matrices. Given $A, B \in \mathbb{R}_{\max}^{n \times p}$, $C \in \mathbb{R}_{\max}^{p \times q}$ and $\alpha \in \mathbb{R}_{\max}$, we have $(A \oplus B)_{ij} = (a_{ij} \oplus b_{ij})$, $(A \otimes C)_{ij} = \left(\bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj}\right)$ and $(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij}$. The partial order relation is also applied to matrices as follows

$$A \succeq B \iff A = A \oplus B$$

for $A, B \in \mathbb{R}_{\max}^{n \times p}$, where $\succeq$ refers to the linear order $\geq$ on $\mathbb{R}^{n \times p}$.

Given $k \in \mathbb{N}$ and $A \in \mathbb{R}_{\max}^{n \times n}$, $A^{\otimes k} = A \otimes \cdots \otimes A$ (k-fold). The matrix $A^{\otimes 0}$ is the n-dimensional identity matrix I_n, which is a special kind of the max-plus version of diagonal matrices[1] $\mathrm{diag}_{\oplus}(\bullet)$ with e on the main diagonal. The absorbing matrix $\mathcal{E}_{n \times m}$ is defined as the $(n \times m)$-dimensional matrix whose entries are ε. The all-e matrix $E_{n \times m}$ follows the same idea, but with its entries equal to e. The Kleene star of a

[1] A max-plus diagonal matrix has its entries outside the main diagonal equal to ε

matrix A is defined as $A^\star = \left(\bigoplus_{k\in\mathbb{N}} A^{\otimes k}\right)$. If A such that $a_{ij} = \varepsilon$ for all $i \leq j$ with $i, j \in \{1, \ldots, n\}$, then $A^{\otimes k} = \mathcal{E}_{n\times n}$ for $k \geq n$, hence $A^\star = \bigoplus_{k=0}^{n-1} A^{\otimes k}$.

A system of linear inequalities $A \otimes x \leq y$, where $A \in \mathbb{R}_{\max}^{m\times n}$, $x \in \mathbb{R}_{\max}^{n}$ and $y \in \mathbb{R}_{\max}^{m}$ admits the greatest solution $\hat{x} = A^\sharp(y)$ given by the following residuation formula

$$(A^\sharp(y))_i = \min_{j=1}^{m} (-a_{ji} + y_j)\,,$$

which is equivalent to $-(A^T \otimes (-y))$. Obviously, if $A \otimes x = y$ admits a solution, then $\hat{x}$ is the greatest solution and $A \otimes \hat{x} = y$ holds[2]. This result is also applied to find the greatest solution of the two-sided equation $A \otimes x = B \otimes x$ where $A, B \in \mathbb{R}_{\max}^{m\times n}$ (see [19] for more details). The method to find this solution, initially shown to terminate for integer-valued matrices in [20] and later extended to real-valued matrices in [21], is presented in the sequel.

The following equivalences hold

$$\begin{aligned} A \otimes x = B \otimes x &\iff A \otimes x \leq B \otimes x \text{ and } B \otimes x \leq A \otimes x \\ &\iff x \leq A^\sharp(B \otimes x) \text{ and } x \leq B^\sharp(A \otimes x) \\ &\iff x \leq A^\sharp(B \otimes x) \ominus B^\sharp(A \otimes x) \\ &\iff x = x \ominus A^\sharp(B \otimes x) \ominus B^\sharp(A \otimes x). \end{aligned}$$

Hence, the greatest fixed-point of

$$\Pi(x) = x \ominus A^\sharp(B \otimes x) \ominus B^\sharp(A \otimes x)$$

is the greatest solution of $A \otimes x = B \otimes x$. Moreover, since A, $A^\sharp$, B and $B^\sharp$ are clearly isotone maps[3] then $\Pi(x)$ is also isotone. Thus, to solve this two-sided equation, it suffices to iterate the sequence

$$\mathcal{I} : x[k+1] = \Pi(x[k])$$

on an initial $x[k]$, namely $x[0]$, until convergence $x[\Bbbk+1] = x[\Bbbk]$ is reached for a specific $\Bbbk \in \mathbb{N}$ (fixed-point iteration). As a consequence, if a *finite* (non-ε entries only) greatest solution $x[\Bbbk]$ of $A \otimes x = B \otimes x$ exists, then $\mathcal{I}$ is able to find it in a finite number of steps such that $x[\Bbbk] \leq x[0]$. This computation is known to have a pseudo-polynomial complexity, i.e., the convergence rate is polynomial according to the distance between $x[\Bbbk]$ and $x[0]$. Conditions are also presented in [19] to ensure that this procedure converges in finite time because $\mathcal{I}$ is likely to run infinitely since it is possible that one or more of the entries of $x[k]$ decrease indefinitely to ε. Nevertheless, the algorithm is efficient (convergence with finite time and with a low number of steps) to handle problems in this work.

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[2] In [18], to check equality $A \otimes x = y$, the following test is considered, with a complexity $\mathcal{O}(nm)$, $A \otimes x = y \iff \bigcup_{i=1}^{n} \operatorname{argmin}_{j\in\{1,\ldots,m\}}(-a_{ji} + y_j) = \{1, \ldots, m\}$.

[3] $A^\sharp$ and $B^\sharp$ are isotone maps but not necessarily linear. Hence, in general $A^\sharp(x) \oplus A^\sharp(y) \neq A^\sharp(x \oplus y)$ for $x, y \in \mathbb{R}_{\max}^{n}$.

2.2 Intervals over max-plus algebra

Interval analysis in the max-plus algebra was originally presented in [22]. Since then, many authors have been interested in the use of intervals within this algebraic framework [23]. A (closed) interval $[x]$ in max-plus algebra is a subset of $\mathbb{R}_{\max}$ of the form

$$[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R}_{\max} \mid \underline{x} \leq x \leq \overline{x}\}$$

with $\underline{x} < \overline{x}$. We denote by $\mathbb{IR}_{\max}$ the set of intervals of $\mathbb{R}_{\max}$. An interval $[x] \subseteq [y]$ if and only if $\underline{y} \leq \underline{x} \leq \overline{x} \leq \overline{y}$. Similarly, $[x] = [y]$ if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$. A value $x \in \mathbb{R}_{\max}$ can be represented by the *degenerated* interval $[x, x]$. The $\oplus$ and $\otimes$ operations exist for intervals: $[\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $[\underline{x}, \overline{x}] \otimes [\underline{y}, \overline{y}] = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$.

An interval matrix in max-plus algebra is a matrix whose elements are intervals. The operations $\oplus$ and $\otimes$ can be extended to interval matrices. Given the interval matrices $[A] = [\underline{A}, \overline{A}], [B] = [\underline{B}, \overline{B}]$ and $[C] = [\underline{C}, \overline{C}]$ of dimensions $(n \times p), (n \times p)$ and $(p \times q)$, then $([A] \oplus [B])_{ij} = [a_{ij}] \oplus [b_{ij}]$ and $([A] \otimes [C])_{ij} = \bigoplus_{k=1}^{p} ([a_{ik}] \otimes [c_{kj}])$. Moreover, respectively the product of $\alpha \in \mathbb{R}_{\max}$ by $[A]$ is given by $\alpha \otimes [A] = [\alpha \otimes \underline{A}, \alpha \otimes \overline{A}]$ and the k-th power of $[A]$ is given by $[A]^{\otimes k} = [\underline{A}^{\otimes k}, \overline{A}^{\otimes k}]$. The Kleene star operation is also defined for intervals matrices, mathematically for $[A]$ we have $[A]^\star = (\bigoplus_{k \in \mathbb{N}} [A]^{\otimes k})$.

2.3 Synchonization and Delay in Discrete Event Systems: Max-plus linear systems

Discrete Event Systems (DES) involve *synchronization* and *concurrence*. Synchronization in manufacturing occurs when multiple resources are needed simultaneously, while concurrence involves making choices among available options within the same timeframe. The max operator is crucial in synchronization modeling for defining temporal alignment.

Synchronization phenomena in Discrete Event Systems (DES) are represented using *timed* models, focusing on sequences of time instants and event occurrences, whereas *logical* models deal with possible event sequences and associated conditions.

One of the existing formalisms for modeling timed systems is to consider *linear* recursive state-space equations within the algebraic framework of $\mathbb{R}_{\max}$. This algebraic structure is well-suited to represent the behavior of synchronization ($\oplus$) and timing information[4] ($\otimes$). By employing appropriate algebraic manipulation and transformation, one obtains the following autonomous[5] Max-Plus Linear (MPL) systems:

$$\mathcal{S}: \begin{cases} x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1), \\ z(k) = C \otimes x(k) \end{cases}$$

where $A_0, A_1 \in \mathbb{R}_{\max}^{n \times n}$ and $C \in \mathbb{R}_{\max}^{p \times n}$. Each event is labeled with an index $i \in \{1, \ldots, n\}$, and $x_i(k) \in \mathbb{R}_{\max}$ represents the time instant of the k-th occurrence of event i. As it can be noticed, $x(k) = (x_1(k), \ldots, x_n(k))^\intercal$ appears in both sides of the

[4] In manufacturing, the timing information may represent the *processing time* of a task (in practice, it is a *delay*).

[5] Any nonautonomous max-plus DES can be transformed into an augmented autonomous one [4, Sec. 2.5] and in this work we consider, without loss of generality, autonomous systems only.

above recursive equation. The *transition* model and the *measurement* function are represented by the pair (A_0, A_1) and C, respectively. The transition model admits an alternative form, given by $x(k) = A \otimes x(k-1)$ with $A = A_0^\star \otimes A_1$ such that the orbit of trajectory of $x(k)$ in this form is equal to the one in $\mathcal{S}$ (see [4] for details).

In this paper, we assume the system $\mathcal{S}$ is uncertain, i.e., the matrices have some entries which are random variables belonging to intervals. Thus, an uncertain MPL (uMPL) system is defined as

$$\mathcal{S}_u : \begin{cases} x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1), \\ z(k) = C(k) \otimes x(k) \end{cases} \tag{1}$$

where $A_0(k) \in [A_0] = [\underline{A}_0, \overline{A}_0] \in \mathbb{IR}_{\max}^{n\times n}$, $A_1(k) \in [A_1] = [\underline{A}_1, \overline{A}_1] \in \mathbb{IR}_{\max}^{n\times n}$ and $C(k) \in [C] = [\underline{C}, \overline{C}] \in \mathbb{IR}_{\max}^{p\times n}$ are nondeterministic matrices. Similarly, the transition model of uMPL systems also admits an alternative form representation by considering $x(k) = A(k) \otimes x(k-1)$ with $A(k) = A_0^\star(k) \otimes A_1(k)$ where $A_0(k) \in [A_0]$ and $A_1(k) \in [A_1]$.

Remark 1. *It is important to note that the equation $x(k) = \mathcal{A}(k) \otimes x(k-1)$, where $\mathcal{A}(k) \in [A_0]^\star \otimes [A_1]$, over-approximates the reachable space of $\mathcal{S}_u$ concerning a given state $x(k-1)$. In other words, this form is conservative since we can compute the bounds of $\mathcal{A}(k)$ which are equal to those of $A(k)$. However, it is likely that some values of $\mathcal{A}(k) \neq A(k) = A_0^\star(k) \otimes A_1(k)$.*

Example 1. *Consider an example of an autonomous uMPL system as given by (1), with*

$$A_0(k) \in A_0 = \begin{pmatrix} \varepsilon & \varepsilon \\ [1,2] & \varepsilon \end{pmatrix}, A_1(k) \in A_1 = \begin{pmatrix} [4,6] & [3,5] \\ [3,7] & [4,5] \end{pmatrix}.$$

It is possible to obtain $x(1) = (5,9)^\intercal$ using the over-approximation of (1), i.e., $x(k) = \mathcal{A}(k) \otimes x(k-1)$. In details,

$$\mathcal{A}(1) = \begin{pmatrix} 4 & 5 \\ 8 & 7 \end{pmatrix} \in [A_0^\star] \otimes [A_1] = \begin{pmatrix} [4,6] & [3,5] \\ [5,8] & [4,7] \end{pmatrix} \text{ and } x(0) = (1,e)^\intercal$$

yield $x(1) = \mathcal{A}(1) \otimes x(0) = (5,9)^\intercal$ However, this state is unfeasible if one considers the exact form of (1) $x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1)$, i.e.,

$$x_1(1) \in [4,6] \otimes x_1(0) \oplus [3,5] \otimes x_2(0) \overset{x(0)=(1,e)^\intercal}{\Longrightarrow} 5 \in [5,7],$$
$$x_2(2) \in [1,2] \otimes x_1(1) \oplus ([3,7] \otimes x_1(0) \oplus [4,5] \otimes x_2(0)) \overset{x_1(1)=5, x(0)=(1,e)^\intercal}{\Longrightarrow} 9 \notin [5,8].$$

2.4 Max-plus systems using disjunctive approach

In [13], the authors represent max-plus systems $y = M \otimes x$, with $y \in \mathbb{R}_{\max}^q$ and $x \in \mathbb{R}_{\max}^n$, using disjunctions with operations in $\mathbb{R}$. If the above systems are bounded, i.e., $\underline{M} \otimes x \leq y \leq \overline{M} \otimes x$ with $\underline{M}, \overline{M} \in \mathbb{R}_{\max}^{q\times n}$ then we obtain the following inequalities for all $i \in \{1, \dots, q\}$:

$$\max(\underline{m}_{i1} + x_1, \dots, \underline{m}_{in} + x_n) \leq y_i \iff x_1 - y_i \leq -\underline{m}_{i1} \text{ and } \dots \text{ and } x_n - y_i \leq -\underline{m}_{in}$$

and

$$y_i \leq \max(\overline{m}_{i1} + x_1, \ldots, \overline{m}_{in} + x_n) \iff y_i - x_1 \leq \overline{m}_{i1} \text{ or } \ldots \text{ or } y_i - x_n \leq \overline{m}_{in},$$

i.e., $\exists g_i \in \{1, \ldots, n\}$ for all $i \in \{1, \ldots, q\}$ such that $\max_{j\in\{1,\ldots,n\}}(\overline{m}_{ij}+x_j)$ is equivalent to $\overline{m}_{ig_i} + x_{g_i}$.
In details, $\underline{M} \otimes x \leq y \iff x \leq \underline{M}^\sharp(y)$, and

$$y \leq \overline{M} \otimes x \iff \bigwedge_{i=1}^{q} \left(\bigvee_{j=1}^{n} (y_i - x_j \leq \overline{m}_{ij}) \right),$$

with $\wedge$ and $\vee$ playing the role of the logic operators AND and OR, respectively. Hence, $\underline{M} \otimes x \leq y$ is represented concisely, which is not the case for $y \leq \overline{M} \otimes x$ since it is represented by the combination of n^q elements[6].

These inequalities can be represented by Difference-Bound Matrices (DBMs) [24], where the entries are the upper bounds of the difference-bound constraints above, i.e, $-\underline{m}_{ij}$ representing $x_j - y_i \leq -\underline{m}_{ij}$ and $\overline{m}_{ij}$ representing $y_i - x_j \leq \overline{m}_{ij}$.

DBMs have some interesting operations such as intersection and union (element-wise min and max, respectively), canonical form (cubic complexity using Floyd-Warshall algorithm) and orthogonal projection. The interested reader is invited to see [13] for more details.

Example 2. *Consider the following bounded max-plus system $y = M \otimes x$ where*

$$M \in [\underline{M}, \overline{M}] = \begin{pmatrix} [4,6] & [3,5] \\ [3,7] & [4,5] \end{pmatrix}.$$

Then, $\underline{M} \otimes x \leq y$ is rewritten as

$$x_1 - y_1 \leq -4 \wedge x_2 - y_1 \leq -3 \wedge x_1 - y_2 \wedge x_2 - y_2 \leq -4,$$

and $y \leq \overline{M} \otimes x$ is rewritten for the combination $(g_1, g_2) = (1, 1) \in \{1, 2\}^2$, as

$$y_1 - x_1 \leq 6 \wedge y_2 - x_1 \leq 7.$$

The following DBM represents the above system for one of 2^2 possible combinations:

$$D^{(1,1)} = \begin{array}{c} \\ \theta \\ y_1 \\ y_2 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccccc} \theta & y_1 & y_2 & x_1 & x_2 \end{array} \\ \begin{pmatrix} 0 & +\infty & +\infty & +\infty & +\infty \\ +\infty & 0 & +\infty & 6 & +\infty \\ +\infty & +\infty & 0 & 7 & +\infty \\ +\infty & -4 & -3 & 0 & +\infty \\ +\infty & -3 & -4 & +\infty & 0 \end{pmatrix} \end{array}.$$

[6] Each row of $y \leq \overline{M} \otimes x$ in the range $\{1, \ldots, q\}$ is represented by $y_i - \overline{m}_{ig_i} \leq x_{g_i}$, where $g_i \in \{1, \ldots, n\}$, hence $(g_1, g_2, \ldots, g_q) \in \{1, \ldots, n\}^q$.

The artificial variable θ is equal to 0 and is used to express an upper bound for a variable, using difference-bound constraints. For instance, $x \leq c$ can be expressed as $x - 0 \leq c$. Moreover, $+\infty$ means that the difference between two variables is unbounded.

3 Support of the posterior PDF

In stochastic filtering, the relevant information is obtained from the posterior PDF. In a set-guaranteed estimation, one is interested in computing its support. Following [12], this support is the set of all possible states $x(k)$ that can be reached from the previous state $x(k-1)$ through the transition model and are consistent with the observed measurement $z(k)$ through the measurement function. Mathematically, the image of $x(k-1)$ w.r.t. $A_0(k) \in [A_0]$ and $A_1(k) \in [A_1]$ is given by

$$\mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\} = \{A_0^\star \otimes A_1 \otimes x(k-1) \in \mathbb{R}^n_{\max} \mid A_0 \in [A_0], A_1 \in [A_1]\}, \quad (2)$$

i.e., the set of all states $x(k)$ that can be reached from $x(k-1)$ through the transition model[7]. We also show how to characterize the inverse image of $z(k)$ w.r.t. $C(k) \in [C]$, formally

$$\mathrm{Im}^{-1}_{[C]}\{z(k)\} = \{x \in \mathbb{R}^n_{\max} \mid \exists C \in [C], C \otimes x = z(k)\}, \quad (3)$$

i.e., the set of all $x(k)$ that can lead to $z(k)$ through the measurement function. Straightforwardly, the support of the posterior PDF is defined as

$$\mathcal{X}_k = \mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\} \cap \mathrm{Im}^{-1}_{[C]}\{z(k)\}. \quad (4)$$

when $x(k-1)$ is considered to be the *prior* knowledge of x. In [11, 12] the authors use difference-bounds matrices (see [25] for an overview), which represent zones, to compute exactly $\mathcal{X}_k$. These disjunctive approaches lack in scalability, since it is necessary to consider an exponential number of combinations for encoding the upper bounds of the transition model and measurement functions (see Subsection 2.4). For further details, please refer to these works.

3.1 A symbolic disjunctive approach

Formal methods have significantly benefited from advancements in solving Boolean satisfiability (SAT) problems. One notable work that exemplifies this progress is the supervisory control of DES [26]. In various applications, multiple problems involve determining the *satisfiability* of formulas in more expressive logics like *first-order logic* w.r.t. background theories. This concept is known as Satisfiability Modulo Theories (SMT) (see [27, 28] for an overview). In SMT, one can verify, for example, if there exist (or for all) certain symbolic variables $\mathtt{x}$ and $\mathtt{y}$ in $\mathbb{R}$ that satisfy a given symbolic formula F. For instance,

$$F : (\mathtt{x} \geq 0) \wedge (\mathtt{y} < 2) \vee (\mathtt{x} - \mathtt{y} < -1),$$

[7] An over-approximation for $\mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\}$ is simply computed as $[\![\mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\}]\!] = \{A \otimes x(k-1) \in \mathbb{R}^n_{\max} \mid A \in [A_0]^\star \otimes [A_1]\}$, i.e., $\mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\} \subseteq [\![\mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\}]\!]$. Please refer to Remark 1.

is tested for satisfiability w.r.t. a set[8]. If a solution exists, it returns values for $\mathtt{x}$ and $\mathtt{y}$ that make each asserted constraint true.

Remark 2. *Difference-bound constraints can be represented as Boolean combinations of atoms $x_i - x_j \leq c$, which form difference-logic formulas. Thus, the SMT approach can incorporate these constraints [12, 13], as briefly presented in Subsection 2.4.*

In [2, 3], max-plus systems have been expressed as SMT formulas. Briefly, we have $y = M \otimes x$ with $m_{ij} \in \mathbb{R}_{\max}$ for $(i,j) \in \{1,\ldots,q\} \times \{1,\ldots,n\}$. It follows from Subsection 2.4 that for each $i \in \{1,\ldots,q\}$ there exists (at least) a $g_i \in \{1,\ldots,n\}$ such that

$$\forall j \in \{1,\ldots,n\} \setminus \{g_i\} \; y_i = m_{ig_i} + x_{g_i} \geq m_{ij} + x_j .$$

Hence, the aforementioned result is equivalent to evaluate the following SMT formula

$$F_i \; : \; \left(\bigwedge_{j \in \mathcal{J}_i} \mathtt{y}_i - \mathtt{m}_{ij} \geq \mathtt{x}_j \right) \wedge \left(\bigvee_{j \in \mathcal{J}_i} \mathtt{y}_i - \mathtt{m}_{ij} = \mathtt{x}_j \right),$$

where $\mathtt{y}_i, \mathtt{x}_j, \mathtt{m}_{ij}$ are symbolic variables and $\mathcal{J}_i \subseteq \{1,\ldots,n\}$ represents the set of indices j such that $m_{ij} \neq \varepsilon$. If each m_{ij} is bounded, then it suffices to add the following symbolic formula

$$B_i \; : \; \left(\bigwedge_{j \in \mathcal{J}_i} (\mathtt{m}_{ij} \geq \underline{m}_{ij}) \wedge (\mathtt{m}_{ij} \leq \overline{m}_j) \right)$$

to F_i, i.e., $F_i \wedge B_i$. Hence, $\bigwedge_{i=1}^{q} F_i \wedge B_i$ *symbolically* represents $y = M \otimes x$ with $\underline{M} \leq M \leq \overline{M}$.

For systems $\mathcal{S}_u$ of (1) let us define for each row of the transition model the following formula:

$$Row_i^{k,k-1} : Conj_i^{k,k-1} \wedge Disj_i^{k,k-1} \wedge Bnd_i^{k,k-1}$$

with

$$Conj_i^{k,k-1} \; : \; \left(\bigwedge_{l \in \mathcal{G}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_l^{(k)} \geq \mathtt{a0}_{il}^{(k)} \right) \wedge \left(\bigwedge_{j \in \mathcal{F}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_j^{(k-1)} \geq \mathtt{a1}_{ij}^{(k)} \right)$$

$$Disj_i^{k,k-1} \; : \; \left(\bigvee_{l \in \mathcal{G}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_j^{(k)} = \mathtt{a0}_{il}^{(k)} \right) \vee \left(\bigvee_{j \in \mathcal{F}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_j^{(k-1)} = \mathtt{a1}_{ij}^{(k)} \right)$$

and

$$Bnd_i^{k,k-1} \; :$$

$$\left(\bigwedge_{l \in \mathcal{G}_i} (\mathtt{a0}_{il}^{(k)} \geq \underline{a}_{0_{il}}) \wedge (\mathtt{a0}_{il}^{(k)} \leq \overline{a}_{0_{il}}) \right) \wedge \left(\bigwedge_{j \in \mathcal{F}_i} (\mathtt{a1}_{ij}^{(k)} \geq \underline{a}_{1_{ij}}) \wedge (\mathtt{a1}_{ij}^{(k)} \leq \overline{a}_{1_{ij}}) \right)$$

[8] The formula F has a solution if $\mathtt{x}, \mathtt{y} \in \mathbb{R}$ but no solution if $\mathtt{x}, \mathtt{y} \in \mathbb{Z}$.

where $\mathtt{x}_1^{(k)},\ldots,\mathtt{x}_n^{(k)}$ and $\mathtt{a1}_{ij}^{(k)},\mathtt{a0}_{il}^{(k)}$ are symbolic variables for each k and $\mathcal{F}_i,\mathcal{G}_i\subseteq\{1,\ldots,n\}$ are, respectively, sets of indices of the i-th rows of $A_1(k),A_0(k)$ that are different ε (i.e., are finite). Hence, $Row_i^{k,k-1}$ is used to represent *symbolically* the transition model of (1) as the following formula:

$$\mathtt{D}^{k,k-1}:\bigwedge_{i=1}^{n} Row_i^{k,k-1}.$$

The following formula represents *symbolically* the measurement function of (1):

$$\mathtt{O}^{k,k}:\bigwedge_{i=1}^{p}\mathtt{O}_i^{k,k}$$

with

$$\begin{aligned}\mathtt{O}_i^{k,k}\ :\ &\left(\bigwedge_{j\in\mathcal{H}_i}\mathtt{z}_i^{(k)}-\mathtt{x}_j^{(k)}\geq\mathtt{c}_{ij}^{(k)}\right)\wedge\left(\bigvee_{j\in\mathcal{H}_i}\mathtt{z}_i^{(k)}-\mathtt{x}_j^{(k)}=\mathtt{c}_{ij}^{(k)}\right)\\ \wedge&\left(\bigwedge_{j\in\mathcal{H}_i}(\mathtt{c}_{ij}^{(k)}\geq\underline{c}_{ij})\wedge(\mathtt{c}_{ij}^{(k)}\leq\overline{c}_{ij})\right)\end{aligned}$$

where $\mathtt{z}_1^{(k)},\ldots,\mathtt{z}_p^{(k)}$ and $\mathtt{c}_{ij}^{(k)}$ are symbolic variables and $\mathcal{H}_i\subseteq\{1,\ldots,n\}$ with the same meaning as for $\mathcal{F}_i$ but for $C(k)$.

Remark 3. *For our purposes, it is considered that* $\mathtt{x}_1^{(k-1)},\ldots,\mathtt{x}_n^{(k-1)}$ *and* $\mathtt{z}_1^{(k)},\ldots,\mathtt{z}_p^{(k)}$ *are known and thus replaced with* $x_1(k-1),\ldots,x_n(k-1)$ *and* $z_1(k),\ldots,z_p(k)$*.*

Symbolically, $\mathcal{X}_k$ of (4) is represented by the following formula:

$$\mathtt{X}^k\ :\ \mathtt{D}^{k,k-1}\wedge\mathtt{O}^{k,k}.$$

Example 3. *Consider* $\mathcal{S}_u$ *of* (1) *with* $A_0(k)\in[A_0]=\begin{pmatrix}\varepsilon & \varepsilon\\ [1,2] & \varepsilon\end{pmatrix}$, $A_1(k)\in[A_1]=\begin{pmatrix}[4,6] & [3,5]\\ [3,7] & [4,5]\end{pmatrix}$ *and* $C(k)\in\begin{pmatrix}[e,1] & \varepsilon\end{pmatrix}$ *with* $x(0)=(1,e)^\intercal$, $x(1)=(6,7)^\intercal$, $C(1)=\begin{pmatrix}e & \varepsilon\end{pmatrix}$ *implying* $z(1)=6$*. In the Figure 1, we depict the computations of* $Im_{[A_0],[A_1]}\{x(0)\}$ *of* (2) *and* $Im_{[C]}^{-1}\{z(1)=C(1)\otimes x(1)\}$ *of* (3)*, such that* $\mathcal{X}_1$ *of* (4) *is the intersection of both and represents the set of all possible* $x(1)$ *that respect both dynamics and measurement. The SMT formula* $\mathtt{D}^{1,0}$ *below represents symbolically all* $x(1)\in Im_{[A_0],[A_1]}\{x(0)\}$,

$$\begin{aligned}&\mathtt{D}^{1,0}:\\ &\{[(\mathtt{x}_1^{(1)}-1\geq\mathtt{a1}_{11}^{(1)})\wedge(\mathtt{x}_1^{(1)}-0\geq\mathtt{a1}_{12}^{(1)}]\wedge[(\mathtt{x}_1^{(1)}-1=\mathtt{a1}_{11}^{(1)})\vee(\mathtt{x}_1^{(1)}-0=\mathtt{a1}_{12}^{(1)})]\\ &\wedge[(\mathtt{a1}_{11}^{(1)}\geq 4)\wedge(\mathtt{a1}_{11}^{(1)}\leq 6)\wedge(\mathtt{a}_{12_0}^{(1)}\geq 3)\wedge(\mathtt{a}_{12_0}^{(1)}\leq 5)]\}\\ &\wedge\end{aligned}$$

$$\{[(\mathtt{x}_2^{(1)} - \mathtt{x}_1^{(1)} \geq \mathtt{a0}_{21}^{(1)}) \wedge (\mathtt{x}_2^{(1)} - 1 \geq \mathtt{a1}_{21}^{(1)}) \wedge (\mathtt{x}_2^{(1)} - 0 \geq \mathtt{a1}_{22}^{(1)})]$$
$$\wedge [(\mathtt{x}_2^{(1)} - \mathtt{x}_1^{(1)} = \mathtt{a0}_{21}^{(1)}) \vee (\mathtt{x}_2^{(1)} - 1 = \mathtt{a1}_{21}^{(1)}) \vee (\mathtt{x}_2^{(1)} - 0 = \mathtt{a1}_{22}^{(1)})]$$
$$\wedge [(\mathtt{a0}_{21}^{(1)} \geq 1) \wedge (\mathtt{a0}_{21}^{(1)} \leq 2) \wedge (\mathtt{a1}_{21}^{(1)} \geq 3) \wedge (\mathtt{a1}_{21}^{(1)} \leq 7) \wedge (\mathtt{a1}_{22}^{(1)} \geq 4) \wedge (\mathtt{a1}_{22}^{(1)} \leq 5)]\}.$$

Following the same procedure, the SMT formula $\mathtt{O}^{1,1}$ *below represents symbolically all* $x(1) \in Im_{[C]}^{-1}\{z(1)\}$*,*

$$\mathtt{O}^{1,1} \ : \ (6 - \mathtt{x}_1^{(1)} \geq \mathtt{c}_{11}^{(1)}) \wedge (6 - \mathtt{x}_1^{(1)} = \mathtt{c}_{11}^{(1)}) \wedge (\mathtt{c}_{11}^{(1)} \geq 0) \wedge (\mathtt{c}_{11}^{(1)} \leq 1),$$

such that it is clear that $(6 - \mathtt{x}_1^{(1)} \geq \mathtt{c}_{11}^{(1)}) \wedge (6 - \mathtt{x}_1^{(1)} = \mathtt{c}_{11}^{(1)})$ *is equivalent to* $(6 - \mathtt{x}_1^{(1)} = \mathtt{c}_{11}^{(1)})$ *in this example.*
We can easily verify that $\mathtt{X}^1 \ : \ \mathtt{D}^{1,0} \wedge \mathtt{O}^{1,1}$ *that symbolically represents* $\mathcal{X}_1$ *is* SAT*, thus* $\mathcal{X}_1 \neq \emptyset$*, by using Z3 solver [17] with the following realizations* $\mathtt{x}_1^{(1)} = 5.5, \mathtt{x}_2^{(1)} = 7$, $\mathtt{a0}_{21}^{(1)} = 1, \mathtt{a1}_{11}^{(1)} = 4.5, \mathtt{a1}_{12}^{(1)} = 3, \mathtt{a1}_{21}^{(1)} = 6, \mathtt{a1}_{22}^{(1)} = 4, \mathtt{c}_{11}^{(1)} = 0.5$ *which are straightforwardly verified by*

$$x(1) = \begin{pmatrix} \varepsilon & \varepsilon \\ \mathtt{a0}_{21}^{(1)} & \varepsilon \end{pmatrix} \otimes x(1) \oplus \begin{pmatrix} \mathtt{a1}_{11}^{(1)} & \mathtt{a1}_{12}^{(1)} \\ \mathtt{a1}_{21}^{(1)} & \mathtt{a1}_{22}^{(1)} \end{pmatrix} \otimes x(0)$$
$$\Rightarrow \begin{pmatrix} 5.5 \\ 7 \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ 1 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} 5.5 \\ 7 \end{pmatrix} \oplus \begin{pmatrix} 4.5 & 3 \\ 6 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ e \end{pmatrix},$$
$$z(1) = \begin{pmatrix} \mathtt{c}_{11}^{(1)} & \varepsilon \end{pmatrix} \otimes x(1) \Rightarrow 6 = \begin{pmatrix} 0.5 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} 5.5 \\ 7 \end{pmatrix}$$

that hold true.

3.2 A concise approach

The previous approach uses the encoding of max-plus systems in standard algebra to take advantage of a powerful method for affine systems. For this reason, we derive in the sequel an equivalent and *concise* method based exclusively on max-plus algebra.

First, let us write the transition model of (1), i.e., $x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1)$ with $A_0(k) \in [\underline{A}_0, \overline{A}_0]$ and $A_1(k) \in [\underline{A}_1, \overline{A}_1]$ as

$$\underline{A}_0 \otimes x(k) \oplus \underline{A}_1 \otimes x(k-1) \leq x(k) \leq \overline{A}_0 \otimes x(k) \oplus \overline{A}_1 \otimes x(k-1),$$

then define $x = x(k)$, $\underline{\zeta} = \underline{A}_1 \otimes x(k-1)$ and $\overline{\zeta} = \overline{A}_1 \otimes x(k-1)$ thus

$$\underline{A}_0 \otimes x \oplus \underline{\zeta} \leq x \text{ and } \overline{A}_0 \otimes x \oplus \overline{\zeta} \geq x.$$

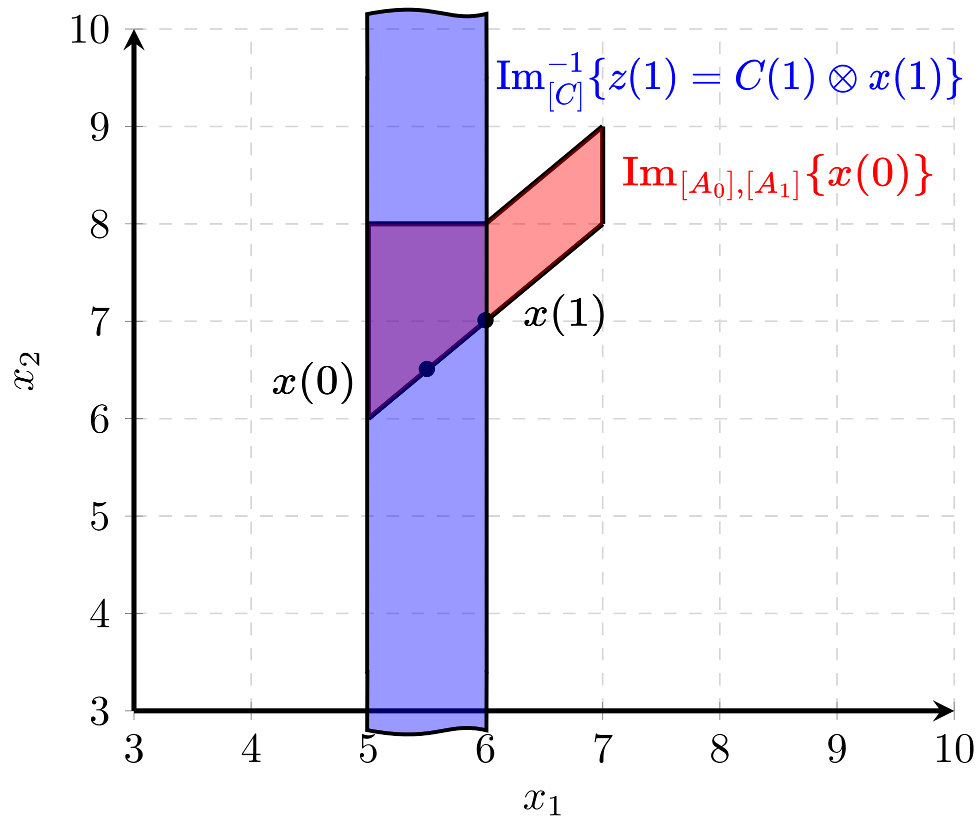


Fig. 1: Exact direct image of $x(0)$ w.r.t. $A_0(1)$ and $A_1(1)$ and inverse image of $z(1)$ w.r.t. $C(1)$ of Example 3.

Now, taking advantage of the partial order relation on this algebraic structure, the two-sided equation below is obtained

$$LD(\underline{\zeta}, \overline{\zeta}) \otimes \begin{pmatrix} x \\ e \end{pmatrix} = UD(\overline{\zeta}) \otimes \begin{pmatrix} x \\ e \end{pmatrix} \tag{5}$$

with $LD(\underline{\zeta}, \overline{\zeta}) := \begin{pmatrix} (\underline{A}_0 \oplus I_n) & \underline{\underline{\zeta}} \\ (\overline{A}_0 \oplus I_n) & \overline{\zeta} \end{pmatrix}$ and $UD(\overline{\zeta}) := \begin{pmatrix} I_n & \mathcal{E}_{n \times 1} \\ \overline{A}_0 & \overline{\zeta} \end{pmatrix}$, such that all $x \in \mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\}$ of (2) satisfy the above equation. For the measurement function of (1), a similar procedure exists and was originally derived in [14]. Briefly, $\underline{C} \otimes x(k) \leq z(k) \leq \overline{C} \otimes x(k)$ is written as

$$LO \otimes \begin{pmatrix} x \\ e \end{pmatrix} = UO \otimes \begin{pmatrix} x \\ e \end{pmatrix} \tag{6}$$

with $LO := \begin{pmatrix} C & z \\ \overline{C} & z \end{pmatrix}$, $UO := \begin{pmatrix} \mathcal{E}_{p\times n} & z \\ \overline{C} & \mathcal{E}_{p\times 1} \end{pmatrix}$, $x = x(k)$ and $z = z(k)$, such that all $x \in \mathrm{Im}^{-1}_{[C]}\{z(k)\}$ of (3) satisfy this two-sided equation.

It is evident that all $x(k) \in \mathcal{X}_k$, as defined in (4), satisfy both (5) and (6) simultaneously. By vertically concatenating the associated matrices, we obtain a single matrix equation that represents all $x(k) \in \mathcal{X}_k$.

Example 4. *Let us recall Example 3. We aim at verifying if there exists a* $x(1)$ *respecting* (4)*, i.e., if* $\mathcal{X}_1 \neq \emptyset$*. This is done by using* (5) *and* (6)*. Thus,*

$$\begin{pmatrix} e & \varepsilon & 5 \\ 1 & e & 4 \\ e & \varepsilon & 7 \\ 2 & e & 8 \end{pmatrix} \otimes \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & 7 \\ 2 & \varepsilon & 8 \end{pmatrix} \otimes \begin{pmatrix} x \\ e \end{pmatrix} \text{ and } \begin{pmatrix} e & \varepsilon & 6 \\ 1 & \varepsilon & 6 \end{pmatrix} \otimes \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon & 6 \\ 1 & \varepsilon & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} x \\ e \end{pmatrix}.$$

Hence, by vertically concatenating both two-sided equations, we can therefore apply the fixed point iteration technique presented in Subsection 2.1, in order to verify if $\mathcal{X}_1 \neq \emptyset$ *and compute the greatest* $x(1)$ *satisfying* (5) *and* (6)*. Starting with* $(100, 100, e)^\intercal$*, the greatest solution* $(6, 8, e)^\intercal$ *is reached in two steps. The sequence of iterations until convergence is* $\{(100, 100, e)^\intercal, (6, 100, e)^\intercal, (6, 8, e)^\intercal\}$*. Thus, the greatest* $x(1) \in \mathcal{X}_1$ *coincides with* $(6, 8)^\intercal$*, which is the upper corner of the intersection region between* $Im_{[A_0],[A_1]}\{x(0)\}$ *of* (2) *and* $Im^{-1}_{[C]}\{z(1) = C(1) \otimes x(1)\}$ *of* (3)*, as depicted in Figure 1.*

4 Feasibility guarantees for set-estimation

In a set-estimation scheme, we aim at computing a value for $x(k)$ within $\mathcal{X}_k$ of (4). Clearly (4) cannot be used, since $x(k-1)$ is unknown. Then, an estimate $\hat{x}(k)$ is computed such that

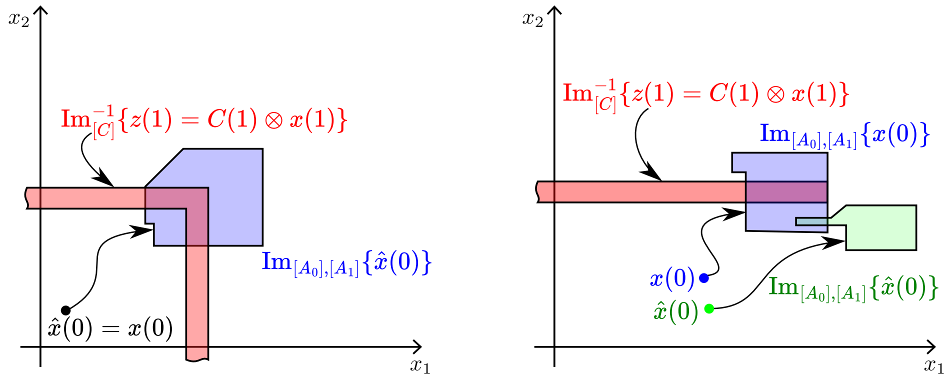
$$\hat{x}(k) \in \hat{\mathcal{X}}_k = \mathrm{Im}_{[A_0],[A_1]}\{\hat{x}(k-1)\} \cap \mathrm{Im}^{-1}_{[C]}\{z(k)\}$$

where $\hat{x}(k-1)$ is the estimate of $x(k)$ at $k-1$. Of course, the success of this approach is related to the distance between $x(k)$ and $\hat{x}(k)$. An estimate $\hat{x}(k) \in \mathrm{Im}_{[A_0],[A_1]}\{\hat{x}(k-1)\}$ may be not in $\mathrm{Im}^{-1}_{[C]}\{z(k) = C(k) \otimes x(k)\}$ since we cannot guarantee that $x(k-1)$ is equal to $\hat{x}(k-1)$ and thus $\hat{\mathcal{X}}_k$ may be empty, as illustrated in Figure 2. In this case, $\hat{x}(k)$ is said to be an *unfeasible* estimation. Based on Section 3, we derive a disjunctive and a concise tests to verify feasibility of $\hat{x}(k)$ at each k and, in the affirmative case, return an estimate.

4.1 Symbolic disjunctive method

In [3], the authors presented a numerical benchmark showing the efficiency of the SMT-based approach for reachability problems.

In a procedural way, consider the *symbolic* formula that represents $\mathcal{X}_k$ of (4). Let us replace $\mathtt{x}_1^{(k-1)}, \ldots, \mathtt{x}_n^{(k-1)}$ with $\hat{x}_1(k-1), \ldots, \hat{x}_n(k-1)$, hence defining $\mathtt{X}^{k|k}$ as the



(a) Illustration of the nonempty case.

(b) Illustration of the empty case.

Fig. 2: Illustrative examples: (a) a nonempty case where $\mathrm{Im}_{[A_0],[A_1]}\{\hat{x}(0)\}$ and $\mathrm{Im}^{-1}_{[C]}\{z(1)\}$ intersect, and (b) an empty case where they are disjoint.

prediction formula. In the same way, we replace $\mathtt{z}_1^{(k)}, \ldots, \mathtt{z}_p^{(k)}$ with $z_1(k), \ldots, z_p(k)$, hence defining $\tilde{\mathtt{X}}_{k|k}$ as the *likelihood* formula. Thus

$$\mathtt{X}^{k|k} : \mathtt{X}^{k|k-1} \wedge \tilde{\mathtt{X}}^{k|k}$$

represents the *correction* formula, i.e., $\hat{\mathcal{X}}_k$. Using Z3 *solver* of [17], we are able to verify if $\mathtt{X}^{k|k}$ is `SAT` and return a *solution* that makes each asserted constraint true, defining then a value for $x(k)$, i.e., an *arbitrary* estimate $\hat{x}(k)$. As part of a filtering algorithm, a recursion is defined, i.e., $\hat{x}(k-1) \leftarrow \hat{x}(k)$ and the solver is called once again. If the solver returns `UNSAT` for some k, then $\hat{x}(k)$ is unfeasible and we stop the filtering procedure.

4.2 Concise fixed-point method

In a procedural way, let us consider (5) with

$$\underline{\zeta} = \underline{A}_1 \otimes \hat{x}(k-1) \text{ and } \overline{\zeta} = \overline{A}_1 \otimes \hat{x}(k-1),$$

hence defining

$$X_{L,k|k-1} \otimes \begin{pmatrix} x \\ e \end{pmatrix} = X_{U,k|k-1} \otimes \begin{pmatrix} x \\ e \end{pmatrix}$$

as the *prediction* equation, with $X_{L,k|k-1} = LD(\underline{\zeta}, \overline{\zeta})$, $X_{U,k|k-1} = UD(\overline{\zeta})$ (see (5)). In the same way, let us consider (6) with $z = z(k)$, hence defining

$$\tilde{X}_{L,k|k} \otimes \begin{pmatrix} x \\ e \end{pmatrix} = \tilde{X}_{U,k|k} \otimes \begin{pmatrix} x \\ e \end{pmatrix}$$

as the *likelihood* equation with $\tilde{X}_{L,k|k} = LO$, $\tilde{X}_{U,k|k} = UO$ (see (6)). Thus,

$$\begin{pmatrix} X_{L,k|k-1} \\ \tilde{X}_{L,k|k} \end{pmatrix} \otimes \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} X_{U,k|k-1} \\ \tilde{X}_{U,k|k} \end{pmatrix} \otimes \begin{pmatrix} x \\ e \end{pmatrix}$$

represents the *correction* equation, i.e., $\hat{\mathcal{X}}_k$. Furthermore, by using the fixed-point iteration algorithm presented in Subsection 2.1, we are able to verify if the previous two-sided equation has solution and compute[9] the *greatest* estimate $\hat{x}(k) \in \hat{\mathcal{X}}_k$. As part of a filtering algorithm, a recursion is defined, i.e., $\hat{x}(k-1) \leftarrow \hat{x}(k)$ and we repeat the procedure. If no solution exists for some k, then $\hat{x}(k)$ is unfeasible and we stop the filtering procedure.

4.3 Numerical simulations

For the numerical simulation's comparison[10], let us consider (1) with $A_0(k)$ in strictly lower form, i.e., $a_{0_{ij}}(k) \neq \varepsilon$ for all $i \leq j$, $i, j \in \{1, \ldots, n\}$ and $A_1(k), C(k)$ be full max-plus matrices, i.e., $a_{1_{ij}}(k) \neq \varepsilon$ for all $i, j \in \{1, \ldots, n\}$ and $c_{ij}(k) \neq \varepsilon$ for all $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$. Every element of these matrices are randomly chosen between the arbitrary bounds 0 and 10 at each k, i.e., the realizations $A_0(k), A_1(k), C(k)$. We suppose that $x(0) = (e, \ldots, e)^\intercal$ and then we obtain the following sequences

$$\{x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1)\}_{k \in \mathbb{N}_{>0}} \text{ and } \{z(k) = C(k) \otimes x(k)\}_{k \in \mathbb{N}_{>0}}.$$

We compare in the sequel the previous approaches to compute feasible estimate $\hat{x}(k)$ for $x(k)$ at each k. If no feasible estimate can be guaranteed, then we stop the simulation.

Table 1 shows the minimum, average and maximum execution times for each call of the estimators for 20 experiments of the disjunctive method $T^{symb}(s)$ and the concise method $T^{mat}(s)$ for $k \in \{1, \ldots, N\}$, where N is the event-horizon. We analyze simulations that are not stopped, i.e., experiments that do not violate the feasibility guarantee of the set-estimation using either approach. Furthermore, we analyze the error-estimation of both approaches. We compute the mean-absolute-percentage-error (MAPE) between $x_i(k)$ and $\hat{x}_i(k)$ for $i \in \{1, \ldots, n\}$, precisely

$$error_i(x_i(k), \hat{x}_i(k)) = \frac{100\%}{N} \sum_{k=1}^{N} \left| \frac{x_i(k) - \hat{x}_i(k)}{x_i(k)} \right|, \ i \in \{1, \ldots, n\}$$

[9] As mentioned in Subsection 2.1, it is possible that no finite solution exists for the two-sided equations (the trivial infinite solution with an all-ε vector is useless). In this case, the fixed-point algorithm does not converge in finite time, and we define the solution as *unfeasible*.

[10] Running `Python` with C++ wrappers for Z3 SMT solver ([17]) and Armadillo ([29]) for fast (sparse) matrix operations in max-plus algebra on a Dell Precision 5530 - 2.6 GHz Intel(R) Core(TM) i7 processor.

for all $i \in \{1, \ldots, n\}$ and then we take the average of the resulting vector, i.e.,

$$error_{avg} = \frac{1}{n} \sum_{i=1}^{n} error_i(x_i(k), \hat{x}_i(k)).$$

We show the minimum, average and maximum values of $error^{symb}_{avg}(\%)$, $error^{mat}_{avg}(\%)$ for each experiment out of 20. As it can be noted, the execution times of both approaches are related to n. However, the disjunctive approach is more affected by p because there are more symbolic constraints to be evaluated by the SMT solver, thus increasing the execution time[11]. In terms of error-estimation, these experiments suggest that the concise method leads to lower error-estimation values. For the last row of Table 1, we evaluate an example with a large n and we only present the results for the concise method because the running time of the disjunctive method exceeds a predefined threshold (timeout).

5 Conclusion

In this work, we have studied two approaches: one developed by the authors and another drawn from the existing literature to provide feasibility guarantees for set-estimation of MPL systems with bounded uncertainties. We *indirectly* characterize reachable sets from previous estimations that respect the measurement output and compute values within these sets. Firstly, we examine a disjunctive approach utilizing SMT techniques. Secondly, we propose a concise method based on solving two-sided equations in max-plus algebra with pseudo-polynomial complexity. The latter method outperforms the former in terms of speed and accuracy. Future works involve integrating probabilistic aspects for additional feasibility certificates and exploring the application of the concise method for *directly* characterizing the reachable sets.

Table 1: Numerical analysis comparison.

n	p	N	$error^{symb}_{avg}(\%)$	$error^{mat}_{avg}(\%)$	$T^{symb}(s)$	$T^{mat}(s)$
5	3	500	{0.40; 0.46; 0.53}	{0.02; 0.03; 0.04}	{0.03; 0.04; 0.116}	{0.005; 0.009; 0.02}
10	5	20	{1.44; 1.62; 1.94}	{0.43; 0.51; 0.57}	{0.12; 0.21; 1.43}	{0.02; 0.03; 0.05}
10	8	20	{1.46; 1.58; 1.85}	{0.44; 0.51 0.58}	{0.14; 0.24; 1.52}	{0.02; 0.03; 0.05}
20	10	5	{2.27; 3.00; 3.84}	{1.54; 1.80; 2.02}	{0.57; 4.90; 28.20}	{0.07; 0.10; 0.12}
100	50	10	{-}	{0.96; 1.01; 1.05}	{-}	{1.99; 2.24; 2.65}

Declarations

The authors declare that this manuscript received no funding.

[11] For the fixed-point algorithm, the execution time grows at a manageable rate (polynomial) with n and p. For the SMT-based approach, NP-hard complexity implies that, in the worst case, the execution time can grow exponentially, making it impractical for large n and p.

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