On the Set-Estimation of Uncertain Max-Plus Linear Systems

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Abstract

The paper focuses on the set-estimation for uncertain Max-Plus Linear systems, with bounded random parameters. This estimation process involves determining the conditional reach set, which is a compact set of all possible states that can be reached from a previous set through the transition model (dynamics) and can lead to the observed measurements through the observation model. In the context of Bayesian estimation theory, this set represents the support of the posterior probability density function of the system's state. We compare two approaches, a disjunctive approach, presented in literature, and a concise approach, presented as a contribution of this paper, to exactly compute this set. Even if both approaches are with an exponential theoretical complexity, it is shown that the concise approach is more efficient.

Key words: Discrete Event Systems, Max-Plus Algebra, Set-Membership State-Estimation, Conditional Reach Sets, Difference-Bound Matrices, Max-plus Polyhedra

1 Introduction

Timed Discrete Event Systems without concurrency but with synchronization can be described by max-plus linear state equations. Timed Event Graphs (TEGs) are widely employed as graphical representations of these systems. TEGs are timed Petri nets where each place has a minimum holding time and are with only one upstream and one downstream transitions. The states considered are the firing dates of TEG transitions. Such systems have found extensive applications in manufacturing plants, telecommunication networks, railway networks and parallel computing [7].

This paper focuses on state estimation for uncertain Max-Plus Linear (uMPL) systems, i.e. when the holding times associated to the TEG places vary in a bounded way. The literature has explored different approaches to tackle this problem:

State observer inspired by Luenberger's work, utilizing residuation theory, is proposed in [19] to estimate the true state from available measurements. The estimation approximates the state from below and is shown to be compatible with the measurements, but no probabilistic guarantee is provided.

Stochastic filtering for uMPL systems in [25,15,14]. These algorithms follow a two-step calculation similar to classical Bayesian filtering. The prediction step utilizes the expectation of max-affine functions, and the correction step refines the prediction using measurement outputs. This approach employs a suboptimal procedure based on interval analysis [21]. It is important to remark that this approach risks convergence to unfeasible states due to over-optimism.

Set-membership filtering also known as set-estimation, has been studied in [11,10] based on previous works in [1]. This approach computes the exact support set of the posterior probability density function (PDF) as a set. It is important to remark that this approach tends to be over-pessimistic by discarding inconsistent states w.r.t. the measurements.

In this study, we address the issue of set estimation by employing max-plus polyhedra [5]. We present a method to calculate the set of reachable states from a prediction (max-plus) polyhedron, which encompasses all potential previous states. Following that, we determine the (max-plus) polyhedron that includes states consistent with the available measurement, referred to as the like-

⋆ This paper was not presented at any IFAC meeting. Corresponding author G. Espindola-Winck.

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lihood (max-plus) polyhedron. Lastly, we employ the likelihood (max-plus) polyhedron to adjust the prediction polyhedron, leading to the estimation (max-plus) polyhedron that characterizes the support of the posterior probability density function (PDF). Furthermore, we demonstrate the superior efficiency of our novel approach compared to Difference-Bound Matrices (DBMs) used in prior studies [1,11,10,27].

The paper is organized as follows: Section 2 recalls the the basic notions on max-plus algebra, intervals, DBM, max-plus polyhedra. Section 3 presents the main contributions of this work and section 4 presents an application: computing the support of the posterior PDF. Numerical examples are given to show the details of the computations. Finally, Section 5 concludes the work and presents some ideas for future works.

2 Preliminaries

2.1 Max-plus algebra

The max-plus algebra, denoted as $\overline{\mathbb{R}}_{\max}$, is a set that includes $\mathbb{R}$ along with the elements $-\infty$ and $+\infty$, i.e. $\mathbb{R}\cup\{-\infty,+\infty\}$. It is equipped with two associative binary operations: $a\oplus b := \max(a,b)$ and $a\otimes b := a+b$. In this algebra, we have $\varepsilon := -\infty$ as the neutral element for $\oplus$, i.e. $a\oplus\varepsilon = \varepsilon\oplus a = a$, which also acts as an absorbing element for $\otimes$, i.e. $a\otimes\varepsilon = \varepsilon\otimes a = \varepsilon$. There is also a neutral element $e := 0$ for the operation $\otimes$ i.e. $a\otimes e = e\otimes a = a$. This algebra is a semiring because it shares similarities with a ring by dropping the requirement that each element must have an additive inverse. However, an inverse for $\otimes$ exists for all $x\in\overline{\mathbb{R}}_{\max}\setminus\{-\infty,+\infty\}$, denoted as $x^{-1} := -x$, such that $x\otimes x^{-1} = x^{-1}\otimes x = e$. The operation $\oplus$ is idempotent, meaning that $a\oplus a = a$. Consequently, the natural order relation $a\preceq b \Leftrightarrow a\oplus b = b$ is defined for elements $a,b\in\overline{\mathbb{R}}_{\max}$, where $\leq$ represents the linear order on $\mathbb{R}$. In the sequel, the symbol $\otimes$ can be omitted in the absence of ambiguity.

The two binary operations in $\overline{\mathbb{R}}_{\max}$ are naturally extended to matrices. Given $A,B\in\overline{\mathbb{R}}_{\max}^{n\times p}$, $C\in\overline{\mathbb{R}}_{\max}^{p\times q}$ and $\alpha\in\overline{\mathbb{R}}_{\max}$, we have $(A\oplus B)_{ij} = (a_{ij}\oplus b_{ij})$, $(A\otimes C)_{ij} = (\bigoplus_{k=1}^{p} a_{ik}\otimes c_{kj})$ and $(\alpha\otimes A)_{ij} = \alpha\otimes a_{ij}$. The natural order relation is also applied to matrices as follows $A\preceq B \Leftrightarrow A\oplus B = B$ for $A,B\in\overline{\mathbb{R}}_{\max}^{n\times p}$, where $\leq$ refers to the partial order on $\mathbb{R}^{n\times p}$. The column-space of $A\in\overline{\mathbb{R}}_{\max}^{n\times p}$, denoted $\texttt{col}(A)\subseteq\overline{\mathbb{R}}_{\max}^{n}$, consists of all possible products $A\otimes x$, for $x\in\overline{\mathbb{R}}_{\max}^{p}$.

Given $k\in\mathbb{N}$ and $A\in\overline{\mathbb{R}}_{\max}^{n\times n}$, $A^{\otimes k} = A\otimes\cdots\otimes A$ (k-fold). The matrix $A^{\otimes 0}$ is the n-dimensional identity matrix I_n, with e on the main diagonal and ε elsewhere. The absorbing matrix $\mathcal{E}_{n\times m}$ is defined as the $(n\times m)$-dimensional matrix whose entries are ε. The all-e matrix $E_{n\times m}$ is such that all entries are equal to e.

A system of linear inequalities $Ax\preceq y$, where $A\in\overline{\mathbb{R}}_{\max}^{m\times n}$, $x\in\overline{\mathbb{R}}_{\max}^{n}$ and $y\in\overline{\mathbb{R}}_{\max}^{m}$ admits the greatest solution $\hat{x} = A^{\sharp}(y)$ given by the following residuation formula $(A^{\sharp}(y))_i = \min_{j=1}^{m}(-a_{ji}+y_j)$, which is equivalent to $-(A^T\otimes(-y))$. Obviously, if $Ax = y$ admits a solution, then $\hat{x}$ is the greatest solution and $A\hat{x} = y$ holds. In [8], to check equality $Ax = y$, i.e. $y\in\texttt{col}(A)$, the following test is considered, with a complexity $\mathcal{O}(nm)$

$$Ax = y \Leftrightarrow \bigcup_{i=1}^{n} \underset{j\in\{1,\ldots,m\}}{\operatorname{argmin}}\,(-a_{ji}+y_j) = \{1,\ldots,m\}. \quad (1)$$

2.2 Interval over the max-plus algebra

Interval analysis in the max-plus algebra was originally presented in [23]. A (closed) interval $[x]$ in max-plus algebra is a subset of $\overline{\mathbb{R}}_{\max}$ of the form $[x] = [\underline{x},\overline{x}] = \{x\in\overline{\mathbb{R}}_{\max} \mid \underline{x}\preceq x\preceq\overline{x}\}$ with $\underline{x}<\overline{x}$. We denote by $\mathbb{I}\overline{\mathbb{R}}_{\max}$ the set of intervals of $\overline{\mathbb{R}}_{\max}$. An interval $[x]\subseteq[y]$ if and only if $\underline{y}\preceq\underline{x}\preceq\overline{x}\preceq\overline{y}$. Similarly, $[x]=[y]$ if and only if $\underline{x}=\underline{y}$ and $\overline{x}=\overline{y}$. A value $x\in\overline{\mathbb{R}}_{\max}$ can be represented by the *degenerated* interval $[x,x]$. The $\oplus$ and $\otimes$ operations exist for intervals: $[\underline{x},\overline{x}]\oplus[\underline{y},\overline{y}] = [\underline{x}\oplus\underline{y},\overline{x}\oplus\overline{y}]$ and $[\underline{x},\overline{x}]\otimes[\underline{y},\overline{y}] = [\underline{x}\otimes\underline{y},\overline{x}\otimes\overline{y}]$. An interval matrix in max-plus algebra is a matrix whose elements are intervals. The operations $\oplus$ and $\otimes$ can be extended to interval matrices. Given the interval matrices $[A] = [\underline{A},\overline{A}]$, $[B] = [\underline{B},\overline{B}]$ and $[C] = [\underline{C},\overline{C}]$ of dimensions $(n\times p)$, $(n\times p)$ and $(p\times q)$, then $([A]\oplus[B])_{ij} = [a_{ij}]\oplus[b_{ij}]$ and $([A]\otimes[C])_{ij} = \bigoplus_{k=1}^{p}([a_{ik}]\otimes[c_{kj}])$. Moreover, the product of $\alpha\in\overline{\mathbb{R}}_{\max}$ by $[A]$ is given by $\alpha\otimes[A] = [\alpha\otimes\underline{A},\alpha\otimes\overline{A}]$ and the k-th power of $[A]$ is given by $[A]^{\otimes k} = [\underline{A}^{\otimes k},\overline{A}^{\otimes k}]$.

2.3 Zones and Difference-Bound Matrices

Definition 1 (Zones [26]) *Zones are used to represent affine invariants. In mathematical terms, given a vector x in $\mathbb{R}^n$, a zone represents the intersection of a finite number of difference-bound constraints. These constraints have the form $x_i - x_j \bowtie \alpha_{ij}$ and $x_i \bowtie \alpha_{ij}$, where $\alpha_{ij}\in\mathbb{R}\cup\{+\infty\}$ and $\bowtie\in\{<,\leq\}$ (with a well-defined ordering[1]). Additionally, the indices i and j represent distinct elements in the range $\{0,\ldots,n\}$, where x_0 is defined as 0. The zone captures the set of all possible values of x that satisfy these difference-bound constraints.*

Zones are represented by Difference-Bound Matrices (DBMs) [12], where the entries are a pair of the upper bound and strictness of the sign of the difference-bound constraint, i.e. $(\alpha_{ij},\bowtie)$.

[1] The symbols $<$ and $\leq$ used in the constraints are assumed to have a total order, i.e. $<$ is strictly less than $\leq$.

DBMs have some interesting operations such as intersection and union (element-wise min and max, respectively), canonical form (cubic complexity using Floyd-Warshall algorithm) and orthogonal projection. The interested reader is invited to see [1] for more details.

2.4 Max-plus polyhedra

A (max-plus) half space [2] is analogous to a classical half space, and is defined as the set of points $x \in \overline{\mathbb{R}}^n_{\max}$ satisfying $(\bigoplus_{j=1}^n a_j \otimes x_j) \oplus b \leq (\bigoplus_{j=1}^n c_j \otimes x_j) \oplus d$, where $a_j, b, c_j, d \in \overline{\mathbb{R}}_{\max}$ (see [16]).

As conventional convex polyhedra, the (convex) max-plus (or *tropical*) polyhedra can be described as an intersection of n max-plus half space. This can be summarized in matrix form as follows $Ax \oplus b \leq Cx \oplus d$, where $A, C \in \overline{\mathbb{R}}^{s\times n}_{\max}$ and $b, d \in \overline{\mathbb{R}}^s_{\max}$, this external representation of the max-plus polyhedron is hereafter called $\mathcal{H}$-form. It is also possible to represent the $\mathcal{H}$-form as the *homogenous* system $Ez \leq Fz$ where $E = (A\ b)$, $F = (C\ d)$, $x \in \overline{\mathbb{R}}^n_{\max}$ and $\alpha \in \overline{\mathbb{R}}_{\max}$, the term $(x^T, \alpha)^T$ refers to the vector $z \in \overline{\mathbb{R}}^{n+1}_{\max}$ whose first n coordinates coincide with x and the latter is α.

As in the conventional context [16, Minkowski-Weyl Th.], a max-plus polyhedron is also internally represented by its $\mathcal{V}$-form, i.e. as the set of points $x \in \overline{\mathbb{R}}^n_{\max}$ which can be written as the affine combination of generators $v^i \in V \subset \overline{\mathbb{R}}^n_{\max}$ (extreme points) and $r^j \in R \subset \overline{\mathbb{R}}^n_{\max}$ (extreme rays) as $x = \left(\bigoplus_{i=1}^p \lambda_i v^i\right) \oplus \left(\bigoplus_{i=1}^q \mu_i r^i\right)$, $\bigoplus_{i=1}^p \lambda_i = e$.

Furthermore, the $\mathcal{V}$-form of max-plus polyhedra admits a *homogenous* matrix representation as follows $\begin{pmatrix} x \\ \alpha \end{pmatrix} = G \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$, $G = \begin{pmatrix} V & R \\ e \dots e & \varepsilon \dots \varepsilon \end{pmatrix}$ with $V = (v^1, \dots, v^p)$ and $R = (r^1, \dots, r^q)$, i.e. seen as generating matrices [3]. If x is a point then $\alpha = e$ otherwise $\alpha = \varepsilon$ because x is a ray. It is worth to mention that by simply analyzing the last row of G, we are able to deduce if a generator, i.e. a column of G, is an extreme point (α-part is e) or an extreme ray (α-part is ε). A max-plus polyhedron is said to be *bounded* if the last row of G is composed of e and the other rows have elements that lie in $\mathbb{R}_{\max} \setminus \{-\infty, +\infty\}$.

It is always possible to go from a $\mathcal{H}$-form to a $\mathcal{V}$-form, and vice versa, with exponential complexity in the dimension n as presented in [5]. We present in the sequel the main results on this complexity analysis.

Theorem 1 ([4,3]) *Let $\mathcal{P} \subseteq \overline{\mathbb{R}}^{n+1}_{\max}$ be a max-plus polyhedron in its homogenous $\mathcal{H}$-form, defined as $\mathcal{P} = \{z \in \overline{\mathbb{R}}^{n+1}_{\max} \mid Ez \leq Fz\}$, where E and F are matrices in $\overline{\mathbb{R}}^{s\times(n+1)}_{\max}$ (with $s \geq 0$). Let $G_0, \dots, G_s$ be the sequence of finite subsets of $\overline{\mathbb{R}}^n_{\max}$ defined as follows $G_0 = \{\epsilon^i\}_{1\leq i\leq n+1}, G_i = \{g \in G_{i-1} \mid E_i g \leq F_i g\} \cup \{(E_i h)g \oplus (F_i g)h \mid g, h \in G_{i-1}, E_i g \leq F_i g, E_i h > F_i h\}$, for all $i \in \{1, \dots, s\}$, where E_i and F_i are the i-th rows of E and F and $\epsilon^i \in \overline{\mathbb{R}}^{n+1}_{\max}$ is a vector whose i-th coordinate is equal to e and the others to ε. Then $\mathcal{P}$ is finitely generated by the set G_s.*

Remark 1 *The complexity of the algorithm derived from Theorem 1 is $\mathcal{O}(s^2 n\beta^2)$, where β represents the maximum number of extreme generators in the intermediate polyhedra represented by G_i. The value of β is bounded by $\mathcal{O}(s^{\lfloor \frac{n}{2} \rfloor})$, i.e. it grows exponentially w.r.t. n for a fixed s [5]. In practical implementations, the double description of max-plus polyhedra [5] can be efficiently handled using `Polymake` [17], which incorporates the functionality of `TPlib` [2]. These tools facilitate the translation of a $\mathcal{H}$-form representation into a $\mathcal{V}$-form representation and vice versa.*

It is important to note that this inductive approach produces redundant generators, which can be eliminated using the following procedure.

Procedure 1 ([9]) *Consider that $V = \{v^1, \dots, v^p\}$ represents the $\mathcal{V}$-form of a max-plus polyhedron. Define $V' = V \setminus \{v^i\}$ then we can use (1) to verify if $v^i \in \mathtt{col}(V')$, i.e. to compute a basis of $\mathtt{col}(V)$. This Procedure 1 has complexity $\mathcal{O}(np^2)$.*

Zones, originally defined over the standard algebra, can also be represented as max-plus polyhedra as described in [24].

Example 1 *Given the difference-bound constraints $x_1 \leq 10$ and $x_1 - x_2 \leq 9$. It is not hard to obtain the following matrix inequality $\begin{pmatrix} e & \varepsilon \\ e & \varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} \leq \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} 10 \\ \varepsilon \end{pmatrix}$, which represents the $\mathcal{H}$-form of a max-plus polyhedron, i.e. as $Ax \oplus b \leq Cx \oplus d$.*

Remark 2 *Every max-plus polyhedron can be expressed as a collection of finitely many DBMs (see [1, Prop. 7]).*

Definition 2 (Polytrope [22]) *A polytrope is a specific type of max-plus polyhedra of $\overline{\mathbb{R}}^n_{\max}$ that represents*

[2] It is worth to recall that every conventional half space is a max-plus half space, but the converse does not hold.
[3] A finite subset of $\overline{\mathbb{R}}^n_{\max}$, for instance $V = \{v^1, \dots, v^r\}$ is equivalently represented by the matrix $V = (v^1, \dots, v^r)$.

a single compact zone of $\mathbb{R}^n$, which is convex both in the standard algebra and in the max-plus algebra.

In [16], the authors demonstrate that polytropes can be represented in a canonical manner. Specifically, they showed that in the $\mathcal{V}$-form, a polytrope can be represented by $n+1$ extreme points. Additionally, in the $\mathcal{H}$-form, the representation involves $n+1$ max-plus half spaces. In the following propositions, we will present evidences to support the validity of these findings.

Proposition 1 *A polytrope of $\overline{\mathbb{R}}^n_{\max}$ is represented by the intersection of $n+1$ max-plus half spaces.*

PROOF. First, a polytrope is a compact zone. Then, all variables are bounded, i.e. $\underline{x}_j \le x_j \le \overline{x}_j$ with $\underline{x}_j < \overline{x}_j$ for $j \in \{1,\dots,n\}$, and are intersected with the constraints $x_i - x_j \le d_{ij}$ for $i \ne j$ where $i,j \in \{1,\dots,n\}$ and $d_{ij} \in \mathbb{R}$. Thus, $\bigwedge_{j=1}^n x_j - \overline{x}_j \le 0 \Leftrightarrow \max_{j=1}^n (x_j - \overline{x}_j) \le 0$, (1 half space) and we have for all $j \in \{1,\dots,n\}$, $\underline{x}_j \le x_j \wedge \bigwedge_{i=1,i\ne j}^n x_i - d_{ij} \le x_j \Leftrightarrow \max(\underline{x}_j, \max_{i=1,i\ne j}^n (x_i - d_{ij})) \le x_j$, (1 half space) which leads to obtain $n+1$ max-plus half spaces. ■

Proposition 2 ([18]) *A polytrope of $\overline{\mathbb{R}}^n_{\max}$ is represented by $n+1$ extreme points given by $g^0 = (\underline{x}_1,\dots,\underline{x}_n)^T$ and $g^k = (\overline{x}_k - d_{k1} \ \dots \ \overline{x}_k - d_{kn})^T$ for $k \in \{1,\dots,n\}$ and with $x_i - x_j \le d_{ij}$, $\underline{x}_j \le x_j \le \overline{x}_j$, $\underline{x}_j < \overline{x}_j$ for $i,j \in \{1,\dots,n\}$ ($d_{ii} = 0$).*

Based on Proposition 2 one obtains that the $\mathcal{V}$-form of the polytrope that represents the interval vector $[x] \in \overline{\mathbb{IR}}^n_{\max}$ is given by $V = (\underline{x} \ x^{(1)} \ \dots \ x^{(n)})$, where $x^{(i)} = (\underline{x}_1, \underline{x}_2, \dots, \overline{x}_i, \underline{x}_{i+1}, \dots, \underline{x}_n)^T$.

The intersection of two max-plus polyhedra is also a max-plus polyhedron which is given in its homogenous $\mathcal{H}$-form as follows: $\mathcal{P}_1 \cap \mathcal{P}_2 = \{z \in \overline{\mathbb{R}}^{n+1}_{\max} \mid (E_1^T \ E_2^T)^T z \le (F_1^T \ F_2^T)^T z\} \equiv \{z \in \overline{\mathbb{R}}^{n+1}_{\max} \mid z \in \mathcal{P}_1 \text{ and } z \in \mathcal{P}_2\}$. where $\mathcal{P}_1 = \{z \in \overline{\mathbb{R}}^{n+1}_{\max} \mid E_1 z \le F_1 z\}$ and $\mathcal{P}_2 = \{z \in \overline{\mathbb{R}}^{n+1}_{\max} \mid E_2 z \le F_2 z\}$ with $\overline{\mathbb{R}}^n_{\max}$ where $E_1, F_1 \in \overline{\mathbb{R}}^{s\times(n+1)}_{\max}$ and $E_2, F_2 \in \overline{\mathbb{R}}^{q\times(n+1)}_{\max}$.

Remark 3 *To compute the $\mathcal{V}$-form of $\mathcal{P}_1 \cap \mathcal{P}_2$, we can invoke Theorem 1. If the $\mathcal{V}$-form of $\mathcal{P}_1$ or $\mathcal{P}_2$ is available, then it is possible to speed up this algorithm by initializing the matrix G_0 with the generators of $\mathcal{P}_1$ or $\mathcal{P}_2$. To an extensive presentation of set-theoretical operations, the reader is invited to consult [6].*

3 Computation of image of a set and the inverse image of a point over uMPL systems

In this section, we recall the disjunctive method and present the concise method to compute the image of a compact set assumed to be a max-plus polyhedron (or equivalently a collection of zones) and the inverse image of a point over uncertain Max-Plus Linear systems.

3.1 Uncertain Max-Plus Linear systems

A (autonomous [4]) Max-Plus Linear (MPL) system [7] is defined as $\mathcal{S} \ : \ \{x(k) = Ax(k-1),\ z(k) = Cx(k)$ where $A \in \overline{\mathbb{R}}^{n\times n}_{\max}$ and $C \in \overline{\mathbb{R}}^{p\times n}_{\max}$. In this paper, we assume the system $\mathcal{S}$ is uncertain, i.e. the matrices have some entries which are random variables belonging to intervals. Thus, an uncertain MPL (uMPL) system is defined as

$$\mathcal{S}_u \ : \ \begin{cases} x(k) = A(k)x(k-1), \\ z(k) = C(k)x(k) \end{cases} \tag{2}$$

where $A(k) \in [A] = [\underline{A}, \overline{A}] \in \overline{\mathbb{IR}}^{n\times n}_{\max}$ and $C(k) \in [C] = [\underline{C}, \overline{C}] \in \overline{\mathbb{IR}}^{p\times n}_{\max}$ are nondeterministic matrices. Hence, $x(k) \in [\underline{A}x(k-1), \overline{A}x(k-1)] \subset \overline{\mathbb{R}}^n_{\max}$. We assume that the system works under FIFO (first in, first out) rule, i.e. $x(k) \ge x(k-1)$ which implies that $A(k) \ge I_n$ for all k.

In practice, many real systems can be described by an implicit equation as

$$x(k) = A_0(k)x(k) \oplus A_1(k)x(k-1). \tag{3}$$

Here, $A_0(k) \in [A_0] = [\underline{A}_0, \overline{A}_0] \in \overline{\mathbb{IR}}^{n\times n}_{\max}$ and $A_1(k) \in [A_1] = [\underline{A}_1, \overline{A}_1] \in \overline{\mathbb{IR}}^{n\times n}_{\max}$. To obtain an explicit representation, we can consider $x(k) = A(k)x(k-1)$, where $A(k) \in [A] = \left(\bigoplus_{k\in\mathbb{N}} [A_0]^{\otimes k}\right)[A_1]$. This representation serves as an over-approximation of the reachable space of (3) w.r.t. a given initial condition $x(0)$.

3.2 Image and inverse image computation

First, we consider a compact set $X_{k-1} \subset \mathbb{R}^n$, which is a single (bounded) max-plus polyhedron, i.e. a collection of finitely many DBMs (see Remark 2), and we define its image w.r.t. to $A(k) \in [A]$ as

$$\mathrm{Im}_{[A]}\{X_{k-1}\} = \{Ax \mid x \in X_{k-1}, A \in [A]\}, \tag{4}$$

i.e. as the set all states x that can be reached from $x(k-1) \in X_{k-1}$ through the dynamics equation. Then we consider the inverse image of $z(k)$ w.r.t. $C(k) \in [C]$, formally

$$\mathrm{Im}^{-1}_{[C]}\{z(k)\} = \{x \in \mathbb{R}^n \mid \exists C \in [C], Cx = z(k)\}, \tag{5}$$

i.e. the set of all x that can lead to $z(k)$ through the observation equation.

[4] Any nonautonomous MPL system can be converted into an autonomous one without loss of generality [7, Sec. 2.5].

3.3 Disjunctive approach

The details of the Piece Wise Affine partition of Max-Plus Linear systems and uncertain Max-Plus Linear systems can be found in [20,1,11,7,27]. In this paper, we consider that uMPL systems are also represented by affine dynamics in the standard algebra within partitions [5] of the state-space.

Following [1,11], the steps below are necessary to compute the image of set $X_{k-1} \subset \mathbb{R}^n$ and the inverse image of the point $z(k)$.

Image of $X_{k-1} \subset \mathbb{R}^n$:
Let $X_l \subset \mathbb{R}^n$ be a DBM, for all state-space partition:
(1) compute the Cartesian product of $\mathbb{R}^n$ and X_l;
(2) intersect the obtained DBM with the DBM generated by the state-space partition and the affine dynamics;
(3) compute the canonical form of the intersection;
(4) project the canonical-form representation over $x(k)$.
Since $X_{k-1} = \bigcup_{l=1}^{N} X_l$ then this procedure is repeated a finite number of times.

As X_{k-1} is represented by the collection of N DBMs, and we calculate the image of each DBM w.r.t. all state-space partitions (bounded by n^n), the total complexity required to compute $\mathrm{Im}_{[A]}\{X_{k-1}\}$ of (4) is $\mathcal{O}(Nn^{n+3})$.

Remark 4 *If we consider system* (3)*, the image of the compact set X_{k-1} (represented as the collection of finitely many DBMs) is given by $Im_{[A_0],[A_1]}\{X_{k-1}\} = \{(\bigoplus_{k\in\mathbb{N}} A_0^{\otimes k})\, A_1 x \mid x \in X_{k-1}, A_0 \in [A_0], A_1 \in [A_1]\}$. The complexity of this computation is $\mathcal{O}(N(2n)^{n+3})$ as demonstrated in [11].*

Inverse image of $z(k)$: Let $\mathcal{Z} \subset \mathbb{R}^n$ be a DBM representing $z(k)$:
(1) compute the Cartesian product of $\mathcal{Z}$ and $\mathbb{R}$;
(2) intersect the obtained DBM with the DBM generated by the measurement-space partition and the affine dynamics;
(3) compute the canonical form of the intersection;
(4) project the canonical-form representation over the variables $x(k-1)$.

Hence, to compute the inverse image [6] of $z(k)$ w.r.t. all measurement-space partitions (bounded by n^p) the complexity amounts to $\mathcal{O}(n^p(p+n)^3)$.

3.4 Concise approach

Based on Remark 2, it is possible to avoid the inherent disjunctive nature of the zones' collection using max-plus polyhedra. Based on this fact, we now study the use of a concise approach to obtain equivalent result with a lower complexity than the disjunctive approach. First, we recall that a bounded linear map is defined as $A : \overline{\mathbb{R}}^n_{\max} \to \overline{\mathbb{R}}^n_{\max}$ such that $\underline{A} \leq A \leq \overline{A}$. Hence $x \mapsto [\underline{A}x, \overline{A}x]$ and the following equivalences hold:

(1) $\alpha[A]x = [A](\alpha x) \Leftrightarrow \alpha A \otimes x = A(\alpha x),\ \forall A \in [A]$,
(2) $[A]x \oplus [A]y = [A](x \oplus y) \Leftrightarrow Ax \oplus Ay = A(x \oplus y),\ \forall A \in [A]$,

for all $\alpha \in \overline{\mathbb{R}}_{\max}$ and $x, y \in \overline{\mathbb{R}}^n_{\max}$. Moreover, let $[A] \circ V$ be the set $\{[A]v^i \mid v^i \in V\}$ for $V \subset \overline{\mathbb{R}}^n_{\max}$, more precisely $\{[A]v^1, \ldots, [A]v^p\}$ with each $[A]v^i$ an interval box of $\mathbb{I}\overline{\mathbb{R}}^n_{\max}$, i.e. a polytrope. For $w \in \overline{\mathbb{R}}^n_{\max}$ we have $w \in [A]v^i \Leftrightarrow w \in \mathcal{P}$, where $\mathcal{P}$ is the max-plus polyhedron in its $\mathcal{V}$-form represented by $\Phi = \{\underline{A}v^i, g^{(1)}(v^i), \ldots g^{(n)}(v^i)\}$ with $g^{(j)}(v^i) = (\underline{A}v^i_1, \underline{A}v^i_2, \ldots, \overline{A}v^i_j, \underline{A}v^i_{j+1}, \ldots, \underline{A}v^i_n)^T$.

Lemma 1 (Image of bounded polyhedra) *Let $A : \overline{\mathbb{R}}^n_{\max} \to \overline{\mathbb{R}}^n_{\max}$ be a bounded linear map such that $A \in [A]$ and $X \subset \overline{\mathbb{R}}^n_{\max}$ be the bounded max-plus polyhedron in its $\mathcal{V}$-form, which is generated by $V = \{v^1, \ldots, v^p\} \subset \overline{\mathbb{R}}^n_{\max}$, such that $x \in X \Leftrightarrow x = \bigoplus_{i=1}^p \lambda_i v^i$, $\bigoplus_{i=1}^p \lambda_i = e$. Then $Im_{[A]}\{X\}$ of* (4) *is given by a max-plus polyhedron in its $\mathcal{V}$-form generated by $\{\Phi_1, \ldots, \Phi_p\}$ where $\Phi_i = \{\phi^1(i), \ldots, \phi^{n+1}(i)\}$ is the set of $n+1$ generators that characterizes a box $[A]v^i$ for $v^i \in V$.*

PROOF. We want to compute the image of all $x \in X$. Then, $x' \in \mathrm{Im}_{[A]}\{X\} \Leftrightarrow x' \in \{Ax \mid x \in X, A \in [A]\}$ is rewritten as $x' = A\left(\bigoplus_{i=1}^p \lambda_i v^i\right) = \bigoplus_{i=1}^p \lambda_i (Av^i)$ $\bigoplus_{i=1}^p \lambda_i = e, A \in [A]$ since $A \in [A]$ then $Av^i \in [A]v^i$ and $Av^i = \bigoplus_{j=1}^{n+1} \beta_j^{(i)} \phi^j(i)$ with $\bigoplus_{i=1}^p \lambda_i = \bigoplus_{j=1}^{n+1} \beta_j^{(i)} = e$ and $\phi^j(i) \in \Phi_i$. Thus, $x' = \bigoplus_{i=1}^p \lambda_i \left(\bigoplus_{j=1}^{n+1} \beta_j^{(i)} \phi^j(i)\right)$, $\bigoplus_{i=1}^p \lambda_i = \bigoplus_{j=1}^{n+1} \beta_j^{(i)} = e$, i.e. $x' = \bigoplus_{k=1}^{p(n+1)} \alpha_k \xi^k$ where $\bigoplus_{k=1}^{p(n+1)} \alpha_k \xi^k = \bigoplus_{i=1}^p \bigoplus_{j=1}^{n+1} \lambda_i \beta_j^{(i)} \phi^j(i)$ with $\alpha_k = \lambda_i \beta_j^{(i)}$, $\xi^k \in \{\Phi_1, \ldots, \Phi_p\}$, $\bigoplus_{k=1}^{p(n+1)} \alpha_k = e$. ■

Since $X \subset \overline{\mathbb{R}}^n_{\max}$ is a bounded max-polyhedron, then its generators v^i (extreme points) lie in $(\overline{\mathbb{R}}_{\max} \setminus \{-\infty, +\infty\})^n$. It is not hard to see that the Av^i, and as consequence $\phi^j(i)$, also lies in $(\overline{\mathbb{R}}_{\max} \setminus \{-\infty, +\infty\})^n$ for all $A \in [A]$. Thus, $\mathrm{Im}_{[A]}\{X\}$ is clearly also a bounded max-plus polyhedron.

Procedure 2 *To compute the image of a bounded max-plus polyhedron X_{k-1}, in its $\mathcal{V}$-form, w.r.t. the dynamics of the uMPL system of* (2)*, the following steps are necessary*

- *Compute the image of X_{k-1}, denoted $Im_{[A]}\{X_{k-1}\}$, based on Lemma 1;*

[5] The number of partitions is bounded by n^n and n^p for the state-space and measurement-space, respectively.
[6] In [10], the authors use disjoint hyperrectangles to represent (5) which is computed with complexity $\mathcal{O}(pn^{p+1})$.

- *Remove the redundant generators of the obtained set (see Procedure 1).*

If X_{k-1} is generated by $V = \{v^1, \ldots, v^p\} \subset \mathbb{R}^n$ in its $\mathcal{V}$-form, then the computation of its image is done with complexity $\mathcal{O}(n^3p^2)$. To prove this result, we verify that this image is characterized by a generating matrix of size $(n+1) \times p(n+1)$. Finally, to remove redundant generators of the set that represents this image, we apply a procedure with complexity $\mathcal{O}(n^3p^2)$ (see Procedure 1).

Remark 5 *The image of a bounded max-plus polyhedron X_{k-1} w.r.t. the dynamics of practical systems* (3)*, formally $Im_{[A_0],[A_1]}\{X_{k-1}\} = \{(\bigoplus_{k\in\mathbb{N}} A_0^{\otimes k})\, A_1 x \mid x \in X_{k-1}, A_0 \in [A_0], A_1 \in [A_1]\}$, can also be handled using the concise approach but in a conservative way. The main idea is to use* (3) *to define a bounded linear map $x \mapsto Hx$, $H \in [H] = ([A_0]\ [A_1])$ and then apply the Procedure 2.*

Example 2 *Let $A \in [A] = \begin{pmatrix} [4,6] & [3,5] \\ [3,7] & [4,5] \end{pmatrix}$ and $X_0 = \{x \mid 0 \le x_1 \le 1, 1 \le x_2 \le 3\}$. Then, to compute the image of X_0 we first write the following $x \in X_0 \Leftrightarrow x = \lambda_1 \begin{pmatrix} e \\ 1 \end{pmatrix} \oplus \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \lambda_3 \begin{pmatrix} e \\ 3 \end{pmatrix}$, $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 = e$. For each generator (extreme point) representing X_0, we calculate its image in the form of a polytrope w.r.t. $[A]$, more precisely $\Phi_1 = \left\{ \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \end{pmatrix} \right\}, \Phi_2 = \left\{ \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \end{pmatrix} \right\}, \Phi_3 = \left\{ \begin{pmatrix} 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 \\ 7 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \end{pmatrix} \right\}$. Finally the image of X_0 is $\{\Phi_1, \Phi_2, \Phi_3\}$(by removing redundant generators - Procedure 1) and the following holds $x' \in Im_{[A]}\{X_0\} \Leftrightarrow x' = \mu_1 \begin{pmatrix} 8 \\ 7 \end{pmatrix} \oplus \mu_2 \begin{pmatrix} 4 \\ 5 \end{pmatrix} \oplus \mu_3 \begin{pmatrix} 5 \\ 8 \end{pmatrix} \oplus \mu_4 \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $\mu_1 \oplus \mu_2 \oplus \mu_3 \oplus \mu_4 = e$. The result is graphically detailed in Figure 1. For the sake of comparison, this max-plus polyhedron is also obtained by the collection of the following two zones $\{x' \mid 4 \le x_1' \le 7, 5 \le x_2' \le 8, -2 \le x_2' - x_1' \le 3\} \cup \{x' \in \mathbb{R}^n \mid 4 \le x_1' \le 8, 5 \le x_2' \le 8, -1 \le x_2' - x_1' \le 3\}$.*

To obtain the inverse image of a point z w.r.t. the observation of the system (2), we recall that $x' \in \mathrm{Im}^{-1}_{[C]}\{z\} \Leftrightarrow \underline{C}x' \le z \le \overline{C}x'$.

Lemma 2 ([13]) *The set $Im^{-1}_{[C]}\{z\}$ of* (5) *is equivalent to the following max-plus polyhedron in its $\mathcal{H}$-form $x' \in Im^{-1}_{[C]}\{z\} \Leftrightarrow \begin{pmatrix} \mathcal{E}_{p\times n} \\ b \end{pmatrix} x' \oplus \begin{pmatrix} E_{p\times 1} \\ \varepsilon \end{pmatrix} \le \begin{pmatrix} d \\ \mathcal{E}_{1\times n} \end{pmatrix} x' \oplus \begin{pmatrix} \mathcal{E}_{p\times 1} \\ e \end{pmatrix}$ where $b = (-\underline{C}^\sharp(z))^T$ and $d = \mathtt{diag}_\oplus(-z)\overline{C}$.*

Example 3 *Let $C(k) \in [C] = \begin{pmatrix} [1,3] & [0,2] \end{pmatrix}$ and $x' = (6,7)^T$ and $C(1) = (2\ 1)$. Then, the set $Im^{-1}_{[C]}\{z = C(1)x'\}$ is represented by the following max-plus polyhedron in its $\mathcal{H}$-form $x' \in Im^{-1}_{[C]}\{z\} \Leftrightarrow \begin{pmatrix} \varepsilon & \varepsilon \\ -7 & -8 \end{pmatrix} x' \oplus \begin{pmatrix} e \\ \varepsilon \end{pmatrix} \le \begin{pmatrix} -5 & -6 \\ \varepsilon & \varepsilon \end{pmatrix} x' \oplus \begin{pmatrix} \varepsilon \\ e \end{pmatrix}$ or equivalently in its $\mathcal{V}$-form such that $x' = \gamma_1 \begin{pmatrix} 7 \\ \varepsilon \end{pmatrix} \oplus \gamma_2 \begin{pmatrix} \varepsilon \\ 8 \end{pmatrix} \oplus \gamma_3 \begin{pmatrix} 5 \\ \varepsilon \end{pmatrix} \oplus \gamma_4 \begin{pmatrix} \varepsilon \\ 6 \end{pmatrix}$, $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4 = e$. In the Figure 1, these extreme points are easily verified. For the sake of comparison, this max-plus polyhedron is also obtained by the collection of the following two zones $\{x' \mid 5 \le x_1' \le 7, x_2' \le 8, x_2' - x_1' \le 1\} \cup \{x' \in \mathbb{R}^n \mid x_1' \le 7, 6 \le x_2' \le 8, 1 \le x_2' - x_1'\}$.*

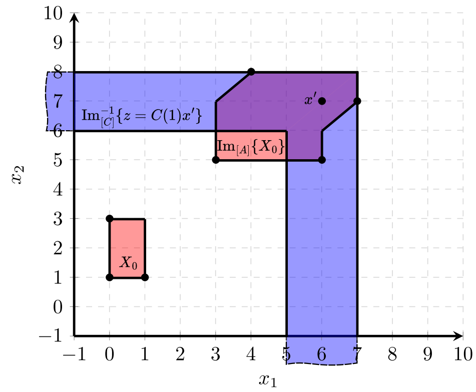

Fig. 1. Computation of $\mathrm{Im}_{[A]}\{X_0\}$ of Example 2 and of $\mathrm{Im}^{-1}_{[C]}\{z\}$ of Example and 3.

4 Application: set-estimation

In a stochastic estimation approach, the uncertain state vector x is characterized by probability density functions (PDFs). Conversely, in set estimation approach, x is characterized by a set X such that $x \in X$. Both approaches are related by the fact that X represents the support of the PDF that represents x. Furthermore, handling sets is easier than computing PDFs for uMPL systems mainly because the Bayesian filtering is not fully addressed (see [25]). Of course, PDFs provide more accuracy rather than simply obtaining their support, but their computation is intractable.

Definition 3 (Conditional reach set) *Let $X_0 \subset \mathbb{R}^n$ be a compact set of initial conditions such that $x(0) \in X_0$. Given that $z(k)$ for $k \ge 0$ is known, then $x(k) \in X_{k|k}$ corresponds to the conditional reach set from $x(k-1) \in X_{k-1|k-1}$ (assuming that $X_{0|0} = X_0$) obtained via the dynamics of* (2)*, which leads to $z(k)$ via the observation of* (2).

The interpretation of Definition 3 is given below

$$x(k) \in X_{k|k} = X_{k|k-1} \cap \tilde{X}_{k|k}, \tag{6}$$

where $X_{k|k-1} = \text{Im}_{[A]}\{X_{k-1|k-1}\}$ as given by (4) is the support of the *prior* PDF $p(x(k)|z(1), \ldots, z(k-1))$ computed thanks to the support of $p(x(k-1)|z(1), \ldots, z(k-1))$, i.e. $X_{k-1|k-1}$, by using the dynamics equation and the *Chapman-Kolmogorov* equation based on all information available at the event step $k-1$. In the correction stage, the support of the *posterior* PDF $p(x(k)|z(1), \ldots, z(k-1), z(k))$ is obtained by correcting the support of the *posterior* PDF, by using the new measurement $z(k)$ and observation equation (measurement *likelihood*), i.e. by intersecting $X_{k|k-1}$ with $\tilde{X}_{k|k} = \text{Im}^{-1}_{[C]}\{z(k)\}$ (given by (5) with complexity $\mathcal{O}(\max(np, p^3))$ due to matrix operations). Therefore, $X_{k|k}$ (which represents the set-estimation version of the Bayes rule) is always as a bounded max-plus polyhedron. This is because $X_{k|k-1}$ is also bounded, and when it is intersected with $\tilde{X}_{k|k}$, the resulting set is always bounded even if $\tilde{X}_{k|k}$ itself is unbounded. In other words, $X_{k|k}$ is a compact set of $\mathbb{R}^n$ representing the support of the *posterior* PDF, which contains all the information about the stochastic state $x(k)$ and is efficiently computed using Remark 3 with the aid of the library [2] (the complexity is exponential in the dimension n as stated in Remark 1).

Example 4 *Let us consider (2) with the following matrices* $A(k) \in [A] = \begin{pmatrix} [1,3] & [3,4] \\ [2,3] & [2,4] \end{pmatrix}$ *and* $C(k) \in [C] = \begin{pmatrix} [1,3] & [1.5, 2.5] \\ [1,1] & [1,3] \end{pmatrix}$. *A simulation of the system yields numerical values for* $x(k)$ *and* $z(k)$ *for* $k \in \{0,1,2,3\}$. *Let us consider the following compact set of initial conditions* $X_{0|0} = X_0 = \{x \in \mathbb{R}^2 \mid 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$ *such that* $x(0) = (0.06,\ 0.02)^T \in X_0$. *This set is the following bounded max-plus polyhedron in its* $\mathcal{V}$*-form (it is an interval, thus a polytrope)* $x \in X_{0|0} \Leftrightarrow x = \lambda_1 \begin{pmatrix} e \\ e \end{pmatrix} \oplus \lambda_2 \begin{pmatrix} e \\ 1 \end{pmatrix} \oplus \lambda_3 \begin{pmatrix} 1 \\ e \end{pmatrix}$, $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 = e$. *The first step consists in computing* $X_{1|0} = Im_{[A]}\{X_{0|0}\}$ *as given by (4) by using Procedure 2. This set corresponds to the* prior *PDF of the state* $p(x(1)|x(0))$. *It is equal to the following bounded max-plus polyhedron in its* $\mathcal{V}$*-form* $x' \in X_{1|0} \Leftrightarrow x' = \mu_1 \begin{pmatrix} 4 \\ 5 \end{pmatrix} \oplus \mu_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \oplus \mu_3 \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mu_1 \oplus \mu_2 \oplus \mu_3 = e$. *Finally, we are able to compute the conditional reach set* $X_{1|1} = X_{1|0} \cap \tilde{X}_{1|1}$ *using the measurement* likelihood, *i.e. the information of* $z(1)$, *which is* $\tilde{X}_{1|1} = Im^{-1}_{[C]}\{z(1) = (6.14, 6.34)^T\}$. $X_{1|1}$ *corresponds to the support of the PDF* $p(x(1)|z(1))$, *i.e. the correction of the support of the* prior *PDF computed in the first step. The following max-plus polyhedron in its* $\mathcal{H}$*-form, represents* $\tilde{X}_{1|1}$ $x' \in \tilde{X}_{1|1} \Leftrightarrow \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ -5.14 & -4.64 \end{pmatrix} x' \oplus \begin{pmatrix} e \\ e \\ \varepsilon \end{pmatrix} \le \begin{pmatrix} -3.14 & -3.64 \\ -5.34 & -3.34 \\ \varepsilon & \varepsilon \end{pmatrix} x' \oplus \begin{pmatrix} \varepsilon \\ \varepsilon \\ e \end{pmatrix}$. *Then, we compute the* $\mathcal{V}$*-form of* $X_{1|1}$ *using the idea of Remark 3, i.e. using the extremes points that define* $X_{1|0}$ *as initial generating set of the algorithm derived from Theorem 1.*

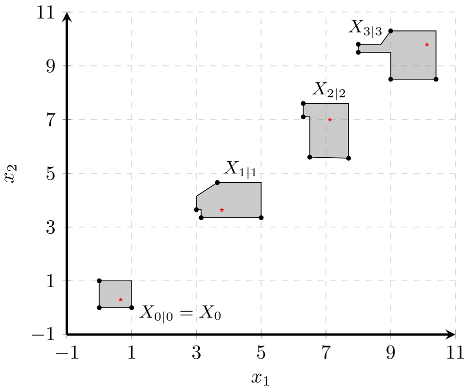

Fig. 2. Calculation of $X_{k|k}$ for $k \in \{1,2,3\}$ of Example 4. The red crosses and the black dots represent the true state-vector $x(k)$ and the extreme points of $X_{k|k}$, respectively.

The conditional reach sets $X_{k|k}$ *for* $k \in \{1,2,3,4\}$ *are depicted in Figure 2. Precisely for* $k = 1$ *we have* $x' \in X_{1|1} \Leftrightarrow x' = \gamma_1 \begin{pmatrix} 5 \\ 3.34 \end{pmatrix} \oplus \gamma_2 \begin{pmatrix} 3.64 \\ 4.64 \end{pmatrix} \oplus \gamma_3 \begin{pmatrix} 3.14 \\ 3.34 \end{pmatrix} \oplus \gamma_4 \begin{pmatrix} 3 \\ 3.64 \end{pmatrix}$, $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4 = e$.

4.1 Complexity analysis

The Table 1 compares the complexity of the disjunctive and concise approaches. As a notation, we have:

- N as the number of DBM to represent $X_{k-1|k-1}$;
- N_1 and N_2 as the number of DBM to represent $X_{k|k-1}$ and $\tilde{X}_{k|k}$, respectively;
- G the number of extreme points that generate the max-plus polyhedron $X_{k-1|k-1}$;
- $p = n$, the dimensions of the state and measurement-spaces are equal.

Proc.	$X_{k\|k-1}$	$\tilde{X}_{k\|k}$	$X_{k\|k}$
Disj.	$\mathcal{O}(Nn^{n+3})$	$\mathcal{O}(n^{n+3})$	$\mathcal{O}(N_1 N_2 n^3)$
Conc.	$\mathcal{O}(G^2 n^3)$	$\mathcal{O}(n^3)$	$\mathcal{O}(n^{n+3})$

Table 1
Comparison of the complexity.

To summarize both approaches have theoretical exponential complexity, nevertheless the efficiency of the concise approach in computing $X_{k|k-1}$ (polynomial time) leads to a more efficient practical complexity to compute $X_{k|k}$.

5 Conclusion

Although computing the *posterior* probability density function (PDF), which is essential in Bayesian formulation, is not feasible in practice for uMPL systems with stochastic processing times, we have presented two approaches to compute its support as a set: the disjunctive approach and the concise approach. Both approaches can be cumbersome in worst-case scenarios, but the concise approach computes the support of the *prior* PDF (prediction state) faster. It is worth noting that this support can be utilized in Particle filter algorithms, as demonstrated in [11,10]. Additionally, this support is useful for validating stochastic filtering algorithms, such as the one proposed in [25,14]. In the future, it could be interesting to develop a more efficient procedure to compute the correction stage, i.e. on obtaining the support of the *posterior* PDF in a faster manner, making it possible to enhance the scalability for large dimensional systems.

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