An overview on control theory for (max,plus) linear systems,

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Angers, along the Loire Valley

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- 1H30 in south west of Paris
- 350000 people, 16th city of France
- 1 State university, with 32000 students,
-  Denis Papin graduated from University of Angers in 1669
-  Carlos Andrey Maia graduated in 2004

Links between UFMG and UA, more than 10 years old

UFMG - UA History in some points

- PhD of Carlos Andrey Maia, in [2001-2004]
- CAPES - Cofecub Project [2008-2012]
- Brafitec Project [2012-2016]
- Vinicius Mariano Gonçalves PhD defense, [2013-2014] in Angers

Alexander Grothendieck 28/03/1928 - 14/11/2014

- Fields medal 1966
- Member of the Nicolas Bourbaki Seminar

Outline

- Introductive examples
- (max,+) algebra in few words
- Discrete event systems in (max,+) algebra
- Optimal control on a finite horizon
- Optimal filtering
- Optimal closed loop control, how to take into account disturbances ?
- Sub observer synthesis
- Interval arithmetic and semiring, robust controller synthesis

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(max,+) algebra in few words

Idempotent Semiring

- Sum $\oplus$, associative,commutative, zero element denoted ε,
- Product $\otimes$, associative, identity element denoted e,
- Product $\otimes$ distributes with respect of sum, $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$,
- Zero element ε is absorbing, $a \otimes \varepsilon = \varepsilon$
- The sum is idempotent, $a \oplus a = a$.
- $a \oplus b = a \Leftrightarrow b \preceq a \Leftrightarrow a \wedge b = b$
  hence a semiring has a complete lattice structure, with $(\varepsilon = \perp)$ as bottom element and $(\top = \bigoplus_{x \in \mathcal{S}} x)$ as top element.

Example :(max,+) algebra, $\overline{\mathbb{Z}}_{max}$, Cuninghame-Green 1962

More

Sum $\oplus$ is the operator max, product $\otimes$ is classical sum $+$, $\varepsilon = -\infty$ and $e = 0$, then :

$$1 \oplus 1 = 1 = max(1,1),$$
$$2 \otimes 1 = 3 = 2 + 1.$$

(max,+) algebra in few words

Residuation Theory (Croisot 1956, Blyth 1972)

It allows to define a kind of inverse for order preserving mapping, defined over ordered sets. The idempotent law $\oplus$ induces that a semi ring has an order structure, *i.e.*, $a \oplus b = a \Leftrightarrow b \preceq a$, then this theory is suitable to consider inversion problem over idempotent semi ring.

Inequality $a \otimes x \preceq b$

Over a complete idempotent semi ring, inequality $a \otimes x \preceq b$ admits a greatest solution , denoted, $x = a \backslash\!\!\!\circ b$.

Example : (max,+) algebra $\mathbb{Z}_{max}$

Inequality $5 \otimes x \preceq 3$ admits a greatest solution $x = 5 \backslash\!\!\!\circ 3 = 3 - 5 = 2$. It achieves equality in the scalar case.

(max,+) algebra in few words

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(max,+) algebra in few words

Fixed point equation $x = ax \oplus b$

More

Theorem : Over a complete idempotent semi ring $\mathcal{T}$, the least solution to $x = ax \oplus b$ is $x = a^* b$ with $a^* = \bigoplus_{i \geq 0} a^i = e \oplus a \oplus a^2 \oplus \ldots$

$*$ is called Kleene star operator.

We will denote $\mathcal{M}$ the mapping defined over $\mathcal{T}$ s.t. $\mathcal{M} : x \mapsto x^*$

Inequality $x^* \preceq a$

More

Theorem : Mapping $\mathcal{M}$ restricted to its image is residuated.
Application : If $a \in \mathrm{Im}\mathcal{M}$ then inequality $x^* \preceq a$ admits a greatest solution $x = a^*$

(max,+) algebra in few words

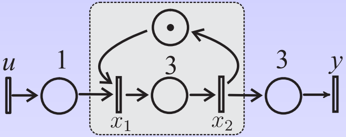
Matrix

Let A, B, C three matrices in $\overline{\mathbb{Z}}_{\max}^{n \times n}$

- $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$
- $(A \otimes B)_{ik} = \bigoplus_{j=1 \ldots n} (A_{ij} \otimes B_{jk})$
- $(A \backslash\!\!\circ B)_{ik} = \bigwedge_{j=1 \ldots n} (A_{ji} \backslash\!\!\circ B_{jk})$, where $A \backslash\!\!\circ B$ is the greatest such $AX \preceq B$

More

TEG Model in $\overline{\mathbb{Z}}_{max}$



Firing Date [Cohen et al., 85]

$x_i(k)$: date of the firing numbered k for the transition labelled i.

For each transition :

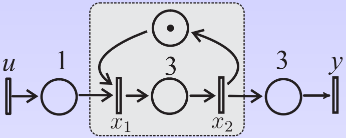
$$\begin{aligned} x_1(k) &= \max(1 + u(k), x_2(k-1)) \\ x_2(k) &= 3 + x_1(k) \\ y(k) &= 3 + x_2(k) \end{aligned}$$

In $\overline{\mathbb{Z}}_{max}$

$\overline{\mathbb{Z}}_{max}$:

$$\begin{aligned} x_1(k) &= 1 \otimes u(k) \oplus x_2(k-1) \\ x_2(k) &= 3 \otimes x_1(k) \\ y(k) &= 3 \otimes x_2(k) \end{aligned}$$

TEG Model in $\overline{\mathbb{Z}}_{max}$



Firing Date [Cohen et al., 85]

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Dynamic Model

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

TEG Model in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

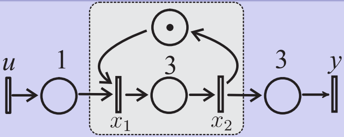
γ transform [Cohen et al,89]

▸ More

γ transform of $x(k)$ is a formal series $x(\gamma) = \bigoplus_{k\in\mathbb{Z}} \gamma^k x(k)$.

γ-operator is like a backward shift operator in the event domain,
$y(\gamma) = \gamma x(\gamma) \Leftrightarrow y(k) = x(k-1) \forall k$.

The previous system in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$:

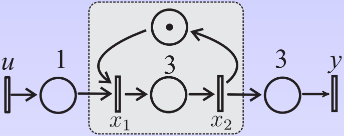


$$\begin{array}{rcl} x_1(\gamma) & = & 1u(\gamma) \oplus \gamma x_2(\gamma) \\ x_2(\gamma) & = & 3x_1(\gamma) \\ y(\gamma) & = & 3x_2(\gamma) \end{array}$$

General case , transfer matrix

$$\left\{ \begin{array}{rcl} x(\gamma) & = & Ax(\gamma) \oplus Bu(\gamma) \\ y(\gamma) & = & Cx(\gamma) \end{array} \right. \quad \Rightarrow \quad y = CA^* Bu = Hu$$

TEG Model in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$



The previous system in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

$$
\begin{aligned}
x(\gamma) &= Ax(\gamma) \oplus Bu(\gamma) &&= \begin{pmatrix} \varepsilon & \gamma \\ 3 & \varepsilon \end{pmatrix} x(\gamma) \oplus \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} u(\gamma) \\
y(\gamma) &= Cx(\gamma) &&= \begin{pmatrix} \varepsilon & 3 \end{pmatrix} x(\gamma)
\end{aligned}
$$

The previous system in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

More

$$
\begin{aligned}
x(\gamma) &= A^*B &&= \begin{pmatrix} (3\gamma)^* & \gamma(3\gamma)^* \\ 3(3\gamma)^* & (3\gamma)^* \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} u(\gamma) \\
y(\gamma) &= CA^*B &&= \begin{pmatrix} 7(3\gamma)^* \end{pmatrix} u(\gamma)
\end{aligned}
$$

Operations over $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

Operations over $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

- $s = s_1 \oplus s_2$ ▸ More
- $s = s_1 \otimes s_2$ ▸ More
- $s = s_1 \wedge s_2$, asymptotic slope $\sigma_\infty(s) = max(\sigma_\infty(s_1), \sigma_\infty(s_2))$
- s^*, asymptotic slope $\sigma_\infty(s^*) = min(\sigma_\infty((p_i)), \sigma_\infty(q_i), \sigma_\infty(r))$
- $s = s_1 \between s_2$, if $\sigma_\infty(s_2) \leq \sigma_\infty(s_1)$ then $\sigma_\infty(s) = \sigma_\infty(s_2)$, else $s = \varepsilon$

Optimal control : IEEE TAC, Cohen et al. 1989

Problem Formulation :

Let z be a desired output. Is it possible to a control input in order to get an output y as close as possible to z while respecting the constraint : $y \preceq z$.

Solution

The optimal input is given by :

$$u_{opt} = (CA^*B) \backslash\!\!\circ\, z$$

It is the greatest input which achieves the inequality :

$$y = (CA^*B)u_{opt} \preceq z$$

In manufacturing setting z corresponds to the customer demands, u the input of raw parts in the system, y the output of processed parts. Optimal control u_{opt} is the one which minimizes the internal stock while ensuring the customer demand is honored.

Optimal control : IEEE TAC, Cohen et al. 1989

Practical computation :

Residuation of matrices of series.... or In (max,plus) algebra it can be formally expressed as :

$$\zeta(k) = A \backslash\!\!\circ\, \zeta(k+1) \wedge C \backslash\!\!\circ\, z(k)$$

$$u(k) = B \backslash\!\!\circ\, \zeta(k)$$

It is a back tracking computation , very reminiscent to the optimal control for classical dynamical systems. $\zeta(k)$ is the adjoint state of the system.

Drawback :

Desired output has to be known a priori, or a receding horizon has to be performed.

Optimal Filtering, model matching problem : JESA, Hardouin et al. 1995

Problem Formulation :

Let H be a matrix of series describing a desired behavior, it is a reference model to achieve. Let CA^*B be the transfer of the system to be controlled, and Pa filter such that $y = CA^*B \otimes P \otimes u$. The optimal filter such that $CA^*BPu \preceq Hu, \ \forall u$ is given by :

$$P_{opt} = (CA^*B) \circ\!\!\!\!\backslash H$$

In manufacturing setting it is the one which delays as much as possible the input while ensuring that the output $y \preceq Hu \forall u$.

Optimal Filtering, model matching problem : JESA, Hardouin et al. 1995

Particular case : Neutral controller

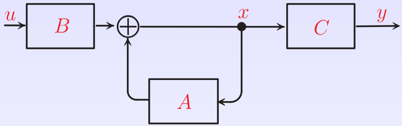
The model to match $H = CA^*B$, this means the output is not modified by the filter P but the input is delayed as much as possible.

Drawback :

it is also on open loop control, it doesn't take into account the possible disturbances acting on the system.

State Feedback controller synthesis

[Cottenceau 1999, Maia 2003, Lhommeau 2003]



system Equation :

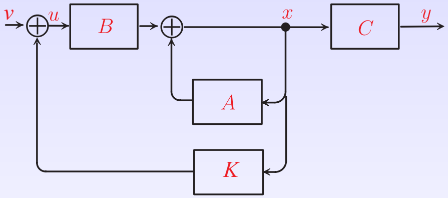
$$\begin{cases} x = Ax \oplus Bu \\ y = Cx \end{cases}$$

input Output transfer function

$$y = CA^*Bu$$

State Feedback controller synthesis

[Cottenceau 1999, Maia 2003, Lhommeau 2003]



Controlled system :

$$\begin{cases} x = Ax \oplus B(v \oplus Kx) \\ y = Cx \end{cases}$$

Closed loop system transfer function :

$$y = C(A \oplus BK)^* Bv$$

Objective :

Compute the greatest controllet K such that : $C(A \oplus BK)^* B \preceq G_{ref}$

State Feedback controller synthesis

Optimal State Feedback Controller :

More

- Objective :

$$C(A \oplus BK)^* B \preceq G_{ref} \Leftrightarrow (A \oplus BK)^* B \preceq A^* B((CA^* B) \diamond\!\!\!\backslash\, G_{ref})$$

- $\hat{\mathcal{G}} = A^* B((CA^* B) \diamond\!\!\!\backslash\, G_{ref})$ is the greatest transfer between u and x while preserving the constraint and belonging to $\mathrm{Im} A^* B$ (i.e. which is achievable).

State Feedback controller synthesis

Proposition :

If $G_{ref} \succeq CA^*B = H$ then the greatest controller K ensuring $(A \oplus BK)\mathcal{G} \subset \mathcal{G}$ is given by :

$$\hat{K} = B \circ\!\!\!\!\setminus \hat{\mathcal{G}} \circ\!\!\!\!/ \hat{\mathcal{G}}$$

Particular case : The neutral controller

If $G_{ref} = CA^*B$, the greatest controller is :

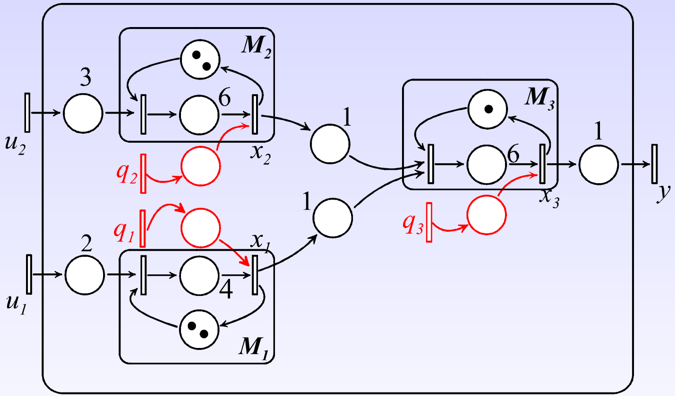
$$\hat{K} = (CA^*B) \circ\!\!\!\!\setminus (CA^*B) \circ\!\!\!\!/ (A^*B)$$

it is the one which delays as much as possible the input while preserving the output.

Application : a kind of disturbances decoupling problem

Lagrange 2002, Lhommeau 2003

Problem statement :



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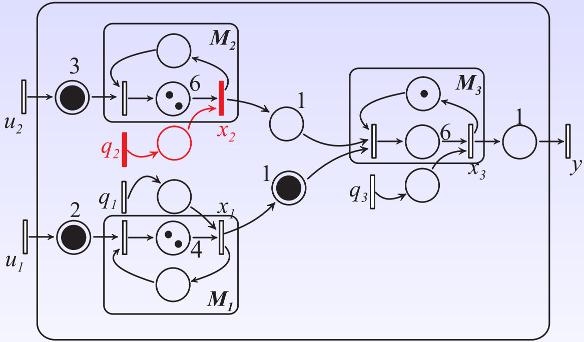

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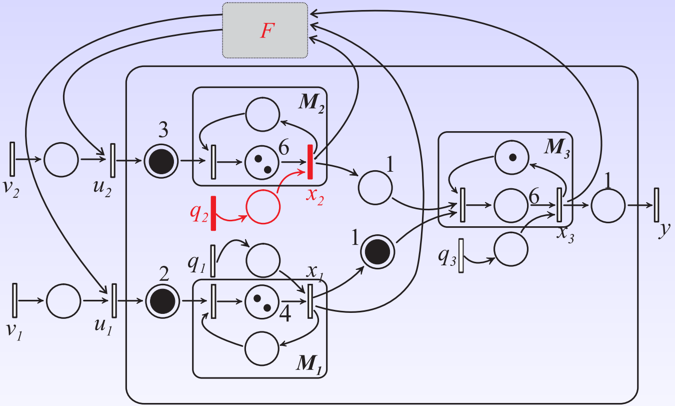

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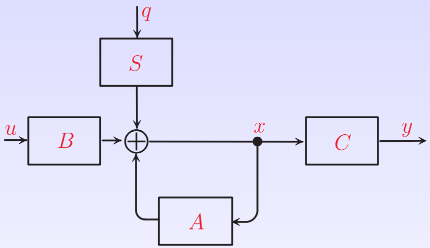

State model :

$$\begin{aligned} x &= Ax \oplus Bu \oplus Sq \\ y &= Cx \end{aligned}$$

Transfer Relation :

$$y = CA^*Bu \oplus CA^*Sq$$

Objective :

Compute the greatest controller $u = Fx$ such that

$$\begin{aligned} y &= C(A \oplus BF)^*Sq \\ &= CA^*Sq \end{aligned}$$

Application : a kind of disturbances decoupling problem

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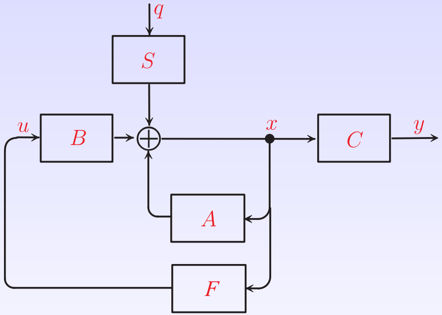

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Application : a kind of disturbances decoupling problem

Objective :

definition

$$(A \oplus BF)^* S \overset{\ker C}{\equiv} A^* S$$

The greatest element of $[A^*S]_C$ [Cohen et al. 96] :

$$\Pi^C(A^*S) = C \backslash CA^*S$$

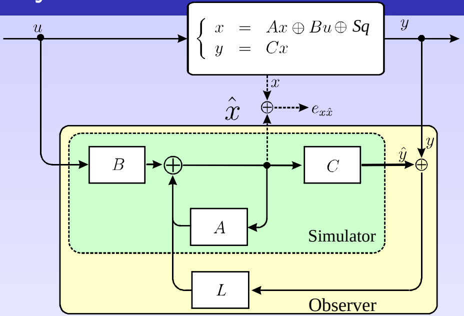
The greatest ideal principal A-invariant ($A\mathcal{V} \subset \mathcal{V}$) included in $[A^*S]_C$:

$$\mathcal{G} = \left\{ x | x \preceq \hat{\mathcal{G}} = A^* \backslash \Pi^C(A^*S) \right\}$$

The greatest controller F such that $(A \oplus BF)\mathcal{G} \subset \mathcal{G}$:

$$\hat{F} = B \backslash \hat{\mathcal{G}} / \hat{\mathcal{G}}$$

Sub Observer Synthesis : Hardouin et al. IEEE TAC 2010

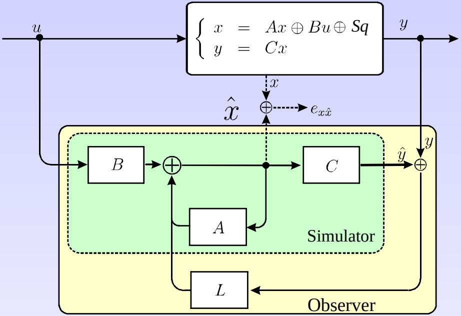


Objective :

Compute the greatest observer matrix L such that

$$\hat{x} \preceq x.$$

Sub Observer Synthesis :



System Equations :

Matrix S

$$
\begin{aligned}
x &= Ax \oplus Bu \oplus Sq = A^* Bu \oplus A^* Sq \\
y &= Cx = CA^* Bu \oplus CA^* Sq.
\end{aligned}
$$

Estimated State Equations :

$$
\begin{aligned}
\hat{x} &= A\hat{x} \oplus Bu \oplus L(\hat{y} \oplus y) \\
\hat{y} &= C\hat{x}.
\end{aligned}
$$

Sub Observer Synthesis :

Constraints Satisfaction :

Compute the greatest observer matrix L such that

$$
\begin{array}{lcll}
(A \oplus LC)^* Bu & \preceq & A^* Bu & \forall u \\
(A \oplus LC)^* LCA^* Sq & \preceq & A^* Sq & \forall q,
\end{array}
$$

Constraints Satisfaction :

Compute the greatest matrix L such that

$$
\begin{array}{lcl}
(A \oplus LC)^* B & \preceq & A^* B \Leftrightarrow L \preceq (A^* B) \phi (CA^* B) \\
(A \oplus LC)^* LCA^* S & \preceq & A^* S \Leftrightarrow L \preceq (A^* S) \phi (CA^* S).
\end{array}
$$

Sub Observer Synthesis :

Optimal Matrix :

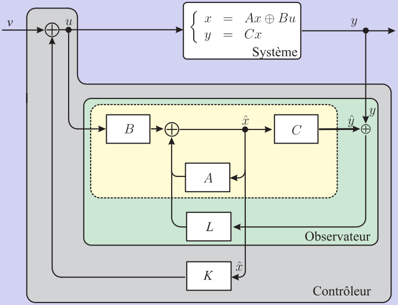
$$L_{opt} = ((A^*B) \phi (CA^*B)) \wedge ((A^*S) \phi (CA^*S))$$

is the greatest such that

$$\hat{x} \preceq x.$$

Control with an observer :

Principe :



Transfer Relation :

$$\begin{aligned} x &= A^*B(K(A \oplus LC)^*B)^*v \\ y &= CA^*B(K(A \oplus LC)^*B)^*v. \end{aligned}$$

Control with an observer :

Objective :

$$\begin{array}{rcl} CA^*B(K(A \oplus LC)^*B)^* & \preceq & G_{ref} \\ \Leftrightarrow A^*B(K(A \oplus LC)^*B)^* & \preceq & A^*B((CA^*B) \lcirclearrowleft G_{ref}) = \hat{\mathcal{G}} \end{array}$$

By considering the principal ideal : $\mathcal{G} = \left\{ x | x \preceq \hat{\mathcal{G}} \right\}$

Controller $\hat{K}$:

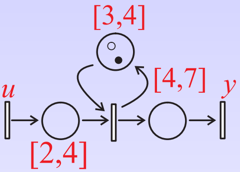
$$\hat{K} = B \lcirclearrowleft \hat{\mathcal{G}} \phi \hat{\mathcal{G}} \phi (A \oplus LC)^*$$

Properties :

Control strategy using both L_{opt} and $\hat{K}$ is better (from the just in time point of view) than the one using only the output [Cottenceau et al. 99].

$(max, +)$-linear systems with uncertain parameters

[Lhommeau 2003]



This TEG can be represented as $\mathrm{I}(\overline{\mathbb{Z}}_{\mathsf{max}}[\![\gamma]\!])$

$$\begin{array}{rcl} x(\gamma) & = & [3\gamma^2, 4\gamma]x(\gamma) \oplus [2,4]u(\gamma) \\ y(\gamma) & = & [4,7]x(\gamma) \end{array}$$

The input/output transfer relation is given by :

$$y(\gamma) = [6(3\gamma^2)^*, 11(4\gamma)^*]u(\gamma) = [\underline{H}, \overline{H}]u(\gamma) = \mathbf{H}u(\gamma)$$

All the transfer relations are included in interval $[\underline{H}, \overline{H}]$.

$(max, +)$-linear systems with uncertain parameters

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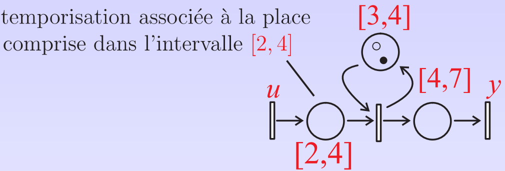

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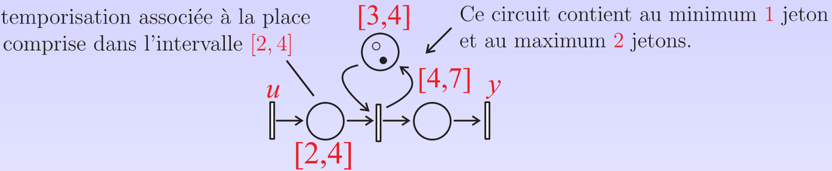

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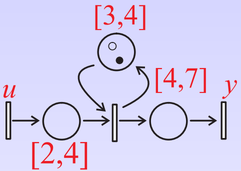

This TEG can be represented as $\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])$

$$\begin{aligned} x(\gamma) &= [3\gamma^2, 4\gamma]x(\gamma) \oplus [2,4]u(\gamma) \\ y(\gamma) &= [4,7]x(\gamma) \end{aligned}$$

The input/output transfer relation is given by :

$$y(\gamma) = [6(3\gamma^2)^*, 11(4\gamma)^*]u(\gamma) = [\underline{H}, \overline{H}]u(\gamma) = \mathbf{H}u(\gamma)$$

All the transfer relations are included in interval $[\underline{H}, \overline{H}]$.

Residuation theory over idempotent semiring

Order relation in $\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])$

$$\mathbf{a} = [\underline{a}, \overline{a}] \preceq_{\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])} \mathbf{b} = [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} \preceq_{\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]} \underline{b} \text{ et } \overline{a} \preceq_{\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]} \overline{b}$$

Residuation over $\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])$

The mapping $\mathbf{L_a} : \mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]) \to \mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]), \mathbf{x} \mapsto \mathbf{a} \overline{\otimes} \mathbf{x}$ is residuated. The residual mapping $\mathbf{L_a^\sharp}$ is given by

$$\mathbf{L_a^\sharp}(\mathbf{b}) = \mathbf{a} \overline{\circ\!\!\!\setminus} \mathbf{b} = [\underline{a} \circ\!\!\!\setminus \underline{b} \wedge \overline{a} \circ\!\!\!\setminus \overline{b}, \overline{a} \circ\!\!\!\setminus \overline{b}].$$

$\mathbf{x}$ is the greatest interval such that :

$$\mathbf{ax} \preceq_{\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])} \mathbf{b}$$

Residuation theory over idempotent semiring

Order relation in $\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])$

$$\mathbf{a} = [\underline{a}, \overline{a}] \preceq_{\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])} \mathbf{b} = [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} \preceq_{\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]} \underline{b} \text{ et } \overline{a} \preceq_{\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]} \overline{b}$$

Residuation over $\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])$

The mapping $\mathbf{L_a} : \mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]) \to \mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]), \mathbf{x} \mapsto \mathbf{a} \overline{\otimes} \mathbf{x}$ is residuated. The residual mapping $\mathbf{L}^\sharp_\mathbf{a}$ is given by

$$\mathbf{L}^\sharp_\mathbf{a}(\mathbf{b}) \quad = \quad \mathbf{a} \overline{\backslash\!\!\!\circ} \mathbf{b} = [\underline{a} \backslash\!\!\!\circ \underline{b} \wedge \overline{a} \backslash\!\!\!\circ \overline{b}, \overline{a} \backslash\!\!\!\circ \overline{b}].$$

$\mathbf{x}$ is the greatest interval such that :

$$\mathbf{ax} \preceq_{\mathrm{I}(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!])} \mathbf{b}$$

Conclusion :

Next works) :

- Application for High-Troughput Screening Systems
- How to deal with a model taking into account negative tokens, in order to take into account the transient behavior
- Control for system with maximal duration constraint, or dually a minimal event
- How to get a hierarchical approach, and also where put the sensors in an efficient way.
- and more ....

Scilab Toolboxes

- http://www.istia.univ-angers.fr/~hardouin
- http://www.scilab.org/contrib/
- http://cermics.enpc.fr/~cohen-g//SED/book-online.html

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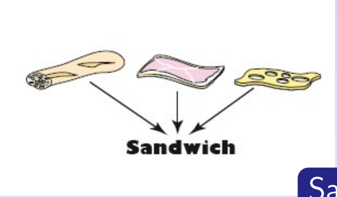
Back



Sandwiches Algebra [Cohen et al.]

1 piece of Bread + 1 slice of ham + 1 slice of cheese is equal to 1 sandwich. Another way of counting !

(max,+) algebra in few words

Back

In classical algebra :

$x = ax + b \Leftrightarrow x = (1-a)^{-1}b$
with the Taylor expansion (MacLaurin Series)
$(1-a)^{-1} \approx 1 + a + a^2 + a^3 + \ldots$

In semiring :

$x = ax \oplus b$ admits $x = a^*b$ as least solution. with $a^* = e \oplus a \oplus a^2 \oplus a^3 \oplus \ldots$

(max,+) algebra in few words

Skecth of proof : if $a \in \mathrm{Im}\mathcal{M}$ then $x^* \preceq a \Leftrightarrow x \preceq a^*$ Back

$a \in \mathrm{Im}\mathcal{M} \Rightarrow a = a^*$ hence $x^* \preceq a^* \Leftrightarrow (e \oplus x \oplus x^2 \oplus \ldots) \preceq a^* \Leftrightarrow x \preceq a^*$

(max,+) algebra in few words

Sum of matrices $A \oplus B = C$

Back

$$\begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \oplus \begin{pmatrix} e & 8 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 3 & 7 \end{pmatrix}$$

Product of matrices $A \otimes B = C$

Back

$$\begin{pmatrix} 2 & 5 \\ \varepsilon & 3 \\ 1 & 8 \end{pmatrix} \otimes \begin{pmatrix} e \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \otimes e \oplus 5 \otimes 1 \\ \varepsilon \otimes e \oplus 3 \otimes 1 \\ 1 \otimes e \oplus 8 \otimes 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix}$$

Residuation of matrices $A \circ\!\!\!\backslash B$ is the greatest solution of $A \otimes X \preceq B$

Back

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \otimes X \preceq \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \circ\!\!\!\backslash \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} = \begin{pmatrix} (1 \circ\!\!\!\backslash 8) \wedge (3 \circ\!\!\!\backslash 9) \wedge (5 \circ\!\!\!\backslash 10) \\ (2 \circ\!\!\!\backslash 8) \wedge (4 \circ\!\!\!\backslash 9) \wedge (6 \circ\!\!\!\backslash 10) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

TEG Model in $\overline{\mathbb{Z}}_{\max}$

$\overline{\mathbb{Z}}_{\max}$

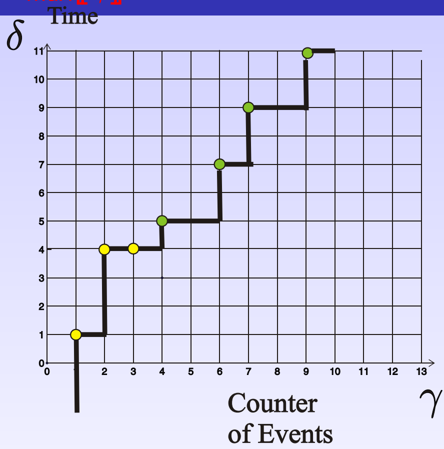
$\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, +)$ is an idempotent semi ring.

$$(\forall a \in \overline{\mathbb{Z}}_{\max}, a \oplus a = a)$$

Back

TEG Model in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

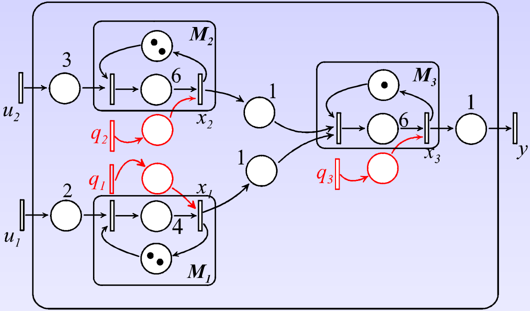
Back



a series in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

$$s = \bigoplus_{k\in\mathbb{Z}} s(k)\gamma^k = 1\gamma \oplus 4\gamma^2 \oplus 5\gamma^4 \oplus 7\gamma^6 \oplus \ldots.$$

Sub Observer Synthesis :



Matrix S and input q :

Back

- vector q represents a vector of exogenous uncontrollable inputs (disturbance) which act on the system through matrix S.
- These disturbances lead to disable the transition firing, that is they decrease system performances and delay tokens output.

TEG Model in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

◂ Back

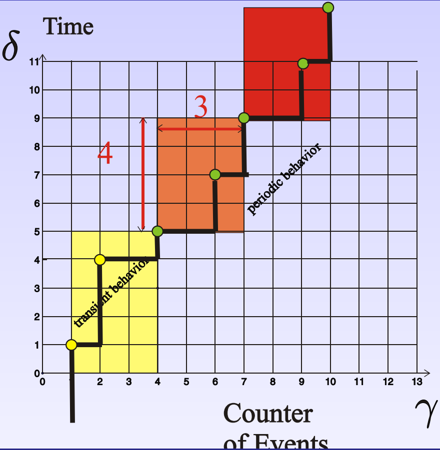
Computation $A^* = E \oplus A \oplus A^2 \oplus A^3 \oplus \dots$

$$A = \begin{pmatrix} e & \varepsilon \\ \varepsilon & e \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & \gamma \\ 3 & \varepsilon \end{pmatrix} \oplus \begin{pmatrix} 3\gamma & \varepsilon \\ \varepsilon & 3\gamma \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & 3\gamma^2 \\ 6\gamma & \varepsilon \end{pmatrix} \oplus \begin{pmatrix} 6\gamma^2 & \varepsilon \\ \varepsilon & 6\gamma^2 \end{pmatrix} \oplus \dots$$

Each entries is a periodic series $A^* = \begin{pmatrix} (3\gamma)^* & \gamma(3\gamma)^* \\ 3(3\gamma)^* & (3\gamma)^* \end{pmatrix}$

TEG Model in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

Back



a periodic series in $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

$$s = (1\gamma \oplus 4\gamma^2) \oplus (5\gamma^4 \oplus 7\gamma^6)(4\gamma^3)^*$$

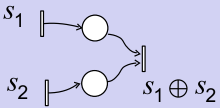
The throughput is denoted by $\sigma_\infty(s) = 3/4$

Operations over $\overline{\mathbb{Z}}_{max}[\![\gamma]\!]$

Sum of series $s_1 \oplus s_2$

◂ Back

- Asymptotic slope $\sigma_\infty(s_1 \oplus s_2) = min(\sigma_\infty(s_1), \sigma_\infty(s_2))$

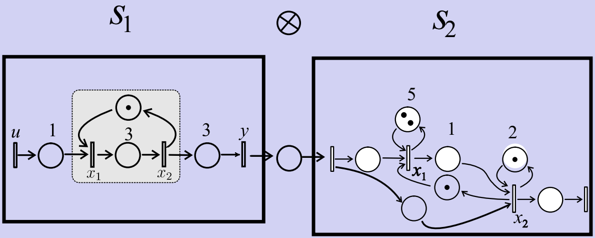


Cauchy product of series $s_1 \otimes s_2$

Back

- $$s_1 \otimes s_2 : (s_1 \otimes s_2)(k) = \bigoplus_{i+j=k} s_1(k) \otimes s_2(k).$$

- Asymptotic slope $\sigma_\infty(s_1 \otimes s_2) = min(\sigma_\infty(s_1), \sigma_\infty(s_2))$



State Feedback controller synthesis

Details :

- Recall :

$$(a \oplus b)^* = a^*(ba^*)^*$$
$$a(ba)^* = (ab)^* a$$

- Let

$$C(A \oplus BK)^* B = CA^*(BKA^*)^* B = CA^* B(KA^* B)^*$$

.

- hence :

$$C(A \oplus BK)^* B \preceq G_{ref} \Leftrightarrow (KA^* B)^* \preceq ((CA^* B) \phi G_{ref})$$
$$\Leftrightarrow (A \oplus BK)^* B \preceq A^* B((CA^* B) \phi G_{ref}).$$

Back

Application : a kind of disturbances decoupling problem

Definition [Davey et al.,1990],[Cohen et al, 1996]

Back

The kernel of a mapping $C : \mathcal{X} \to \mathcal{Y}$, denoted $\ker C$, is defined by the following equivalence relation

$$x \overset{\ker C}{\equiv} x' \quad \Longleftrightarrow \quad C(x) = C(x').$$