Control of uncertain (max,+)-linear systems in order to decrease uncertainty

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Abstract: This paper deals with the control of uncertain (max,+)-linear systems, more precisely those of which parameters are not exactly known but assumed to belong to an interval. In this context, we aim at synthesizing a controller in order to reduce the uncertainty at the output of the controlled system. This study is possible thanks to the residuation theory and relies on solutions of fixed-point equations.

Keywords: Discrete Event Dynamic Systems, (max,+)-algebra, interval of systems, residuation theory, fixed-point equation theory, optimal control.

1. INTRODUCTION

The theory of (max,+)-algebra enables the study of Discrete Event Dynamic Systems (DEDS) characterized by delay and synchronization phenomena such as production systems, computing networks and transportation systems (see Baccelli et al. (1992) and Heidergott et al. (2006)). Such systems can be described by linear models, thanks to the particular algebraic structure called *idempotent semiring* (or *dioid*) and the residuation theory enables to deal with their control. For instance, some model matching problems are solved by the way of different control structures (open-loop or close-loop structures) as presented in Cottenceau et al. (2001) and Maia et al. (2005). These results assume that the model is perfectly known.

Over the last past years, more and more studies have been done for systems described by intervals which contain their behavior.

- Firstly, due to some uncertain or variable parameters (see Lhommeau et al. (2004) and Di Loreto et al. (2009)), the transfer of the system may fluctuate. In spite of such variations, the system can be framed by an interval describing its upper and lower behaviors.
- Secondly, it is shown in Boutin et al. (2009) that when a system has some ressource sharing or routing problems these phenomena are not managed by (max,+)-linear properties. Nevertheless, it is sometimes possible to handle an interval the bounds of which are two (max,+)-linear systems framing the real system.
- Finally, a (max,+)-linear system of high dimension can invalidate toolboxes which handle it both in storage capacity and in computing time. As shown in Le Corronc et al. (2009), the real system can be included in an interval easier to compute thanks to algorithms of linear complexity and restricted data-processing representation.

All these works share a feature that a system is not exactly modeled but is described by an interval containing its behaviors. The lower and the upper bounds of the interval correspond to (max,+)-linear systems and illustrate the extreme behaviors of the system. Fig. 1 illustrates that kind of context. For all inputs u, a system of which the transfer h belongs to an interval $[\ \underline{h}\ ,\ \overline{h}\ ]$ has an output y included in an interval too. In other words, if $h \in [\ \underline{h}\ ,\ \overline{h}\ ]$ then $\forall u,\ y \in [\ \underline{h}u\ ,\ \overline{h}u\ ]$.

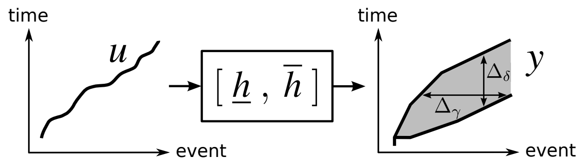


Fig. 1. Uncertain (max,+)-linear system $[\ \underline{h}\ ,\ \overline{h}\ ]$ with an exact input u and an output y included in an interval. Maximal distances of this interval in event and time domains are respectively Δ_γ and Δ_δ.

Thus for a given input u, output y is only known with an approximation, *i.e.* an uncertainty linked to the size of the interval and defined by the distances Δ_γ for the event domain and Δ_δ for the time domain.

This paper puts forward that by adding an upstream corrector p to an uncertain system then the controlled system hp also belongs to an interval $[\ \underline{h}p\ ,\ \overline{h}p\ ]$ which has at worst the same uncertainty. The synthesis of such a controller is given here in order to decrease this uncertainty. This control problem is given as a research of a fixed point of an isotone mapping.

In order to introduce this work, the paper is organized as follows. Section 2 recalls some algebraic tools required for the DEDS study through idempotent semiring, residuation theory and fixed point of isotone mapping theory. In the third section, the modelling of (max,+)-linear systems is presented, as well as interval of such systems. Finally, in the fourth section, the controller p reducing the uncertainty of the interval is proposed and some examples are given.

2. ALGEBRAIC PRELIMINARIES

For this section, interested reader is invited to peruse (Baccelli et al., 1992, Chap 4).

2.1 Dioid theory

Definition 1. An idempotent semiring $\mathcal{D}$ is a set endowed with two inner operations denoted $\oplus$ and $\otimes$. The sum $\oplus$ is associative, commutative, idempotent (*i.e.* $\forall a \in \mathcal{D}, a \oplus a = a$) and admits a neutral element denoted ε. The product $\otimes$ is associative, distributes over the sum and accepts e as neutral element.

An idempotent semiring is said to be complete if it is closed for infinite sums and if the product distributes over infinite sums too. In this case, the greatest element of $\mathcal{D}$ is denoted T (for *Top*) and represents the sum of all its elements ($T = \bigoplus_{x \in \mathcal{D}} x$). Finally, a subset $\mathcal{D}_{sub}$ of a semiring $\mathcal{D}$ is called a subsemiring if $\varepsilon \in \mathcal{D}_{sub}$, $e \in \mathcal{D}_{sub}$ and if $\mathcal{D}_{sub}$ is closed for $\oplus$ and $\otimes$.

Due to the idempotency of addition, an order relation can be associated with $\mathcal{D}$ by the following equivalences: $\forall a, b \in \mathcal{D}$, $a \succeq b \iff a = a \oplus b$ and $b = a \wedge b$. Because of the lattice properties of a complete idempotent semiring, $a \oplus b$ is the least upper bound of $\mathcal{D}$ whereas $a \wedge b$ is its greatest lower bound.

Example. The set $\overline{\mathbb{Z}}_{max} = (\mathbb{Z} \cup \{-\infty, +\infty\})$ endowed with the max operator as sum $\oplus$ and the classical sum as product $\otimes$, is a complete idempotent semiring where $\varepsilon = -\infty$, $e = 0$ and $T = +\infty$. On $\overline{\mathbb{Z}}_{max}$, the greatest lower bound $\wedge$ takes the sense of the min operator.

Example. The set of formal series with two commutatives variables γ and δ, Boolean coefficients in $\{\varepsilon, e\}$ and exponents in $\mathbb{Z}$, is a complete idempotent semiring denoted $\mathbb{B}[\![\gamma, \delta]\!]$ where $\varepsilon = \bigoplus_{k,t \in \mathbb{Z}} \varepsilon \gamma^k \delta^t$ (null series) and $e = \gamma^0 \delta^0$. The series $s \in \mathbb{B}[\![\gamma, \delta]\!]$ is written in a single way by $s = \bigoplus_{n,t \in \mathbb{Z}} s(n,t) \gamma^n \delta^t$ where $s(n,t) = e$ (presence of the monomial) or ε (absence of the monomial).

Example. The quotient of $\mathbb{B}[\![\gamma, \delta]\!]$ by the equivalence $(\gamma \oplus \delta^{-1})^*$ provides the complete idempotent semiring $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. By considering this equivalence in a graphical point of view, an element $\gamma^n \delta^t$ of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is represented by a southeast cone with coordinates (n, t).

2.2 Residuation theory

Residuation is a general notion in lattice theory which allows to define "pseudo-inverse" of some isotone maps. In particular, the residuation theory provides optimal solutions to inequalities $f(x) \preceq b$, where f is an order-preserving mapping (*i.e.*, $a \preceq b \Rightarrow f(a) \preceq f(b)$) defined over ordered sets.

Definition 2. Let $f : \mathcal{D} \to \mathcal{C}$ be an isotone mapping, where $\mathcal{D}$ and $\mathcal{C}$ are complete idempotent semirings. Mapping f is said residuated if $\forall b \in \mathcal{C}$, the greatest element denoted $f^\sharp(b)$ of subset $\{x \in \mathcal{D} | f(x) \preceq b\}$ exists and belongs to this subset. Mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique isotone mapping such that $f \circ f^\sharp \preceq \mathsf{Id}_\mathcal{C}$ and $f^\sharp \circ f \succeq \mathsf{Id}_\mathcal{D}$, where $\mathsf{Id}_\mathcal{C}$ and $\mathsf{Id}_\mathcal{D}$ are respectively the identity mappings on $\mathcal{C}$ and $\mathcal{D}$.

Example. Mapping $L_a : x \mapsto a \otimes x$ defined over $\mathcal{D}$ is residuated. Its residual is usually denoted $L_a^\sharp : x \mapsto a \backslash\!\!\circ x$ and called *left quotient*. Therefore, $a \backslash\!\!\circ b$ is the greatest solution to inequality $a \otimes x \preceq b$, *i.e.* $a \backslash\!\!\circ b = \hat{x} = \bigoplus \{x \mid a \otimes x \preceq b\}$. Several properties of L_a are given in appendix A.

Then, connected to the residuation theory, the mapping restriction allows to have some projectors from a set to another set.

Definition 3. Let $\mathsf{Id}_{|\mathcal{D}_{sub}} : \mathcal{D}_{sub} \mapsto \mathcal{D}, x \mapsto x$ be the canonical injection from a complete subsemiring into a complete semiring. Injection $\mathsf{Id}_{|\mathcal{D}_{sub}}$ is residuated and its residual is a projector denoted $\mathsf{Pr}_{\mathcal{D}_{sub}}$.

2.3 Fixed point of isotone mapping

Whereas residuation theory provides optimal solutions to inequalities $f(x) \preceq b$, the fixed point theory enables to find greatest finite solutions to equations $f(x) = x$, where f is an isotone mapping defined over a complete idempotent semiring $\mathcal{D}$.

Definition 4. Let $\mathcal{F}_f = \{x \in \mathcal{D} \mid f(x) = x\}$ be the set of fixed points of an isotone mapping f defined over $\mathcal{D}$. Respectively, let $\mathcal{P}_f = \{x \in \mathcal{D} \mid f(x) \succeq x\}$ be the set of post-fixed points which can be interpreted in $\mathcal{F}_f$ as following equivalence shows: $f(x) \succeq x \Leftrightarrow f(x) \wedge x = x$.

Theorem 5. (Knaster-Tarski) Let $\mathcal{F}_f$ be a complete lattice. The greatest fixed solution $\hat{y}$ of $\mathcal{F}_f$ is given by:

$$\hat{y} = \lim_{n \to \infty} f^n(T)$$

where $f^{n+1} = f \circ f^n$ and $f^0 = \mathsf{Id}_\mathcal{D}$.

In order to obtain this solution, the following theorem puts forward a method to compute it in a recurrent way.

Theorem 6. Let f be an isotone mapping defined over $\mathcal{D}$ and let us recall that $\mathcal{F}_f$ is the set of fixed points of f. Now consider the following iterative scheme:

Let $x_0 = T$,
do $x_{n+1} = f(x_n)$,
until $x_{m+1} = x_m$ for $m \in \mathbb{N}$.

If function f admits a fixed point $x \in \mathcal{F}_f$ and $x \neq \varepsilon$, then the previous algorithm converges toward the greatest fixed point $\hat{y} = x_m$.

Proof. Firstly, as $x_{m+1} = x_m$, $x_m = f(x_m)$ and so x_m belongs to the set $\mathcal{F}_f$. Secondly, it is necessary to show that x_m is the greatest solution of $\mathcal{F}_f$. Let $x' \in \mathcal{F}_f$, since $x_0 = T$, $x_0 \succeq x'$. Finally, if $x_m \succeq x'$ then $x_{m+1} \succeq x'$: $x_{m+1} = f(x_m) \succeq f(x') = x'$ (thanks to the isotony of f and knowing that $x' \in \mathcal{F}_f$).

It is also possible to use this algorithm in order to find the greatest fixed point less than a given value of $\mathcal{D}$. In that case, the following corollary is given.

Corollary 7. Let $h : \mathcal{D} \mapsto \mathcal{D}$ be an isotone mapping and $val \in \mathcal{D}$; let f defined by $f : \mathcal{D} \mapsto \mathcal{D}, x \mapsto h(x) \wedge val$. If f admits a fixed point $x \in \mathcal{F}_f$, then the algorithm of theorem 6 converges toward the greatest fixed point of h less than val, that is $\hat{y} = x_m \preceq val$.

3. SYSTEM MODELLING

3.1 Models of (max,+)-linear systems

The complete idempotent semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ enables to model DEDS which involve synchronization and delay phenomena. Indeed, equivalence $(\gamma \oplus \delta^{-1})^*$ for all $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ series is well suited to describe the weakly increasing nature of DEDS. A monomial $\gamma^n\delta^t \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is interpreted as follows: *the n^{th} event occurs at earliest at time t*. These systems can thus be modeled by the following input/output relation:

$$y = hu = CA^*Bu \qquad (1)$$

where $A \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times n}$, $B \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\times p}$ and $C \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{q\times n}$ while n, p and q refer respectively to the state vector size, the input vector (u) size and the output one (y). In this equation, $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$ with $A^0 = e$ and h is the transfer function of the system [1] .

Criterion 8. According to (Baccelli et al., 1992, Theorem 5.39) and Gaubert (1992), a (max,+)-linear system defined by equation (1) is necessarily such that h is periodic and causal [2] which is denoted $h \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\mathsf{caus}}$. In other words, the system has no anticipation neither on event domain nor on time domain.

Definition 9. The canonical injection $\mathsf{Id}_{|\mathsf{caus}} : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\mathsf{caus}} \mapsto \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is residuated and its residual is denoted $\mathsf{Pr}_{\mathsf{caus}} : \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!] \mapsto \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Formally, series $\mathsf{Pr}_{\mathsf{caus}}(s)$ is the greatest causal series less than or equal to s:

$$\mathsf{Pr}_{\mathsf{caus}}(s) = \mathsf{Pr}_{\mathsf{caus}}\Big(\bigoplus_{i\in\mathbb{N}} f(n_i,t_i)\gamma^{n_i}\delta^{t_i}\Big) = \bigoplus_{i\in\mathbb{N}} g(n_i,t_i)\gamma^{n_i}\delta^{t_i}$$

where

$$g(n_i,t_i) = \begin{cases} f(n_i,t_i) & \text{if } (n_i,t_i) \geq (0,0), \\ \varepsilon & \text{otherwise.} \end{cases}$$

3.2 Interval of (max,+)-linear system

The systems considered here are not exact (max,+)-linear systems but described by intervals $[\ \underline{h}\ ,\ \overline{h}\ ]$ containing in a guaranteed way their behavior. In these intervals, $\underline{h}$ and $\overline{h}$ are periodic and causal series of the idempotent semiring $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\mathsf{caus}}$ and represent respectively the lower and the upper behaviors of the system. Hence, as illustrated in Fig. 2, for a given input u, output y evolves in an area framed by the interval $[\ \underline{h}u\ ,\ \overline{h}u\ ]$.

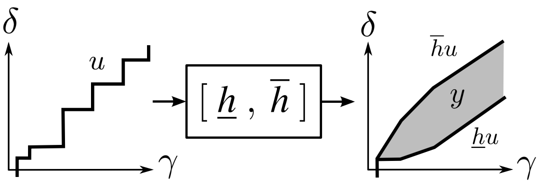


Fig. 2. Uncertain (max,+)-linear system $[\ \underline{h}\ ,\ \overline{h}\ ]$ with an exact input u and an output y included in interval $[\ \underline{h}u\ ,\ \overline{h}u\ ]$.

Remark 10. Currently, a toolbox called MinMaxGD (see Cottenceau et al. (2000)) enables to handle series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Elementary operations of (max,+)-linear systems, such as addition $\oplus$, product $\otimes$ or residuation $\circ\!\!\!\!\diagdown$, are proposed in this library. However, some correlations between system elements can reveal that the computations are expensive both in storage capacity and in computing time. In such a case, another toolbox called Container-MinMaxGD (see Le Corronc et al. (2009)) enables a more effective handling of interval, the bounds of which have convexity properties. Thus, starting from a known system, *i.e.* a system with known matrices C, A, B (see equation (1)), the toolbox is able to compute an interval $[\ \underline{h}\ ,\ \overline{h}\ ]$ that contains $h = CA^*B$ even if the matrices have high dimensions.

3.3 Maximal uncertainty of the interval

According to the following theorem coming from the second order theory of (max,+)-linear systems detailed in MaxPlus (1991), the distance between the bounds of an interval $[\ \underline{h}\ ,\ \overline{h}\ ]$ of (max,+)-linear system can be measured easily.

Theorem 11. (MaxPlus (1991)) Let u and v be two series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ and $u \preceq v$. The series $v \circ\!\!\!\!\diagdown u$ is called the correlation of u over v and contains the maximal distances between u and v in the event domain γ as well as in time domain δ. More precisely, monomial $\gamma^\nu\delta^0$ of series $v \circ\!\!\!\!\diagdown u$ gives the maximal event distance ν between u and v also denoted by:

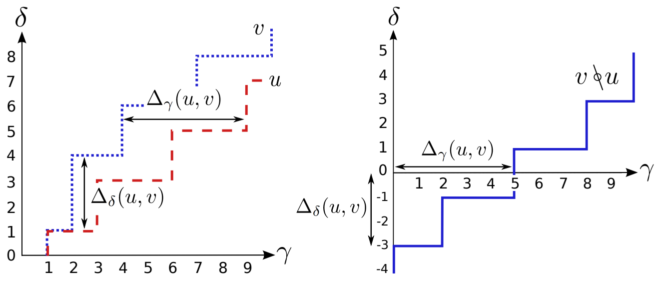
$$\nu = \Delta_\gamma(u,v) = \min\{n \mid \gamma^n v \preceq u\},$$

whereas $\gamma^0\delta^{-\tau}$ provides the maximal time distance τ denoted by:

$$\tau = \Delta_\delta(u,v) = \min\{t \mid \delta^t u \succeq v\}.$$

Remark 12. It is possible that $v \circ\!\!\!\!\diagdown u = \varepsilon$. In such a case, distances $\Delta_\gamma(u,v)$ and $\Delta_\delta(u,v)$ are infinite.

Example. Let $u = \gamma^1\delta^1 \oplus \gamma^3\delta^3(\gamma^3\delta^2)^*$ and $v = \gamma^1\delta^1 \oplus \gamma^2\delta^4 \oplus \gamma^4\delta^6(\gamma^3\delta^2)^*$ be two series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (see Fig. 3a). The correlation of u over v is the following series (see Fig. 3b): $v \circ\!\!\!\!\diagdown u = \boldsymbol{\gamma^0\delta^{-3}} \oplus \gamma^2\delta^{-1} \oplus \boldsymbol{\gamma^5\delta^1}(\gamma^3\delta^2)^*$. The maximal event distance is $\Delta_\gamma(u,v) = 5$, whereas the maximal time distance is $\Delta_\delta(u,v) = 3$.



(a) Series u and v (b) Series $v \circ\!\!\!\!\diagdown u$

Fig. 3. Maximal distances in event and time domains between two series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

Then, thanks to theorem 11, the maximal uncertainty at the output of an uncertain system belonging to $[\ \underline{h}\ ,\ \overline{h}\ ]$ is given by the following proposition.

Proposition 13. Let $[\ \underline{h}\ ,\ \overline{h}\ ]$ with $\underline{h} \preceq \overline{h}$, be an interval of (max,+)-linear system. The computation of correlation $\overline{h} \circ\!\!\!\!\diagdown \underline{h}$ provides the maximal distances [3] $\Delta_\gamma(\underline{h},\overline{h})$ and

[1] Which corresponds to the impulse response of the system.

[2] The exponents of all the monomials of h are positives or null.

[3] These distances are called the maximal uncertainty of $[\ \underline{h}\ ,\ \overline{h}\ ]$.

$\Delta_\delta(\underline{h}, \overline{h})$. Moreover for any input u these distances are the maximal distances of interval $[\ \underline{h}u\ ,\ \overline{h}u\ ]$ in which output y evolves in a guaranteed way. Formally:

$$\Delta_\gamma(\underline{h}, \overline{h}) = \max_{\forall u \in \mathcal{D}} \{\Delta_\gamma(\underline{h}u, \overline{h}u)\},$$
$$\Delta_\delta(\underline{h}, \overline{h}) = \max_{\forall u \in \mathcal{D}} \{\Delta_\delta(\underline{h}u, \overline{h}u)\}.$$

Proof. According to theorem 11, correlation $(\overline{h}u) \circ\!\!\!\!\setminus (\underline{h}u)$ represents the maximal event and time distances between the bounds of $[\underline{h}u\ ,\ \overline{h}u]$. Then, the following inequality is shown:

$$\begin{array}{rcll} (\overline{h}u) \circ\!\!\!\!\setminus (\underline{h}u) & = & (u\overline{h}) \circ\!\!\!\!\setminus (u\underline{h}) & \text{since } \otimes \text{ commutative}, \\ & = & \overline{h} \circ\!\!\!\!\setminus (u \circ\!\!\!\!\setminus (u\underline{h})) & \text{see } (A.2), \\ & \succeq & \overline{h} \circ\!\!\!\!\setminus \underline{h} & \text{see } (A.1). \end{array}$$

Hence, $\forall u$:

$$\Delta_\gamma(\underline{h}u, \overline{h}u) \ \leq\ \Delta_\gamma(\underline{h}, \overline{h}),$$
$$\Delta_\delta(\underline{h}u, \overline{h}u) \ \leq\ \Delta_\delta(\underline{h}, \overline{h}).$$

Remark 14. A particular input u which leads to equality in these equations is given by $u = e$.

Example. Let $\underline{h} = \gamma^1\delta^0 \oplus \gamma^4\delta^1(\gamma^2\delta^1)^*$ and $\overline{h} = \gamma^1\delta^0 \oplus \gamma^2\delta^3 \oplus \gamma^4\delta^5(\gamma^2\delta^1)^*$ be two series of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ forming an interval of (max,+)-linear system and $u = \gamma^0\delta^0(\gamma^2\delta^3)^*$ be its input. The computation of $(\overline{h}u) \circ\!\!\!\!\setminus (\underline{h}u)$ and $\overline{h} \circ\!\!\!\!\setminus \underline{h}$ with the toolbox MinMaxGD is given below.

```
// Script for the example with Scilab/MinMaxGD
// lh and uh = lower and upper bounds of h
lh = series([1 0],[4 1],[2 1])
uh = series([1 0;2 3],[4 5],[2 1])
u = series(eps,e,[2 3])
lhu = lh * u
uhu = uh * u
correlation1 = uhu \ lhu
correlation2 = uh \ lh
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The results are:

$$\begin{array}{l} \texttt{correlation1} \ = \ (\overline{h}u) \circ\!\!\!\!\setminus (\underline{h}u) \ = \ \boldsymbol{\gamma^0\delta^{-3}} \oplus \boldsymbol{\gamma^1\delta^0}(\gamma^2\delta^1)^*, \\ \texttt{correlation2} \ = \ \overline{h} \circ\!\!\!\!\setminus \underline{h} \ = \ \gamma^0\delta^{-4}(\gamma^2\delta^1)^* \\ \qquad = \ \boldsymbol{\gamma^0\delta^{-4}} \oplus \gamma^2\delta^{-3} \oplus \gamma^4\delta^{-2} \oplus \gamma^6\delta^{-1} \oplus \boldsymbol{\gamma^8\delta^0}(\gamma^2\delta^1)^*. \end{array}$$

so $(\overline{h}u) \circ\!\!\!\!\setminus (\underline{h}u) \succeq \overline{h} \circ\!\!\!\!\setminus \underline{h}$. As regards to distances in event and time domains:

$$\Delta_\gamma(\underline{h}u, \overline{h}u) = 1 \quad \leq \quad \Delta_\gamma(\underline{h}, \overline{h}) = 8,$$
$$\Delta_\delta(\underline{h}u, \overline{h}u) = 3 \quad \leq \quad \Delta_\delta(\underline{h}, \overline{h}) = 4.$$

4. REDUCING THE UNCERTAINTY OF A (MAX,+)-LINEAR SYSTEM THANKS TO A CONTROLLER

According to the previous section, when a system belongs to an interval $[\ \underline{h}\ ,\ \overline{h}\ ]$, the maximal uncertainty over its output y can be found for all inputs u thanks to correlation $\overline{h} \circ\!\!\!\!\setminus \underline{h}$. This section shows that the use of a (max,+)-linear controller $p \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ placed upstream of the system can reduce this uncertainty or even cancel it completely.

4.1 Computation of the uncertainty with a controller

First of all, if a controller p is placed upstream of an uncertain system $[\ \underline{h}\ ,\ \overline{h}\ ]$, as illustrated in Fig. 4, the controlled system becomes the interval $[\ \underline{h}p\ ,\ \overline{h}p\ ]$ and its output y is therefore included in the interval $\underline{h}pv \preceq y \preceq \overline{h}pv$.

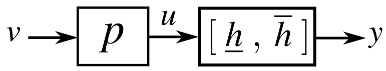

Fig. 4. Controlled system.

Then, the maximal uncertainty over the output of the controlled system $[\ \underline{h}p\ ,\ \overline{h}p\ ]$ can be computed in the same way as in proposition 13.

Proposition 15. Let $[\ \underline{h}p\ ,\ \overline{h}p\ ]$ with $\underline{h}p \preceq \overline{h}p$, be an interval of a (max,+)-linear controlled system. Correlation $(\overline{h}p) \circ\!\!\!\!\setminus (\underline{h}p)$ provides the maximal distances [4] $\Delta_\gamma(\underline{h}p, \overline{h}p)$ and $\Delta_\delta(\underline{h}p, \overline{h}p)$. Moreover, for any input v these distances are the maximal distances of the interval $[\underline{h}pv\ ,\ \overline{h}pv]$ in which the output y evolves in a guaranteed way. Formally:

$$\Delta_\gamma(\underline{h}p, \overline{h}p) = \max_{\forall v \in \mathcal{D}} \{\Delta_\gamma(\underline{h}pv, \overline{h}pv)\},$$
$$\Delta_\delta(\underline{h}p, \overline{h}p) = \max_{\forall v \in \mathcal{D}} \{\Delta_\delta(\underline{h}pv, \overline{h}pv)\}.$$

Finally, the uncertainty over the controlled system output is smaller than the maximal uncertainty of $[\ \underline{h}\ ,\ \overline{h}\ ]$, *i.e.* the one of the uncontrolled system:

$$\Delta_\gamma(\underline{h}p, \overline{h}p) \ \leq\ \Delta_\gamma(\underline{h}, \overline{h}),$$
$$\Delta_\delta(\underline{h}p, \overline{h}p) \ \leq\ \Delta_\delta(\underline{h}, \overline{h}).$$

Proof. The proof is immediate by applying proposition 13 for the interval $[\ \underline{h}p\ ,\ \overline{h}p\ ]$.

Remark 16. In the case where $\Delta_\gamma(\underline{h}, \overline{h})$ and $\Delta_\delta(\underline{h}, \overline{h})$ are infinite distances [5], for some p, $\Delta_\gamma(\underline{h}p, \overline{h}p)$ and $\Delta_\delta(\underline{h}p, \overline{h}p)$ may be finite distances. Indeed, the controller enables to slow down the production rate of the controlled system until it reaches the slowest behavior of the system alone, behavior given by series $\overline{h}$.

4.2 Reduction of the uncertainty

According to proposition 15, a controller can reduce the size of the box inside of which the output function evolves. It is then interesting to look for a controller p such that $\Delta_\gamma(\underline{h}p, \overline{h}p)$ and $\Delta_\delta(\underline{h}p, \overline{h}p)$ are no greater than a fixed ν_0 for the former and a fixed τ_0 for the latter. Among the controllers which allow these constraints to be achieved, it is relevant to compute the greatest one, *i.e.* the optimal controller in regards of the just in time criterion. Hence, the objective can be stated as follows:

$$\hat{p} = \bigoplus \{p \mid \Delta_\gamma(\underline{h}p, \overline{h}p) \leq \nu_0 \text{ and } \Delta_\delta(\underline{h}p, \overline{h}p) \leq \tau_0\}$$
$$\Leftrightarrow \qquad \hat{p} = \bigoplus \{p \mid (\overline{h}p) \circ\!\!\!\!\setminus (\underline{h}p) \succeq \gamma^0\delta^{-\tau_0} \oplus \gamma^{\nu_0}\delta^0\} \qquad (2)$$

where $\gamma^0\delta^{-\tau_0} \oplus \gamma^{\nu_0}\delta^0 \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is a polynomial and by considering the following equivalences:

$$\Delta_\gamma(\underline{h}p, \overline{h}p) \leq \nu_0 \quad \Leftrightarrow \quad (\overline{h}p) \circ\!\!\!\!\setminus (\underline{h}p) \succeq \gamma^{\nu_0}, \qquad (3)$$

[4] These distances are called the maximal uncertainty of $[\ \underline{h}p\ ,\ \overline{h}p\ ]$.

[5] In other word, $\underline{h}$ and $\overline{h}$ have two different asymptotic slopes.

$$\Delta_\delta(\underline{h}p, \overline{h}p) \leq \tau_0 \quad \Leftrightarrow \quad (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \delta^{-\tau_0}. \tag{4}$$

According to the results of section 2.3, this objective can be reworded by the following proposition and its corollary.

Proposition 17. The greatest controller $\hat{p}$ as defined in equation (2) corresponds to the greatest finite fixed point of the equation:

$$p = p \wedge \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}p) \wedge \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}p). \tag{5}$$

This equation can be written as $x = x \wedge f(x)$ where $f(x) = \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}x) \wedge \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}x)$ is an isotone mapping.

Proof.

- Firstly, in the event domain, the constraint is written as follows:

$$\begin{array}{lll} & (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \gamma^{\nu_0} & \text{see (3)}, \\ \Leftrightarrow & \underline{h}p \succeq \gamma^{\nu_0}\overline{h}p & \text{see } (A.3), \\ \Leftrightarrow & \gamma^{-\nu_0}\underline{h}p \succeq \overline{h}p & \text{since } \gamma^{-\nu_0} \otimes \gamma^{\nu_0} = e, \\ \Leftrightarrow & \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}p) \succeq p & \text{see } (A.5) \text{ and } (A.1), \\ \Leftrightarrow & p = p \wedge \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}p) & \text{see definition 4.} \end{array}$$

- Secondly, in the time domain, the objective becomes:

$$\begin{array}{lll} & (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \delta^{-\tau_0} & \text{see (4)}, \\ \Leftrightarrow & \underline{h}p \succeq \delta^{-\tau_0}\overline{h}p & \text{see } (A.3), \\ \Leftrightarrow & \delta^{\tau_0}\underline{h}p \succeq \overline{h}p & \text{since } \delta^{\tau_0} \otimes \delta^{-\tau_0} = e, \\ \Leftrightarrow & \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}p) \succeq p & \text{see } (A.5) \text{ and } (A.1), \\ \Leftrightarrow & p = p \wedge \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}p) & \text{see definition 4.} \end{array}$$

- Thirdly, by considering these both constraints, the controller p has to satisfy:

$$p = p \wedge \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}p) \wedge \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}p).$$

- Finally, since operators $\otimes$ and $\wedge$ are isotone and thanks to property (A.5) of the left quotient, function $f(x) = \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}x) \wedge \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}x)$ is an isotone mapping.

Remark 18. The greatest controller $\hat{p}$ as defined in proposition 17 can be found by applying to equation (5) the algorithm given in theorem 6. However, this theorem points out that the initial value is $x_0 = T$ and because of property (A.4) of the left quotient, the first iteration of the algorithm will compute:

$$f(x_0) = f(T) = \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}T) \wedge \overline{h} \circ\!\!\!\!\backslash (\gamma^{-\nu_0}\underline{h}T) = T.$$

Hence, it is necessary to involve corollary 7 which looks for the greatest fixed point, so the greatest controller, less than an initial value denoted val. Moreover, since the controller p is a (max,+)-linear system, it must be causal (see criterion 8), *i.e.* it must verify equality $p = \mathsf{Pr}_{\mathsf{caus}}(p)$. The problem given in equation (2) becomes:

$$\hat{p} = \bigoplus\{p \mid (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \gamma^0\delta^{-\tau_0} \oplus \gamma^{\nu_0}\delta^0, \; p = \mathsf{Pr}_{\mathsf{caus}}(p), p \preceq val\}. \tag{6}$$

Corollary 19. The greatest solution to equation (6) is the greatest fixed series:

$$p = p \wedge \overline{h}(\gamma^{-\nu_0}\underline{h}p) \wedge \overline{h} \circ\!\!\!\!\backslash (\delta^{\tau_0}\underline{h}p) \wedge \mathsf{Pr}_{\mathsf{caus}}(p) \wedge val \tag{7}$$

where val is a causal series of $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\mathsf{caus}}$.

Proof. According to definition 9 of projection $\mathsf{Pr}_{\mathsf{caus}}$, $\forall p$, $\mathsf{Pr}_{\mathsf{caus}}(p) \preceq p$. Then, in order to obtain $p = \mathsf{Pr}_{\mathsf{caus}}(p)$, the controller p has to satisfy:

$$\mathsf{Pr}_{\mathsf{caus}}(p) \succeq p \quad \Leftrightarrow \quad p = p \wedge \mathsf{Pr}_{\mathsf{caus}}(p).$$

Finally, it is interesting to consider the particular case $\nu_0 = 0$ and $\tau_0 = 0$ which means that the objective is to obtain a controlled system without uncertainty over the output. In this case the controller p has to satisfy:

$$\underline{h}p = \overline{h}p \;\Leftrightarrow\; \begin{cases} \Delta_\gamma(\underline{h}p, \overline{h}p) \leq 0 \\ \Delta_\delta(\underline{h}p, \overline{h}p) \leq 0 \end{cases} \;\Leftrightarrow\; \begin{cases} (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \gamma^0 \\ (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \delta^0 \end{cases}$$

and the greatest controller $\hat{p}$ is now given by:

$$\hat{p} = \bigoplus\{p \mid (\overline{h}p) \circ\!\!\!\!\backslash (\underline{h}p) \succeq \gamma^0\delta^0, p = \mathsf{Pr}_{\mathsf{caus}}(p), p \preceq val\}. \tag{8}$$

This objective is reworded by the following proposition.

Proposition 20. The greatest controller $\hat{p}$ as defined in equation (8) corresponds to the greatest finite fixed point of the equation:

$$p = p \wedge \overline{h} \circ\!\!\!\!\backslash (\underline{h}p) \wedge \mathsf{Pr}_{\mathsf{caus}}(p) \wedge val. \tag{9}$$

This equation can be written as $x = x \wedge f(x)$ where $f(x) = \overline{h} \circ\!\!\!\!\backslash (\underline{h}x) \wedge \mathsf{Pr}_{\mathsf{caus}}(x) \wedge val$ is an isotone mapping.

Proof. The proof takes back the one of proposition 17.

5. APPLICATION

Let us see an example of a controller which cancels the uncertainty over the output of a controlled system. In order to compute such a controller, we consider the uncertain SISO TEG [6] given in Fig. 5. This TEG may represent a manufacturing system where the tokens (black dots) mean that the ressource is available whereas the delays in bracket give the interval in which the place temporization evolve.

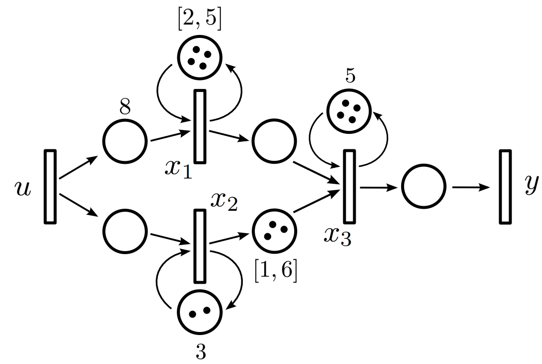

Fig. 5. Example of a uncertain SISO TEG.

The system is subject to time variations which means that processing times are not exactly known but only with minimum and maximum bounds. Therefore, the system is modeled by the interval $[\,\underline{h}\,,\,\overline{h}\,]$ where $\underline{h} \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ represents its lower behavior (*i.e.* all the minimum delays are considered), and $\overline{h} \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ represents its upper behavior (*i.e.* all the maximum one are considered). After computations, these extreme behaviors are given by:

$$\underline{h} = \gamma^0\delta^8(\gamma^4\delta^5)^* \;\oplus\; (\gamma^{35}\delta^{49} \oplus \gamma^{45}\delta^{64})(\gamma^4\delta^6)^*$$

[6] Single Input Single Output system represented thanks to subclass Time Event Graphs of Timed Petri Nets in which each place has exactly one upstream and one downstream transition.

and

$$\overline{h} = \gamma^0\delta^8(\gamma^4\delta^5)^* \ \oplus \ (\gamma^{15}\delta^{24} \oplus \gamma^{25}\delta^{39})(\gamma^4\delta^6)^*.$$

According to proposition 13, for all inputs u the maximal uncertainty of interval $[\ \underline{h}\ ,\ \overline{h}\ ]$ is:

$$\Delta_\gamma(\underline{h}, \overline{h}) = 4 \quad \text{and} \quad \Delta_\delta(\underline{h}, \overline{h}) = 5.$$

In order to find the greatest controller $\hat{p}$ such that the uncertainty of the controlled system is canceled:

$$\hat{p} = \bigoplus\{p \mid (\overline{h}p) \circ\!\!\!\backslash (\underline{h}p) \succeq \gamma^0\delta^0,\ p = \mathsf{Pr_{caus}}(p),\ p \preceq val\}$$

the algorithm of theorem 6 is applied to equation (9):

$$p = p \wedge \overline{h} \circ\!\!\!\backslash (\underline{h}p) \wedge \mathsf{Pr_{caus}}(p) \wedge val.$$

As regards the series val, the correlation $\underline{h} \circ\!\!\!\backslash \underline{h}$ which corresponds to the optimal neutral precompensator is picked in this example (this choice being left with the needs for the real application). Therefore, the controller found will be the greatest controller such that $\hat{p} \preceq \underline{h} \circ\!\!\!\backslash \underline{h}$.

Thus, the computation of this controller with the toolbox MinMaxGD is given below.

```
// Script for the example with Scilab/MinMaxGD
// lh and uh = lower and upper bounds of h
lh = series(eps,[0 8],[4 5])
    + series(eps,[35 49;45 64],[4 6]);
uh = series(eps,[0 8],[4 5])
    + series(eps,[15 24;25 39],[4 6]);
val = prcaus(lh\lh)
p = s_top
fixedPoint = %F;
while fixedPoint == %F
  pPrevious = p;
  p = (uh\(lh*p)) ^ prcaus(p) ^ val
  if pPrevious == p
    fixedPoint = %T;
  end
end
pOptimal = p
lhp = lh * pOptimal
uhp = uh * pOptimal
correlation = uhp \ lhp
```

The results obtained in four iterations, are:

$$\begin{aligned}
\texttt{pOptimal} &= \gamma^4\delta^1 \oplus \ldots \oplus \gamma^{49}\delta^{62}(\gamma^2\delta^3)^*,\\
\texttt{lhp} = \texttt{uhp} &= \gamma^4\delta^9 \oplus \ldots \oplus \gamma^{49}\delta^{70}(\gamma^2\delta^3)^*,\\
\texttt{correlation} &= (\overline{h}p) \circ\!\!\!\backslash (\underline{h}p) \ = \ \boldsymbol{\gamma^0\delta^0} \oplus \ldots \oplus \gamma^{29}\delta^{37}(\gamma^2\delta^3)^*.
\end{aligned}$$

As regards to distances in event and time domains:

$$\begin{aligned}
\Delta_\gamma(\underline{h}p, \overline{h}p) = 0 \quad &\leq \quad \Delta_\gamma(\underline{h}, \overline{h}) = 4,\\
\Delta_\delta(\underline{h}p, \overline{h}p) = 0 \quad &\leq \quad \Delta_\delta(\underline{h}, \overline{h}) = 5.
\end{aligned}$$

6. CONCLUSION

This paper has introduced the control of uncertain (max,+)-linear systems, *i.e.* the behavior of which is described by an interval. The uncertainty at the output of these systems can be easily measured and thanks to a controller placed upstream of them, this uncertainty can decrease or even be completely removed.

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Appendix A. PROPERTIES OF RESIDUATED MAPPING L_A

The following equations introduced properties about the residuated mapping L_a. Interested reader will find the proofs in (Baccelli et al., 1992, p.182-185) and (Gaubert, 1992, §5.3).

$$a \circ\!\!\!\backslash (ax) \quad \succeq \quad x \tag{A.1}$$

$$(ab) \circ\!\!\!\backslash x \quad = \quad b \circ\!\!\!\backslash (a \circ\!\!\!\backslash x) \tag{A.2}$$

$$b \circ\!\!\!\backslash a \succeq x \quad \Leftrightarrow \quad a \succeq xb \tag{A.3}$$

$$a \circ\!\!\!\backslash T \quad = \quad T \tag{A.4}$$

Moreover, $\forall x, y, a \in \mathcal{D}$:

$$x \preceq y \Rightarrow \begin{cases} a \circ\!\!\!\backslash x \preceq a \circ\!\!\!\backslash y & (x \mapsto a \circ\!\!\!\backslash x \text{ is isotone}),\\ x \circ\!\!\!\backslash a \succeq y \circ\!\!\!\backslash a & (x \mapsto x \circ\!\!\!\backslash a \text{ is antitone}). \end{cases} \tag{A.5}$$