Performance Analysis of Linear Systems over Semiring with Additive Inputs

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***Abstract*— This paper deals with the computation of a maximal flow in single input single output $(max, +)$ linear systems. Assuming known a system composed of some sub-systems - each one being described by a transfer function- and some secondary inputs interfering with the principal flow on consecutive sub-systems, the computation of a maximal principal output is addressed. Transfer functions, inputs and outputs are represented by periodical series in a semiring of formal series, namely $\mathbb{N}_{min}[\![\delta]\!]$. Previously, it is shown that the Hadamard product of such series allows to compute the addition of inputs, and that this product is both residuated and dually residuated. These properties are used to compute the maximal principal output. An example concludes the paper and allows to illustrate the efficiency of the proposed approach.**

Keywords: Discrete Event Systems, Timed Event Graphs, $(max, +)$ algebra, Residuation Theory.

I. INTRODUCTION

Timed event graphs are Timed Petri Nets of which each place has one and only one upstream transition and one and only one downstream transition. TEGs enable to depict systems characterized by synchronization and delay phenomena in a graphical way. These phenomena are often found in manufacturing systems such as assembly lines, but also in transportation networks subject to connection and in computer networks. TEG behavior can be modelled by a dynamic linear model in the $(max, +)$ semiring, by associating for each transition labelled x_i a "dater" function $x_i(k)$ which represents the k-th firing date of this transition. Dually it is possible to consider a counter function $x_i(t)$ which depicts the number of firing occurred up to time t and then to obtain a dynamic linear model in the $(min, +)$ semiring. The methodology to build these models is exhaustively proposed in [2].

These mathematical models are used for the performance evaluation of manufacturing systems, transportation networks [12] and computer networks [14]. For these linear systems a control theory has been constructed in an analogous way to the control theory for classical linear systems. Non-exhaustively, we can cite the identification methods [4], [21], [17] and, among the control structures, the model predictive control [22], the optimal control [6][19], and the closed loop control [16], [7], [15], [11]. Some graph algorithms for both the shortest path problem and for the maximum flow problem can also be depicted in these particular semirings (see [9]).

This paper deals with the maximal output computation in a system composed of some sub-systems in which secondary inputs interfere in an additive way. These results are based on the Hadamard product of series, which is both residuated and dually residuated.

Section II recalls useful algebraic tools. In particular, it gives necessary and sufficient conditions for a monotonic mapping f to be residuated.

Section III presents some semirings of formal power series and their Hadamard product. It is shown that this product is residuated and dually residuated if the co-domain is restricted to a given subset. Section IV briefly recalls how to model a TEG in a semiring, and the set of input series which leads to the lowest output of the system is given. This output, called impulse response, corresponds to the maximal instantaneous number of tokens which can be put out of the corresponding TEG. A discussion about practical computation of the residuals of the Hadamard product concludes the section.

Section V is devoted to the performance analysis of a single input single output (SISO) system subject to interfering inputs which act in an additive way. These inputs are not disturbances in the sense of the one studied in [15], but flows added to the system. This means that the system is no longer a TEG because some places have more than one input transition and others have more than one output transition. Nevertheless, assuming known these additive inputs, the lowest system output achievable and the greatest system input leading to this output are computed. An illustrative example concludes the paper.

II. ALGEBRAIC PRELIMINIARIES

This section aims at recalling some algebraic properties of idempotent semirings and to present some semirings of formal series used afterwards.

Definition 1 (Idempotent Semiring): An idempotent semiring is a set $\mathcal{S}$ endowed with two inner operations denoted $\oplus$ and $\otimes$. The sum is associative, commutative, idempotent (*i.e.* $\forall a \in \mathcal{S}, a \oplus a = a$) and admits a neutral element denoted ε. The product is associative, distributes over the sum and admits a neutral element denoted e. The element ε is absorbing for the product. When product $\otimes$ is commutative, the semiring is said to be commutative. As in the classical algebra, symbol $\otimes$ is often omitted.

Definition 2 (Order Relation): An order relation can be associated with $\mathcal{S}$ by the following equivalence: $\forall a, b \in \mathcal{S}, a \succeq b \iff a = a \oplus b$. Therefore, ε is the bottom element of $\mathcal{S}$.

Definition 3 (Complete Idempotent Semiring): Semiring $\mathcal{S}$ is complete if it is closed for infinite sums and if the product distributes over infinite sums too. In particular $\mathsf{T} = \bigoplus_{x \in \mathcal{S}} x$ is the greatest element of $\mathcal{S}$ (T is called the top element of $\mathcal{S}$). The greatest lower bound of every subset $\mathcal{C}$ of a complete semiring $\mathcal{S}$ always exists, and $a \wedge b$

denotes the greatest lower bound between a and b. $\mathcal{S}$ is said distributive if it is complete and if for all subset $\mathcal{C}$ of $\mathcal{S}$, $(\bigwedge_{x\in\mathcal{C}} x)\oplus a = \bigwedge_{x\in\mathcal{C}}(x\oplus a)$.

Definition 4 (Sub Semiring): A subset $\mathcal{C}$ of a semiring $\mathcal{S}$ is called a sub semiring of $\mathcal{S}$ if $\varepsilon\in\mathcal{C}$ and $e\in\mathcal{C}$ and if $\mathcal{C}$ is closed for $\oplus$ and $\otimes$.

Example 1 ($\overline{\mathbb{Z}}_{min}$,$\mathbb{N}_{min}$): Set $\overline{\mathbb{Z}}_{min}=\mathbb{Z}\cup\{-\infty,+\infty\}$, endowed with the $\min$ operator as sum and the classical sum (operation $+$) as product, is a complete idempotent semiring, where $\varepsilon=+\infty$, $e=0$ and $\mathsf{T}=-\infty$. According to definition 3, one has $5\oplus 3=3$ hence $3\succeq 5$. The order relation of $\overline{\mathbb{Z}}_{min}$ is the $reversed$ order of $\mathbb{Z}$. In this particular semiring the product distributes over $\wedge$, $i.e.$, $(a\wedge b)\otimes c=(a\otimes c)\wedge(b\otimes c)$. According to definition 4, $\mathbb{N}_{min}=\mathbb{N}\cup\{+\infty\}$ endowed with the same operators is a sub semiring of $\overline{\mathbb{Z}}_{min}$, which will be also considered afterwards.

Theorem 1 ([2] 4.5.3): Over a complete idempotent semiring $\mathcal{S}$, the implicit equation $x=ax\oplus b$ admits $x=a^*b$ as least solution, where $a^*=\bigoplus_{i\in\mathbb{N}}a^i$ (Kleene star operator) with $a^0=e$.

The residuation theory provides, under some assumptions, optimal solutions to inequalities such as $f(x)\preceq b$ (resp. $f(x)\succeq b$), where f is an order-preserving mapping defined over ordered sets. Some theoretical results are recalled below. Complete presentations are given in [3] [2].

Definition 5 (Isotone mapping): A mapping f defined over ordered sets is isotone if $a\preceq b\Rightarrow f(a)\preceq f(b)$.

Definition 6 (Residuated and dually residuated mappings): Let $f:\mathcal{E}\to\mathcal{F}$ an isotone mapping, where $(\mathcal{E},\preceq)$ and $(\mathcal{F},\preceq)$ are ordered sets. Mapping f is said residuated if for all $y\in\mathcal{F}$, the least upper bound of subset $\{x\in\mathcal{E}|f(x)\preceq y\}$ exists and belongs to this subset. It is then denoted $f^\sharp(y)$. Mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique isotone mapping such that

$$f\circ f^\sharp\preceq \mathsf{Id}_\mathcal{F} \text{ and } f^\sharp\circ f\succeq \mathsf{Id}_\mathcal{E}, \tag{1}$$

where $\mathsf{Id}_\mathcal{F}$ (respectively $\mathsf{Id}_\mathcal{E}$) is the identity mapping on $\mathcal{F}$ (respectively on $\mathcal{E}$). Mapping f is said dually residuated if for all $y\in\mathcal{F}$, the greatest lower bound of subset $\{x\in\mathcal{E}|f(x)\succeq y\}$ exists and belongs to this subset. It is then denoted $f^\flat(y)$. Mapping $f^\flat$ is called the dual residual of f. When f is dually residuated, $f^\flat$ is the unique isotone mapping such that

$$f\circ f^\flat\succeq \mathsf{Id}_\mathcal{F} \text{ and } f^\flat\circ f\preceq \mathsf{Id}_\mathcal{E}. \tag{2}$$

If $\exists x\in\mathcal{E}$ such that $f(x)=y$, then $f^\sharp(y)$ (respectively $f^\flat(y)$) yields the greatest solution (respectively the lowest solution).

Theorem 2 ([2]): Let $f:\mathcal{E}\to\mathcal{F}$ where $\mathcal{E}$ and $\mathcal{F}$ are complete idempotent semirings of which bottom (respectively top) elements are denoted $\varepsilon_\mathcal{E}$ (respectively $\mathsf{T}_\mathcal{E}$) and $\varepsilon_\mathcal{F}$ (respectively $\mathsf{T}_\mathcal{F}$). Mapping f is residuated iff $f(\varepsilon_\mathcal{E})=\varepsilon_\mathcal{F}$ and $\forall\mathcal{A}\subset\mathcal{E}$ $f(\bigoplus_{x\in\mathcal{A}}x)=\bigoplus_{x\in\mathcal{A}}f(x)$. And, mapping f is dually residuated iff $f(\mathsf{T}_\mathcal{E})=\mathsf{T}_\mathcal{F}$ and $\forall\mathcal{A}\subset\mathcal{E}$ $f(\bigwedge_{x\in\mathcal{A}}x)=\bigwedge_{x\in\mathcal{A}}f(x)$.

Corollary 1: Mappings $L_a:x\mapsto ax$ and $R_a:x\mapsto xa$ defined over a complete idempotent semiring $\mathcal{S}$ are both residuated. Their residuals are usually denoted respectively $(L_a)^\sharp: x\mapsto a\backslash\!\!\circ x$ and $(R_a)^\sharp:x\mapsto x/\!\!\circ a$ in (max,+) literature.

Proof: According to definition 3, if $\mathcal{S}$ is a complete idempotent semiring then the product distributes over infinite sums and ε is absorbing, therefore the requirements of theorem 2 are satisfied. ■

Definition 7 (Restricted mapping): Let $f:\mathcal{E}\to\mathcal{F}$ a mapping and $\mathcal{A}\subseteq\mathcal{E}$. We will denote $f_{|\mathcal{A}}:\mathcal{A}\to\mathcal{F}$ the mapping defined by $f_{|\mathcal{A}}=f\circ\mathsf{Id}_{|\mathcal{A}}$ where $\mathsf{Id}_{|\mathcal{A}}:\mathcal{A}\to\mathcal{E}, x\mapsto x$ is the canonical injection. Identically, let $\mathcal{B}\subseteq\mathcal{F}$ with $\mathsf{Im}f\subseteq\mathcal{B}$. Mapping ${}_{\mathcal{B}|}f:\mathcal{E}\to\mathcal{B}$ is defined by $f=\mathsf{Id}_{|\mathcal{B}}\circ{}_{\mathcal{B}|}f$, where $\mathsf{Id}_{|\mathcal{B}}:\mathcal{B}\to\mathcal{F}, x\mapsto x$ is the canonical injection.

III. Semiring of Power Series

Definition 8 (Formal power series): A formal power series in p (commutative) variables, denoted z_1 to z_p, with coefficients in a semiring $\mathcal{S}$, is a mapping s defined from $\mathbb{Z}^p$ or $\mathbb{N}^p$ in $\mathcal{S}$: $\forall k=(k_1,...,k_p)\in\mathbb{N}^p$ or $\mathbb{Z}^p$, $s(k)$ represents the coefficient of $z_1^{k_1}...z_p^{k_p}$ and $(k_1,...,k_p)$ are the exponents. Another equivalent representation is

$$s(z_1,...,z_2)=\bigoplus_{k\in\mathbb{Z}^p}s(k)z_1^{k_1}...z_p^{k_p}.$$

Definition 9 (Support, degree, valuation): The support $supp(s)$ of a series s in p variables is defined as

$$supp(s)=\{k\in\mathbb{Z}^p|s(k)\neq\varepsilon\}.$$

The degree $deg(s)$ (respectively valuation $val(s)$) is the upper bound (respectively lower bound) of $supp(s)$.

The set of formal series endowed with the following sum and Cauchy product:

$$s\oplus s':(s\oplus s')(k)=s(k)\oplus s'(k), \tag{3}$$

$$s\otimes s':(s\otimes s')(k)=\bigoplus_{i+j=k}s(i)\otimes s'(j), \tag{4}$$

is a semiring denoted $\mathcal{S}[\![z_1,...,z_p]\!]$. If $\mathcal{S}$ is complete, $\mathcal{S}[\![z_1,...,z_p]\!]$ is complete. A series with a finite support is called a polynomial, and a monomial if there is only one element.

The greatest lower bound of series is given by :

$$s\wedge s':(s\wedge s')(t)=s(t)\wedge s'(t). \tag{5}$$

A. Semirings $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ and $\mathbb{N}_{min}[\![\delta]\!]$

The particular semiring of formal power series with coefficients in $\overline{\mathbb{Z}}_{min}$ and exponents in $\mathbb{Z}$, denoted $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$, is now considered. A series $s\in\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is defined as follows:

$$s=\bigoplus_{t\in\mathbb{Z}}s(t)\delta^t,$$

where $s(t)\in\overline{\mathbb{Z}}_{min}$. The sequence $\{s(t)\}\forall t\in\mathbb{Z}$ represents a trajectory and series s is called the δ-transform of this trajectory which is analogous to the z-transform used to represent discrete-time trajectories in classical system theory.

Usually sequence $\{s(t)\}$ represents a counter of events , hence number of events $s(t)$ is greater or equal than $s(t-1)$, $i.e.$ that this is a non decreasing trajectory. According to the order in $\overline{\mathbb{Z}}_{min}$ (see example 1), sequence $\{s(t)\}$ satisfies the monotonicity property : $\forall t, s(t) \preceq s(t-1) \Leftrightarrow s(t-1) = s(t-1) \oplus s(t)$. Hence, thanks to theorem 1, the following equivalence holds true

$$s = s \oplus \delta^{-1} s \iff s = (\delta^{-1})^* s.$$

This means that the sequences belong to semiring $(\delta^{-1})^* \overline{\mathbb{Z}}_{min}[\![\delta]\!]$. In this complete semiring $\varepsilon = (\delta^{-1})^* \otimes (+\infty \delta^{-\infty})$, $e = (\delta^{-1})^* \otimes (0\delta^0)$, and $\mathsf{T} = (\delta^{-1})^* \otimes (-\infty \delta^{+\infty})$. Afterwards all the series are assumed to be non decreasing, in order to simplify notations the semiring will be simply denoted $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ ($(\delta^{-1})^*$ will be omitted). Due to the monotonicity property of trajectories, the following calculation rules between monomials of $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ come :

$$n\delta^t \oplus n\delta^{t'} = n\delta^{\max(t,t')}, \tag{6}$$

$$n\delta^t \oplus n'\delta^t = \min(n, n')\delta^t. \tag{7}$$

Definition 10: The Hadamard product of series of $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is defined as follows :

$$s \odot s' : (s \odot s')(t) = s(t) \otimes s'(t).$$

Series $s \odot s'$ describes the classical sum of counter since $(s \odot s')(t) = s(t) + s'(t)$. The series $e_\odot = 0\delta^{+\infty}$ is the neutral element of this product, and ε is absorbing for this law ($s \odot \varepsilon = \varepsilon$). Afterwards the following mapping will be also considered

$$\Pi_a : \overline{\mathbb{Z}}_{min}[\![\delta]\!] \to \overline{\mathbb{Z}}_{min}[\![\delta]\!], x \mapsto a \odot x.$$

Proposition 1: The Hadamard product of series of $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ satisfies the following distributivity properties:

$$(s_1 \oplus s_2) \odot s_3 = (s_1 \odot s_3) \oplus (s_2 \odot s_3) \tag{8}$$

$$(s_1 \wedge s_2) \odot s_3 = (s_1 \odot s_3) \wedge (s_2 \odot s_3). \tag{9}$$

Proof: According to definitions 3 and 10, the first statement leads to

$$\begin{aligned} ((s_1 \oplus s_2) \odot s_3)(t) &= (s_1 \oplus s_2)(t) \otimes s_3(t) \\ &= (s_1(t) \oplus s_2(t)) \otimes s_3(t) \\ &= (s_1(t) \otimes s_3(t)) \oplus (s_2(t) \otimes s_3(t)), \end{aligned}$$

since $\otimes$ distributes over $\oplus$ in $\overline{\mathbb{Z}}_{min}$. Therefore by considering definition 10 again, the following equalities hold true,

$$\begin{aligned} ((s_1 \oplus s_2) \odot s_3)(t) &= (s_1 \odot s_3)(t) \oplus (s_2 \odot s_3)(t) \\ &= ((s_1 \odot s_3) \oplus (s_2 \odot s_3))(t). \end{aligned}$$

By considering the same arguments (distributivity of $\otimes$ over $\wedge$ in $\overline{\mathbb{Z}}_{min}$, see example 1) equality 9 is obtained. ∎

This proposition (equation (8)) implies that mapping Π_a is a $\oplus$-morphism (*i.e.* $\Pi_a(s_1 \oplus s_2) = \Pi_a(s_1) \oplus \Pi_a(s_2)$ and $\Pi_a(\varepsilon) = \varepsilon$), then it is an isotone mapping.

Proposition 2: The mapping $\Pi_a : x \mapsto a \odot x$ defined over $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ is residuated. The residual will be denoted $(\Pi_a)^\sharp : x \mapsto a \odot^\sharp x$. $(\Pi_a)^\sharp(b)$ is the greatest series x of $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ such that $a \odot x \preceq b$.

Proof: First, series ε is absorbing for the Hadamard product, then $\Pi_a(\varepsilon) = \varepsilon$ and the distributivity of $\odot$ over $\oplus$ leads to $\forall \mathcal{C} \subseteq \overline{\mathbb{Z}}_{min}[\![\delta]\!]$, $\Pi_a(\bigoplus_{x \in \mathcal{C}} x) = \bigoplus_{x \in \mathcal{C}} \Pi_a(x)$, therefore theorem 2 yields the result. ∎

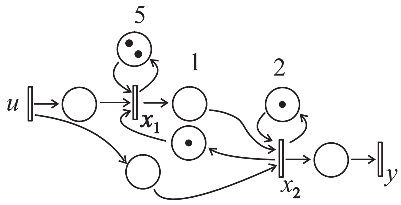

Fig. 1. A Single Input Single Output.

All the previous results stay valid in semiring $\mathbb{N}_{min}[\![\delta]\!]$, which is defined as the set of formal power series with coefficient in $\mathbb{N}_{min}$ and exponent in $\mathbb{N}$. A series $s \in \mathbb{N}_{min}[\![\delta]\!]$ is defined as follows :

$$s = \bigoplus_{t \in \mathbb{N}} s(t)\delta^t,$$

where $s(t) \in \mathbb{N}_{min}$. Semiring $\mathbb{N}_{min}[\![\delta]\!]$ is a sub semiring of $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ and the top element is $\bigoplus_{x \in \mathbb{N}_{min}[\![\delta]\!]} x = \mathsf{T}_{\mathbb{N}_{min}[\![\delta]\!]} = 0\delta^{+\infty}$. Just remark that the top element is then equal to the neutral element of the Hadamard product (see definition 10).

Proposition 3: Let a be a series of $\mathbb{N}_{min}[\![\delta]\!]$ and $\mathcal{C}_a = \{y | y \preceq a\}$ be a subset of $\mathbb{N}_{min}[\![\delta]\!]$. The mapping ${}_{\mathcal{C}_a|}\Pi_a : \mathbb{N}_{min}[\![\delta]\!] \to \mathcal{C}_a, x \mapsto a \odot x$ is dually residuated. The dual residual will be denoted $({}_{\mathcal{C}_a|}\Pi_a)^\flat : x \mapsto a \odot^\flat x$. If $b \in \mathcal{C}_a$ then $({}_{\mathcal{C}_a|}\Pi_a)^\flat(b)$ is the lowest series of $\mathbb{N}_{min}[\![\delta]\!]$ such that $a \odot x \succeq b$.

Proof: First just note that $\mathsf{Im}\Pi_a \subseteq \mathcal{C}_a$ indeed Π_a is an isotone mapping and $\forall s \in \mathbb{N}_{min}[\![\delta]\!], s \preceq e_\odot = \mathsf{T}_{\mathbb{N}_{min}[\![\delta]\!]}$, therefore $\forall s \in \mathbb{N}_{min}[\![\delta]\!], \Pi_a(s) \preceq \Pi_a(e_\odot) = a$. Furthermore, the top element of $\mathcal{C}_a$ is $\mathsf{T}_{\mathcal{C}_a} = a$ and $\Pi_a(\mathsf{T}_{\mathbb{N}_{min}[\![\delta]\!]}) = \Pi_a(e_\odot) = a = \mathsf{T}_{\mathcal{C}_a}$. According to the distributivity property (see equation (9)) the following equality holds true, $\forall \mathcal{A} \subseteq \mathbb{N}_{min}[\![\delta]\!], \Pi_a(\bigwedge_{x \in \mathcal{A}} x) = \bigwedge_{x \in \mathcal{A}} \Pi_a(x)$. The requirements of theorem 2 are then satisfied, and yields the result. ∎

IV. TIMED EVENT GRAPH DESCRIPTION

Timed event graphs can be seen as linear discrete event dynamical systems in some semirings [6] [2]. For instance, by associating to each transition x_i a "counter" function $\{x_i(t)\}_{t \in \mathbb{N}}$, in which $x_i(t)$ is equal to the number of firing for transition x_i to time t, it is possible to obtain a linear state representation in $\overline{\mathbb{Z}}_{min}$. As in conventional system theory, output $\{y(t)\}_{t \in \mathbb{N}}$ of a SISO TEG is then expressed as a $(min, +)$ convolution of its input $\{u(t)\}_{t \in \mathbb{N}}$ by its impulse response[1] $\{h(t)\}_{t \in \mathbb{N}}$. Counter $\{x_i(t)\}_{t \in \mathbb{N}}$ can be represented by a formal series in $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$. For instance, considering the TEG drawn in figure 1, counters u, x_1 and x_2 are related as follows over $\overline{\mathbb{Z}}_{min}$:

$$x_1(t) = 2 \otimes x_1(t-5) \oplus 1 \otimes x_2(t) \oplus u(t).$$

Their respective δ-transforms, expressed over $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$, are then related as:

$$x_1 = 2\delta^5 x_1 \oplus 1x_2 \oplus u.$$

[1]which is the output due to an infinity of input firings at date zero [18].

Consequently, by considering state vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, the following representation over $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ can be obtained :

$$
\begin{aligned}
x &= \begin{pmatrix} 2\delta^5 & 1 \\ \delta & 1\delta^2 \end{pmatrix} x \oplus \begin{pmatrix} e \\ e \end{pmatrix} u \\
y &= \begin{pmatrix} \varepsilon & e \end{pmatrix} x.
\end{aligned}
$$

In a general way, TEG model can be expressed as:

$$
\begin{aligned}
x &= Ax \oplus Bu \\
y &= Cx,
\end{aligned}
$$

where $x \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]^n$ with n the number of internal transitions, $u \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]^p$ with p the number of input transitions and $y \in \overline{\mathbb{Z}}_{min}[\![\delta]\!]^q$ with q the number of output transitions. Matrices A, B and C are of appropriate size with their entries in $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$. According to theorem 1, this state system leads to a transfer relation $y = CA^*Bu$, then in $\overline{\mathbb{Z}}_{min}[\![\delta]\!]$ semiring the transfer matrix of the TEG depicted in figure 1, is given by :

$$CA^*B = (\delta \oplus 1\delta^3)(2\delta^5)^*. \tag{10}$$

Entries of the transfer matrix are periodic series [1] which are usually represented by[2] $p \oplus q(\nu\delta^\tau)^*$. The asymptotic slope[3] of a periodic series $s = p \oplus q(\nu\delta^\tau)^*$ denoted $\sigma_\infty(s)$ is defined as the ratio $\sigma_\infty(s) = \frac{\nu}{\tau}$.

For a SISO system, input[4] $u = e$ yields output $y = (CA^*B)e = CA^*B$ which is called the impulse response of the system. This output is the lowest which can be achieved, $i.e.$ the maximal number of tokens which can come out of the system at each time t. Thanks to corollary 1, it is possible to compute the greatest input u which leads to this lowest output. This greatest input is given by :

$$u = \bigoplus_{\{x|(CA^*B)u \preceq (CA^*Be)\}} x = (CA^*B) \backslash\!\!\!\circ (CA^*Be). \tag{11}$$

This input represents the minimal number of tokens which are necessary to achieve the lowest output.

Algorithms and software tools[5] are available in order to handle such periodic series (see [10] and [8] for algorithms). In particular, the last version allows to compute Hadamard product and its residuals (see propositions 2 and 3). Practical computations can be obtained by considering the following remark.

Remark 1: Let s and s' be two series of $\mathbb{N}_{min}[\![\delta]\!]$, let s" be a series defined as follows :

$$s" : s"(t) = s(t) - s'(t).$$

Series s" is not necessarily a monotonic series. Series $s \odot^\sharp s'$ can be obtained from s" by considering the greatest monotonic series lower than or equal to s", and dually $s \odot^\flat s'$ can be obtained by considering the lowest monotonic series greater than or equal to s".

[2] p is a polynomial that represents the transient and q is a polynomial that represents a pattern which is repeated each τ time units and each ν firings of the transition

[3] Asymptotic slope in a manufacturing context can be viewed as the production rate of the system.

[4] series $e = 0\delta^0 \oplus 1\delta^0 \oplus \ldots$ represents an infinity of tokens at time $t = 0$.

[5] Note that another library which handle ultimately pseudo periodic functions is under development, see [5].

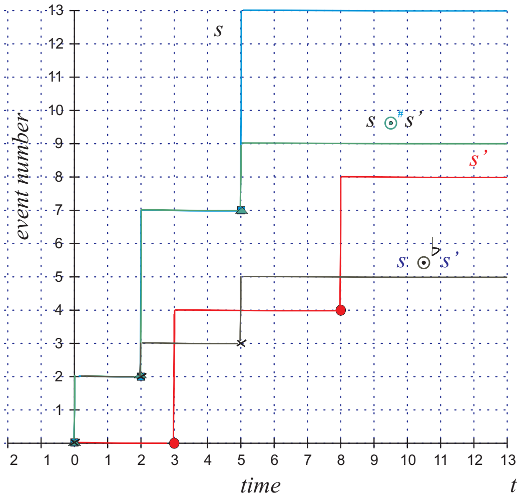

Fig. 2. Series s, s', $s \odot^\sharp s'$ and $s \odot^\flat s'$.

Hereafter asymptotic slope resulting from operations between series is given. If $\nu, \nu' \neq 0$ and $\tau, \tau' \neq 0$, then

$$
\begin{aligned}
\sigma_\infty(s \oplus s') &= \min(\sigma_\infty(s), \sigma_\infty(s')), \\
\sigma_\infty(s \otimes s') &= \min(\sigma_\infty(s), \sigma_\infty(s')), \\
\sigma_\infty(s \odot s') &= \sigma_\infty(s) + \sigma_\infty(s'),
\end{aligned}
$$

if $s' \preceq s$ then $\sigma_\infty(s \odot^\flat s') = \sigma_\infty(s) - \sigma_\infty(s')$,
if $\sigma_\infty(s) \leq \sigma_\infty(s')$ then $\sigma_\infty(s \odot^\sharp s') = \sigma_\infty(s) - \sigma_\infty(s')$,
if $\sigma_\infty(s) \leq \sigma_\infty(s')$ then $\sigma_\infty(s' \backslash\!\!\!\circ s) = \sigma_\infty(s)$, else $s' \backslash\!\!\!\circ s = \varepsilon$.

Example 2: Let $s' = \delta^3 \oplus 4\delta^8 \oplus 8\delta^{+\infty}$ and $s = e \oplus 2\delta^2 \oplus 7\delta^5 \oplus 13\delta^{+\infty}$ be two series representing counters of events. Series s' can be read as no event has occurred until time $t = 3$, 4 events have occurred until time $t = 8$, and 8 events until time $t = +\infty$, that means that the following events have never occurred. Figure 2 proposes a graphical representation of s, s', $s \odot^\sharp s' = e \oplus 2\delta^2 \oplus 7\delta^5 \oplus 9\delta^{+\infty}$ and $s \odot^\flat s' = e \oplus 2\delta^2 \oplus 3\delta^5 \oplus 5\delta^{+\infty}$. Furthermore, it can be checked that $(s \odot^\sharp s') \odot s' = e \oplus 2\delta^2 \oplus 7\delta^3 \oplus 11\delta^5 \oplus 13\delta^8 \oplus 17\delta^{+\infty}$ is lower than s and dually that $(s \odot^\flat s') \odot s' = e \oplus 2\delta^2 \oplus 3\delta^3 \oplus 7\delta^5 \oplus 9\delta^8 \oplus 13\delta^{+\infty}$ is greater than s.

V. Maximal flow for linear systems

The problem addressed now is to compute the lowest output of a system made up of several SISO sub-systems, in the presence of cross-inputs, and the greatest input allowing to achieve this lowest output. First, the case of one interfering input is considered and algorithms, which generalize the approach, are given in a second step.

A. One interfering input on a SISO system

Figure 3 depicts the system studied. Two inputs α_1 and α_2 put tokens in a system which is characterized by a transfer relation denoted β, then the system output is given by :

$$y = \beta \otimes (\alpha_1 \odot \alpha_2).$$

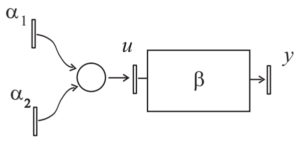

Fig. 3. A system with two convergent inputs.

Trajectories α_1, α_2, u and y are depicted by series of $\mathbb{N}_{min}[\![\delta]\!]$. Input α_2 and output $y = \beta u$ are assumed to be known. The problem considered is to compute the greatest input α_1 which must be added to α_2 in order to achieve output y. Furthermore the flows are assumed to be blindly multiplexing, roughly speaking this means that the worst case must be considered for input α_1 (see [20] for a discussion in the network calculus context).

Then, this problem consists in computing the greatest α_1 such that $\beta(\alpha_1 \odot \alpha_2) \preceq y$. Thanks to corollary 1 and proposition 2, this input is given by :

$$\alpha_1 = \bigoplus_{\{x \mid \beta(x \odot \alpha_2) \preceq y\}} x = (\beta \mathbin{\backslash\!\!\circ} y) \odot^{\sharp} \alpha_2. \tag{12}$$

As said previously, for a SISO system, the best output (the lowest series) is given by the impulse response $y = \beta$, and the greatest input allowing to achieve this output is given by : $\beta \mathbin{\backslash\!\!\circ} \beta$ (see equation (11)). Therefore, by considering equation (12), the greatest series α_1 which leads to the lowest output $y = \beta$ is given by :

$$\alpha_1 = (\beta \mathbin{\backslash\!\!\circ} \beta) \odot^{\sharp} \alpha_2. \tag{13}$$

This bound characterizes the minimal number of tokens which must be added to α_2 to obtain output $y = \beta$, it is not necessary to introduce more tokens, they would not be processed by the system, in other words α_1 is the maximal flow which can be added to this system.

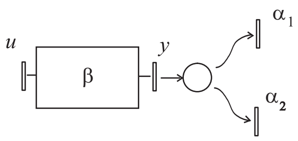

Fig. 4. A system with two divergent outputs.

Figure 4 presents a dual problem to the previous one. The output of a system $y = \beta u$ is assumed to be known, a part of the output flow is devoted to output α_2. This output is assumed to be known and satisfying condition $\alpha_2 \succeq y$. The problem arising is to know what maximal flow α_1 can be achieved. This problem can be expressed as the computation of the lowest series α_1 which is such that $(\alpha_1 \odot \alpha_2) \succeq y$, formally:

$$\alpha_1 = \bigwedge_{\{x \mid (\alpha_2 \odot x) \succeq y\}} x.$$

Conditions about dual residuation of the Hadamard product being fulfilled (see proposition 3), the lowest series is given by :

$$\alpha_1 = y \odot^{\flat} \alpha_2. \tag{14}$$

It characterizes the maximal flow which can go towards output α_1 while preserving output α_2.

B. Several interfering inputs on several SISO sub-systems

Now, let us consider a principal flow $\alpha_1^{(s_1,e_n)}$, crossing sub-systems $\beta_1, ..., \beta_n$ in that order. Let $\alpha^{(s_d,.)}$ be the input interfering with $\alpha_1^{(s_1,e_n)}$, in the front of sub-system β_d with $d \in \{1,..,n\}$ and let $\alpha^{(.,e_q)}$ be the output leaving the system after sub-system β_q with $q \in \{1,..,n\}$.

For each stage i, the system input is denoted u_i and the system output is denoted $y_i = \beta_i u_i$. Secondary inputs $\alpha^{(s_i,.)}$ and outputs $\alpha^{(.,e_i)}$ are assumed to be known.

On each node, the flows are linked by a kind of monotonic version of the Kirchhoff law, which can be expressed as follows:

$$y_i \odot \alpha^{(s_{i+1},.)} = u_{i+1} \odot \alpha^{(.,e_i)}. \tag{15}$$

The following condition is assumed to be fulfilled $\alpha^{(.,e_i)} \succeq (y_i \odot \alpha^{(s_{i+1},.)})$, it means that the flow leaving the node is lower than or equal to the flow coming in the node. Therefore the lowest input u_{i+1} satisfying equality (15) is given by (see equation (14)):

$$u_{i+1} = (y_i \odot \alpha^{(s_{i+1},.)}) \odot^{\flat} \alpha^{(.,e_i)}. \tag{16}$$

This signal represents, at each time t, the maximal number of tokens which can go towards u_{i+1}, while preserving output $\alpha^{(.,e_i)}$.

Dually, the greatest output y_i satisfying equation (15) is given by :

$$y_i = (u_{i+1} \odot \alpha^{(.,e_i)}) \odot^{\sharp} \alpha^{(s_{i+1},.)}. \tag{17}$$

This signal represents, at each time t, the minimal number of tokens which are necessary to satisfy equality (15).

The signal due to the principal flow in the front of system β_i is denoted α_{1i}. By considering, in a first step, that the principal flow is characterized by an impulse input, *i.e.* $\alpha_{11} = e$, it is possible to compute recursively system inputs u_i, system outputs y_i and to obtain the lowest signal $\alpha_{1(n+1)}$ characterizing the maximal instantaneous flow which can go towards this output (see forward algorithm 1 based on equation (14)).

Algorithm 1: Forward computation of the lowest principal output $\alpha_{1(n+1)}$

Data: Series β_i, $\alpha^{(s_i,.)}$, $\alpha^{(.,e_i)}$.
Result: Series u_i, y_i, $\alpha_{1(n+1)}$.
begin
    $u_1 = \alpha^{(s_1,.)} \odot e$;
    $y_1 = \beta_1 u_1$;
    for $i = 2$ *to* $i = n$ **do**
        $u_i = (y_{i-1} \odot \alpha^{(s_i,.)}) \odot^{\flat} \alpha^{(.,e_{i-1})}$;
        $y_i = \beta_i u_i$;
    $\alpha_{1(n+1)} = y_n \odot^{\flat} \alpha^{(.,e_n)}$.
end

Conversely, by considering the backward algorithm 2, based on equation (12), it is possible to compute the greatest

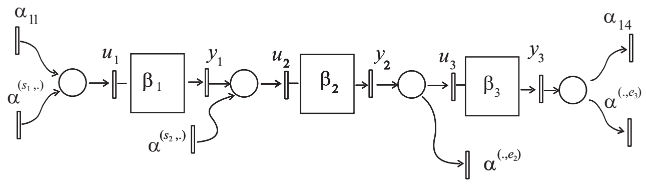

Fig. 5. A system with two interfering inputs.

input α_{11} allowing to satisfy the lowest output $\alpha_{1(n+1)}$. This signal characterizes the minimal number of tokens which must be introduced in the system to obtain output $\alpha_{1(n+1)}$, it is not necessary to introduce more tokens, they would not be processed by the system.

Algorithm 2: Backward computation of the greatest principal input α_{11}.

Data: Series β_i, u_i, $\alpha^{(s_i,.)}$, $\alpha^{(.,e_i)}$, $\alpha_{1(n+1)}$.
Result: Series α_{11}.
begin
  $y_n = \alpha_{1(n+1)} \odot \alpha^{(.,e_n)}$;
  $u_n = \beta_n \circ\!\!\!\!\backslash\, y_n$;
  for $i = n-1$ *to* $i = 1$ **do**
    $y_i = (u_{i+1} \odot \alpha^{(.,e_i)}) \odot^\sharp \alpha^{(s_{i+1},.)}$;
    $u_i = \beta_i \circ\!\!\!\!\backslash\, y_i$;
  $\alpha_{11} = u_1 \odot^\sharp \alpha^{(s_1,.)}$.
end

Figure 5 depicts the system studied to illustrate the results. The system is composed of three sub-systems of which transfer are assumed to be given by :
$\beta_1 = (\delta \oplus 1\delta^3)(2\delta^5)^*$, $\beta_2 = \delta^5(2\delta^6)^*$, $\beta_3 = \delta^7(8\delta^5)^*$.
Secondary inputs and outputs trajectories are assumed to be known. The upstream input of system 1 is given by $\alpha^{(s_1,.)} = e \oplus 2\delta^{10} \oplus 4\delta^{19} \oplus 5\delta^{+\infty}$, the upstream input of system 2 is given by $\alpha^{(s_2,.)} = e \oplus 1\delta^9 \oplus 2\delta^{15} \oplus 3\delta^{+\infty}$, the downstream output of system 2 is given by $\alpha^{(.,e_2)} = \delta^6 \oplus 2\delta^{18} \oplus 4\delta^{26} \oplus 5\delta^{+\infty}$ and the downstream output of system 3 is given by $\alpha^{(.,e_3)} = \delta^{30} \oplus 1\delta^{33} \oplus 2\delta^{39} \oplus 3\delta^{+\infty}$. Algorithm 1 allows to compute the lowest series for the principal flow: $\alpha_{14} = (\delta^{36} \oplus 1\delta^{37})(2\delta^6)^*$.
Algorithm 2 yields the greatest series allowing to achieve α_{14} while preserving secondary outputs:
$\alpha_{11} = \delta^6 \oplus (1\delta^{24} \oplus 2\delta^{28})(2\delta^6)^*$.

VI. Conclusion

In this paper is computed the maximal flow which can be added to a system composed of many $(max, +)$ linear subsystems with exogenous inputs interfering in additive way. The next step will be to compute the maximal flow in a network of $(max, +)$ systems. Usually the networks considered are with constant capacity. Therefore this results would be a generalization of the classical case (see [9]). An avenue is to formalize this problem such as a constraint satisfaction problem (see [13]).

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