Comparing Disjunctive and Concise Approaches for Set-Guaranteed Estimation in Max-Plus Linear Systems

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Abstract: This study compares an existing method with a novel approach for state estimation of Max-Plus Linear systems with bounded uncertainties. Traditional stochastic filtering does not apply to this system class, despite computable posterior probability density function (PDF) support. Existing literature suggests a limited scalability disjunctive approach using difference-bound matrices. To overcome this, we study an alternative method recently investigated in Mufid et al. (2022) using Satisfiability Modulo Theory (SMT) techniques, which are known to be NP-hard. We propose a concise method that utilizes a pseudo-polynomial time algorithm using max-plus algebra. We evaluate its efficiency against SMT techniques through numerical experiments involving sparse matrix multiplications for enhanced computational speed.

Keywords: Max-Plus Linear Systems, State-Estimation, Satisfiability Modulo Theories, Fixed point algorithm

1. INTRODUCTION

Max-plus algebra theory is suitable in analyzing Discrete Event Systems (DES) with delay and synchronization. These phenomena are found in production systems, computing networks, and transportation systems (see Baccelli et al. (1992); Heidergott et al. (2006) for an overview). This theory employs an algebraic structure known as idempotent semiring, enabling the description of these systems as linear models. Thus, Max-Plus Linear (MPL) systems can be defined through recursive state-space equations, where states represent event-times (time instants) within the system, forming a timetable trajectory. Residuation theory (Blyth and Janowitz (1972)) further aids in addressing crucial issues in control theory: controllability, observability, stabilization, and feedback synthesis (see Hardouin et al. (2018)).

In problems involving model parameter uncertainties, deterministic considerations are common, disregarding probabilistic aspects. However, in filtering problems affected by random processes influencing model parameters, addressing probabilistic aspects becomes crucial. Stochastic Max-Plus Linear (sMPL) systems handle this by defining MPL systems with matrices containing random variable entries. State-estimation in the Bayesian approach involves computing the *posterior* probability density function (PDF) using available measurements. While the Kalman filter and its extensions are practical for filtering with additive Gaussian noise, they are unsuitable for MPL systems due to their nonlinear discontinuities (see Mendes et al. (2019); Winck et al. (2022c) for details). For such systems, we can apply other stochastic filtering strategies as the Sequential Monte-Carlo (SMC) method, also known as Particle Filter but with numerical difficulties related to the generation of the particles (see Candido et al. (2013); Candido et al. (2020)). This work focuses on systems where uncertain parameters can vary within known intervals, namely uncertain MPL (uMPL) systems, i.e., sMPL systems with bounded random variables.

In this work, we study an *indirect* computation of the support of the *posterior* PDF for uMPL systems. This computation is referred to as set-estimation. In Candido et al. (2018), the authors use the works of Adzkiya et al. (2015) on difference-bound matrices, in Winck et al. (2022b) they use max-plus polyhedra (Allamigeon et al. (2013)) and in Winck et al. (2022a) they use residuation theory (Blyth and Janowitz (1972)).

Contribution: we propose the concise approach using a fixed-point algorithm (with sparse matrix operations in max-plus algebra) and we compare it with the disjunctive Satisfiability Modulo Theory (SMT) approach of Mufid et al. (2022) using Z3 solver of De Moura and Bjørner (2008) to estimate (if it is feasible) the state of uMPL systems.

The paper is organized as follows: section 2 recalls the the basic notions of MPL systems. Section 3 presents the *indirect* computation of the set of all states that can be reached from a previous state through the transition model and that can lead to the measurement output through the measurement function by using the disjunctive and concise approaches. Section 4 presents the application: proving the feasibility guarantee of set-estimation. Numerical simulations are performed to compare the two approaches. Finally, section 5 concludes the work and presents some ideas for future works.

2. PRELIMINARIES

2.1 Max-plus algebra

A set $\mathcal{D}$ forms a *dioid* or *idempotent semiring* if it satisfies certain algebraic properties. These properties include the associativity, commutativity and idempotency of the sum $\oplus$, as well as the associativity and left and right distributivity of the product $\otimes$ w.r.t $\oplus$. The dioid $\mathcal{D}$ also includes a null element, ε such that $\forall a \in \mathcal{D}, a \oplus \varepsilon = a$ and an identity element e, such that $\forall a \in \mathcal{D}, a \otimes e = e \otimes a = a$. A partial order relation

$$a \succeq b \Leftrightarrow a = a \oplus b$$

is defined for elements $a, b \in \mathcal{D}$. This order relation makes $\mathcal{D}$ to be a partially ordered set such that each pair of elements a, b admits the lowest upper bound $\sup\{a, b\}$ which coincides with $a \oplus b$. Hence, a dioid is in particular a *sup-semilattice*. Furthermore, the sum and the left and right products preserve this relation, i.e., if $a \succeq b$ then $a \oplus c \succeq b \oplus c$, $a \otimes c \succeq b \otimes c$ and $c \otimes a \succeq c \otimes b$. A dioid $\mathcal{D}$ is complete if it is closed for infinite sums and the left and right distributivity of the product extend to infinite sums. In practice, for $\mathcal{D}$ to be complete, the top element, denoted $\top$, exists and is equal to the sum of all elements of $\mathcal{D}$, i.e., $\top = \bigoplus_{a \in \mathcal{D}} a$, such that $\forall a \in \mathcal{D}, a \oplus \top = \top$. This element respects the absorbing rule, i.e., $\varepsilon \otimes \top = \varepsilon$. For a complete dioid, an inner operation representing the lower bound of the operands, denoted, $\overline{\oplus}$ automatically exists. The partial order relation can be expressed as $a \succeq b \Leftrightarrow a = a \oplus b \Leftrightarrow b = a \overline{\oplus} b$ where $a \overline{\oplus} b = \inf\{a, b\}$ is the greatest lower bound of a, b.

The max-plus algebra, denoted as $\overline{\mathbb{R}}_{\max}$, is a set that includes $\mathbb{R}$ along with the elements $\varepsilon = -\infty$, $\top = +\infty$ and $e = 0$, i.e., $\mathbb{R} \cup \{-\infty, +\infty\}$, with the two binary operations $a \oplus b := \max\{a, b\}$ and $a \otimes b := a + b$. This algebra is an example of a complete dioid. This dioid is linearly ordered w.r.t. $\oplus$ and the order $\succeq$ in this set coincides with the usual linear order $\geq$. Furthermore, in this dioid, the operation $a \overline{\oplus} b$ coincides with $\min\{a, b\}$. In the sequel, the symbol $\otimes$ can be omitted in the absence of ambiguity.

The two binary operations in $\overline{\mathbb{R}}_{\max}$ are naturally extended to matrices. Given $A, B \in \overline{\mathbb{R}}_{\max}^{n \times p}$, $C \in \overline{\mathbb{R}}_{\max}^{p \times q}$ and $\alpha \in \overline{\mathbb{R}}_{\max}$, we have $(A \oplus B)_{ij} = (a_{ij} \oplus b_{ij})$, $(A \otimes C)_{ij} = (\bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj})$ and $(\alpha \otimes A)_{ij} = \alpha \otimes a_{ij}$. The partial order relation is also applied to matrices as follows $A \geq B \Leftrightarrow A = A \oplus B$ for $A, B \in \overline{\mathbb{R}}_{\max}^{n \times p}$, where $\geq$ refers to the linear order on $\mathbb{R}^{n \times p}$.

Given $k \in \mathbb{N}$ and $A \in \overline{\mathbb{R}}_{\max}^{n \times n}$, $A^{\otimes k} = A \otimes \cdots \otimes A$ (k-fold). The matrix $A^{\otimes 0}$ is the n-dimensional identity matrix I_n, which is a special kind of the max-plus version of diagonal matrices[1] $\mathrm{diag}_{\oplus}(\bullet)$ with e on the main diagonal. The absorbing matrix $\mathcal{E}_{n \times m}$ is defined as the $(n \times m)$-dimensional matrix whose entries are ε. The all-e matrix $E_{n \times m}$ follows the same idea, but with its entries equal to e. The Kleene star of a matrix A is defined as $A^* = (\bigoplus_{k \in \mathbb{N}} A^{\otimes k})$.

A system of linear inequalities $A \otimes x \leq y$, where $A \in \overline{\mathbb{R}}_{\max}^{m \times n}$, $x \in \overline{\mathbb{R}}_{\max}^{n}$ and $y \in \overline{\mathbb{R}}_{\max}^{m}$ admits the greatest solution $\hat{x} = A^{\sharp}(y)$ given by the following residuation formula $(A^{\sharp}(y))_i = \min_{j=1}^{m} (-a_{ji} + y_j)$, which is equivalent to $-(A^T \otimes (-y))$. Obviously, if $A \otimes x = y$ admits a solution, then $\hat{x}$ is the greatest solution and $A \otimes \hat{x} = y$ holds. This result is also applied to find the greatest solution of the two-sided equation $A \otimes x = B \otimes x$ where $A, B \in \overline{\mathbb{R}}_{\max}^{m \times n}$. The following equivalences hold

$$\begin{aligned} A \otimes x = B \otimes x &\Leftrightarrow A \otimes x \leq B \otimes x \text{ and } B \otimes x \leq A \otimes x \\ &\Leftrightarrow x \leq A^{\sharp}(B \otimes x) \text{ and } x \leq B^{\sharp}(A \otimes x) \\ &\Leftrightarrow x \leq A^{\sharp}(B \otimes x) \overline{\oplus} B^{\sharp}(A \otimes x) \\ &\Leftrightarrow x = x \overline{\oplus} A^{\sharp}(B \otimes x) \overline{\oplus} B^{\sharp}(A \otimes x). \end{aligned}$$

Hence, the greatest fixed-point of

$$\Pi(x) = x \overline{\oplus} A^{\sharp}(B \otimes x) \overline{\oplus} B^{\sharp}(A \otimes x)$$

is the greatest solution of $A \otimes x = B \otimes x$. Moreover, since A, $A^{\sharp}$, B and $B^{\sharp}$ are clearly isotone maps[2] then $\Pi(x)$ is also isotone. Thus, to solve this two-sided equation, it suffices to iterate the sequence $\mathcal{I} : x[k+1] = \Pi(x[k])$ on an initial $x[k]$, namely $x[0]$, until convergence $x[\Bbbk + 1] = x[\Bbbk]$ is reached for a specific $\Bbbk \in \mathbb{N}$ (fixed-point iteration). As a consequence, if a *finite* (non-ε entries only) greatest solution $x[\Bbbk]$ of $A \otimes x = B \otimes x$ exists, then $\mathcal{I}$ is able to find it in a finite number of steps such that $x[\Bbbk] \leq x[0]$. This computation is known to have a pseudo-polynomial complexity, i.e., the convergence rate is polynomial according to the distance between $x[\Bbbk]$ and $x[0]$. Conditions are also presented in Cuninghame-Green and Butkovic (2003) to ensure that this procedure converges in finite time because $\mathcal{I}$ is likely to run infinitely since it is possible that one or more of the entries of $x[k]$ decrease indefinitely to ε. Nevertheless, the algorithm seems to be efficient (convergence with finite time and with a low number of steps) to handle problems in this work.

2.2 Intervals over max-plus algebra

Interval analysis in the max-plus algebra was originally presented in Litvinov and Sobolevskii (2001). A (closed) interval $[x]$ in max-plus algebra is a subset of $\overline{\mathbb{R}}_{\max}$ of the form $[x] = [\underline{x}, \overline{x}] = \{x \in \overline{\mathbb{R}}_{\max} \mid \underline{x} \leq x \leq \overline{x}\}$ with $\underline{x} < \overline{x}$. We denote by $\mathbb{I}\overline{\mathbb{R}}_{\max}$ the set of intervals of $\overline{\mathbb{R}}_{\max}$. An interval $[x] \subseteq [y]$ if and only if $\underline{y} \leq \underline{x} \leq \overline{x} \leq \overline{y}$. Similarly, $[x] = [y]$ if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$. A value $x \in \overline{\mathbb{R}}_{\max}$ can be represented by the *degenerated* interval $[x, x]$. The $\oplus$ and $\otimes$ operations exist for intervals: $[\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $[\underline{x}, \overline{x}] \otimes [\underline{y}, \overline{y}] = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$.

An interval matrix in max-plus algebra is a matrix whose elements are intervals. The operations $\oplus$ and $\otimes$ can be extended to interval matrices. Given the interval matrices $[A] = [\underline{A}, \overline{A}]$, $[B] = [\underline{B}, \overline{B}]$ and $[C] = [\underline{C}, \overline{C}]$ of dimensions $(n \times p)$, $(n \times p)$ and $(p \times q)$, then $([A] \oplus [B])_{ij} = [a_{ij}] \oplus [b_{ij}]$ and $([A] \otimes [C])_{ij} = \bigoplus_{k=1}^{p} ([a_{ik}] \otimes [c_{kj}])$. Moreover, the product of $\alpha \in \overline{\mathbb{R}}_{\max}$ by $[A]$ is given by $\alpha \otimes [A] = [\alpha \otimes \underline{A}, \alpha \otimes \overline{A}]$ and the k-th power of $[A]$ is given by $[A]^{\otimes k} = [\underline{A}^{\otimes k}, \overline{A}^{\otimes k}]$. The Kleene star operation is also defined for intervals matrices, mathematically for $[A]$ we have $[A]^* = (\bigoplus_{k \in \mathbb{N}} [A]^{\otimes k})$.

[1] A max-plus diagonal matrix has its entries outside the main diagonal equal to ε

[2] $A^{\sharp}$ and $B^{\sharp}$ are isotone maps but not necessarily linear. Hence, in general $A^{\sharp}(x) \oplus A^{\sharp}(y) \neq A^{\sharp}(x \oplus y)$ for $x, y \in \overline{\mathbb{R}}_{\max}^{n}$.

2.3 Max-plus linear systems

Discrete Event Systems (DES) involve *synchronization* and *concurrence*. Synchronization in manufacturing occurs when multiple resources are needed simultaneously, while concurrence involves making choices among available options within the same timeframe. The *max* operator is crucial in synchronization modeling for defining temporal alignment.

Synchronization phenomena in Discrete Event Systems (DES) are represented using *timed* models, focusing on sequences of time instants and event occurrences, whereas *logical* models deal with possible event sequences and associated conditions.

One of the existing formalisms for modeling timed systems is to consider *linear* recursive state-space equations within the algebraic framework of $\overline{\mathbb{R}}_{\max}$. This algebraic structure is well-suited to represent the behavior of synchronization ($\oplus$) and timing information [3] ($\otimes$). By employing appropriate algebraic manipulation and transformation, one obtains the following autonomous [4] Max-Plus Linear (MPL) systems:

$$\mathcal{S}: \begin{cases} x(k) = A_0 x(k) \oplus A_1 x(k-1), \\ z(k) = Cx(k) \end{cases}$$

where $A_0, A_1 \in \overline{\mathbb{R}}_{\max}^{n\times n}$ and $C \in \overline{\mathbb{R}}_{\max}^{p\times n}$. Each event is labeled with an index $i \in \{1,\ldots,n\}$, and $x_i(k) \in \overline{\mathbb{R}}_{\max}$ represents the time instant of the k-th occurrence of event i. As it can be noticed, $x(k) = (x_1(k),\ldots,x_n(k))^\intercal$ appears in both sides of the above recursive equation. The *transition* model and the *measurement* function are represented by the pair (A_0, A_1) and C, respectively. The transition model admits an alternative form, given by $x(k) = Ax(k-1)$ with $A = A_0^* A_1$ such that the orbit of trajectory of $x(k)$ in this form is equal to the one in $\mathcal{S}$ (see Baccelli et al. (1992) for details).

In this paper, we assume the system $\mathcal{S}$ is uncertain, i.e., the matrices have some entries which are random variables belonging to intervals. Thus, an uncertain MPL (uMPL) system is defined as

$$\mathcal{S}_u: \begin{cases} x(k) = A_0(k)x(k) \oplus A_1(k)x(k-1), \\ z(k) = C(k)x(k) \end{cases} \qquad (1)$$

where $A_0(k) \in [A_0] = [\underline{A}_0, \overline{A}_0] \in \overline{\mathbb{IR}}_{\max}^{n\times n}$, $A_1(k) \in [A_1] = [\underline{A}_1, \overline{A}_1] \in \overline{\mathbb{IR}}_{\max}^{n\times n}$ and $C(k) \in [C] = [\underline{C}, \overline{C}] \in \overline{\mathbb{IR}}_{\max}^{p\times n}$ are nondeterministic matrices. Similarly, the transition model of uMPL systems also admits an alternative form representation by considering $x(k) = A(k)x(k-1)$ with $A(k) = A_0^*(k)A_1(k)$ where $A_0(k) \in [A_0]$ and $A_1(k) \in [A_1]$. The "exact" bounds of $A(k)$ are unknown.

Remark 1. It is important to note that the equation $x(k) = A(k)x(k-1)$, where $A(k) \in [A_0]^*[A_1]$, over-approximates the reachable space of $\mathcal{S}_u$ concerning a given state $x(k-1)$. In other words, this form is conservative since we can compute the "rough" bounds of $A(k)$.

[3] In manufacturing, the timing information represents the *processing time* of a task (in practice, it is a *delay*).

[4] Any nonautonomous max-plus DES can be transformed into an augmented autonomous one (Baccelli et al., 1992, Sec. 2.5) and in this work we consider, without loss of generality, autonomous systems only.

2.4 Max-plus systems using disjunctive approach

In Adzkiya et al. (2015), the authors represent max-plus systems $y = M \otimes x$, with $y \in \overline{\mathbb{R}}_{\max}^q$ and $x \in \overline{\mathbb{R}}_{\max}^n$, using disjunctions with operations in $\mathbb{R}$. If the above systems are bounded, i.e., $\underline{M} \otimes x \le y \le \overline{M} \otimes x$ with $\underline{M}, \overline{M} \in \overline{\mathbb{R}}_{\max}^{q\times n}$ then we obtain the following inequalities for all $i \in \{1,\ldots,q\}$:

$$\max(\underline{m}_{i1} + x_1, \ldots, \underline{m}_{in} + x_n) \le y_i \Leftrightarrow$$
$$x_1 \le y_i - \underline{m}_{i1} \text{ and } \ldots \text{ and } x_n \le y_i - \underline{m}_{in}$$

and

$$y_i \le \max(\overline{m}_{i1} + x_1, \ldots, \overline{m}_{in} + x_n) \Leftrightarrow$$
$$y_i - \overline{m}_{i1} \le x_1 \text{ or } \ldots \text{ or } y_i - \overline{m}_{in} \le x_n.$$

In details, $\underline{M} \otimes x \le y \Leftrightarrow x \le \underline{M}^\sharp(y)$, and

$$y \le \overline{M} \otimes x \Leftrightarrow \bigwedge_{i=1}^{q} \left(\bigvee_{j=1}^{n} (y_i - \overline{m}_{ij} \le x_j) \right),$$

with $\wedge$ and $\vee$ playing the role of the logic operators AND and OR, respectively. Hence, $\underline{M} \otimes x \le \underline{y}$ is represented concisely, which is not the case for $y \le \overline{M} \otimes x$ since it is represented by the combination of n^q elements.

3. SUPPORT OF THE POSTERIOR PDF

In stochastic filtering, the relevant information is obtained from the posterior PDF. In a set-guaranteed estimation, one is interested in computing its support. Following Candido et al. (2018), this support is the set of all possible states $x(k)$ that can be reached from the previous state $x(k-1)$ through the transition model and are consistent with the observed measurement $z(k)$ through the measurement function. Mathematically, the image of $x(k-1)$ w.r.t. to $A_0(k) \in [A_0]$ and $A_1(k) \in [A_1]$ is given by

$$\text{Im}_{[A_0],[A_1]}\{x(k-1)\} = \{A_0^* A_1 x(k-1) \in \overline{\mathbb{R}}_{\max}^n \mid A_0 \in [A_0], A_1 \in [A_1]\}, \qquad (2)$$

i.e., the set of all states $x(k)$ that can be reached from $x(k-1)$ through the transition model [5]. We also show how to characterize the inverse image of $z(k)$ w.r.t. $C(k) \in [C]$, formally

$$\text{Im}_{[C]}^{-1}\{z(k)\} = \{x \in \overline{\mathbb{R}}_{\max}^n \mid \exists C \in [C], Cx = z(k)\}, \qquad (3)$$

i.e., the set of all $x(k)$ that can lead to $z(k)$ through the measurement function. Straightforwardly, the support of the posterior PDF is defined as

$$\mathcal{X}_k = \text{Im}_{[A_0],[A_1]}\{x(k-1)\} \cap \text{Im}_{[C]}^{-1}\{z(k)\}. \qquad (4)$$

In Candido et al. (2018); Candido et al. (2020) the authors use difference-bounds matrices (see Miné (2007) for an overview), which represent zones, to compute exactly $\mathcal{X}_k$. These disjunctive approaches lack in scalability, since it is necessary to consider an exponential number of combinations for encoding the upper bounds of the transition model and measurement functions (see Subsection 2.4). For further details, please refer to these works.

[5] An over-approximation for $\text{Im}_{[A_0],[A_1]}\{x(k-1)\}$ is simply computed as $[\![\text{Im}_{[A_0],[A_1]}\{x(k-1)\}]\!] = \{Ax(k-1)\overline{\mathbb{R}}_{\max}^n \mid A \in [A_0]^*[A_1]\}$, i.e., $\text{Im}_{[A_0],[A_1]}\{x(k-1)\} \subseteq [\![\text{Im}_{[A_0],[A_1]}\{x(k-1)\}]\!]$. Please refer to Remark 1.

3.1 A symbolic disjunctive approach

Formal methods have significantly benefited from advancements in solving Boolean satisfiability (SAT) problems. One notable work that exemplifies this progress is the supervisory control of DES (Shoaei et al. (2014)). In various applications, multiple problems involve determining the *satisfiability* of formulas in more expressive logics like *first-order logic* w.r.t. background theories. This concept is known as Satisfiability Modulo Theories (SMT) (see Barrett and Tinelli (2018); Kroening and Strichman (2016) for an overview). In SMT, one can verify, for example, if there exist (or for all) certain symbolic variables $\mathtt{x}$ and $\mathtt{y}$ in $\mathbb{R}$ that satisfy a given symbolic formula F. For instance, $F : (\mathtt{x} \geq 0) \wedge (\mathtt{y} < 2) \vee (\mathtt{x} - \mathtt{y} < -1)$, is tested for satisfiability w.r.t. a set [6]. If a solution exists, it returns values for $\mathtt{x}$ and $\mathtt{y}$ that make each asserted constraint true.

Remark 2. Difference-bound constraints can be represented as Boolean combinations of *atoms* $x_i - x_j \leq c$, which form difference-logic formulas. Thus, the SMT approach has the same expressiveness as the difference-bound matrix approach in Adzkiya et al. (2015); Candido et al. (2018).

In Mufid et al. (2020, 2022), max-plus systems have been expressed as SMT formulas. Briefly, we have $y = M \otimes x$ with $m_{ij} \in \overline{\mathbb{R}}_{\max}$ for $(i, j) \in \{1, \ldots, q\} \times \{1, \ldots, n\}$. It follows from Subsection 2.4 that for each $i \in \{1, \ldots, q\}$ there exists (at least) a $g_i \in \{1, \ldots, n\}$ such that $\forall j \in \{1, \ldots, n\} \setminus \{g_i\}$ $y_i = m_{ig_i} + x_{g_i} \geq m_{ij} + x_j$. Hence, the aforementioned result is equivalent to evaluate the following SMT formula

$$F_i \;:\; \left(\bigwedge_{j \in \mathcal{J}_i} \mathtt{y}_i - \mathtt{m}_{ij} \geq \mathtt{x}_j \right) \wedge \left(\bigvee_{j \in \mathcal{J}_i} \mathtt{y}_i - \mathtt{m}_{ij} = \mathtt{x}_j \right),$$

where $\mathtt{y}_i, \mathtt{x}_j, \mathtt{m}_{ij}$ are symbolic variables and $\mathcal{J}_i \subseteq \{1, \ldots, n\}$ represents the set of indices j such that $m_{ij} \neq \varepsilon$. If each m_{ij} is bounded, then it suffices to add the following symbolic formula

$$B_i \;:\; \left(\bigwedge_{j \in \mathcal{J}_i} (\mathtt{m}_{ij} \geq \underline{m}_{ij}) \wedge (\mathtt{m}_{ij} \leq \overline{m}_j) \right)$$

to F_i, i.e., $F_i \wedge B_i$. Hence, $\bigwedge_{i=1}^{q} F_i \wedge B_i$ *symbolically* represents $y = M \otimes x$ with $\underline{M} \leq M \leq \overline{M}$.

For systems $\mathcal{S}_u$ of (1) let us define for each row of the transition model the following formula:

$$Row_i^{k,k-1} : Conj_i^{k,k-1} \wedge Disj_i^{k,k-1} \wedge Bnd_i^{k,k-1}$$

with

$Conj_i^{k,k-1}$:

$$\left(\bigwedge_{l \in \mathcal{G}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_l^{(k)} \geq \mathtt{a0}_{il}^{(k)} \right) \wedge \left(\bigwedge_{j \in \mathcal{F}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_j^{(k-1)} \geq \mathtt{a1}_{ij}^{(k)} \right)$$

$Disj_i^{k,k-1}$:

$$\left(\bigvee_{l \in \mathcal{G}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_j^{(k)} = \mathtt{a0}_{il}^{(k)} \right) \vee \left(\bigvee_{j \in \mathcal{F}_i} \mathtt{x}_i^{(k)} - \mathtt{x}_j^{(k-1)} = \mathtt{a1}_{ij}^{(k)} \right)$$

[6] The formula F has a solution if $\mathtt{x}, \mathtt{y} \in \mathbb{R}$ but no solution if $\mathtt{x}, \mathtt{y} \in \mathbb{Z}$.

and

$$Bnd_i^{k,k-1} \;:\; \left(\bigwedge_{l \in \mathcal{G}_i} (\mathtt{a0}_{il}^{(k)} \geq \underline{a}_{0_{il}}) \wedge (\mathtt{a0}_{il}^{(k)} \leq \overline{a}_{0_{il}}) \right) \wedge \left(\bigwedge_{j \in \mathcal{F}_i} (\mathtt{a1}_{ij}^{(k)} \geq \underline{a}_{1_{ij}}) \wedge (\mathtt{a1}_{ij}^{(k)} \leq \overline{a}_{1_{ij}}) \right)$$

where $\mathtt{x}_1^{(k)}, \ldots, \mathtt{x}_n^{(k)}$ and $\mathtt{a1}_{ij}^{(k)}, \mathtt{a0}_{il}^{(k)}$ are symbolic variables for each k and $\mathcal{F}_i, \mathcal{G}_i \subseteq \{1, \ldots, n\}$ are, respectively, sets of indices of the i-th rows of $A_1(k), A_0(k)$ that are different ε (i.e., are finite). Hence, $Row_i^{k,k-1}$ is used to represent *symbolically* the transition model of (1) as the following formula: $\mathtt{D}^{k,k-1} : \bigwedge_{i=1}^{n} Row_i^{k,k-1}$. The following formula represents *symbolically* the measurement function of (1): $\mathtt{O}^{k,k} : \bigwedge_{i=1}^{p} \mathtt{O}_i^{k,k}$ with

$\mathtt{O}_i^{k,k}$:

$$\left(\bigwedge_{j \in \mathcal{H}_i} \mathtt{z}_i^{(k)} - \mathtt{x}_j^{(k)} \geq \mathtt{c}_{ij}^{(k)} \right) \wedge \left(\bigvee_{j \in \mathcal{H}_i} \mathtt{z}_i^{(k)} - \mathtt{x}_j^{(k)} = \mathtt{c}_{ij}^{(k)} \right) \wedge \left(\bigwedge_{j \in \mathcal{H}_i} (\mathtt{c}_{ij}^{(k)} \geq \underline{c}_{ij}) \wedge (\mathtt{c}_{ij}^{(k)} \leq \overline{c}_{ij}) \right)$$

where $\mathtt{z}_1^{(k)}, \ldots, \mathtt{z}_p^{(k)}$ and $\mathtt{c}_{ij}^{(k)}$ are symbolic variables and $\mathcal{H}_i \subseteq \{1, \ldots, n\}$ with the same meaning as for $\mathcal{F}_i$ but for $C(k)$. *Symbolically*, $\mathcal{X}_k$ of (4) is represented by the following formula:

$$\mathtt{X}^k \;:\; \mathtt{D}^{k,k-1} \wedge \mathtt{O}^{k,k}.$$

3.2 A concise approach

The previous approach uses the encoding of max-plus systems in standard algebra to take advantage of a powerful method for affine systems. For this reason, we derive in the sequel an equivalent and *concise* method based exclusively on max-plus algebra.

First, let us write the transition model of (1) as

$$\underline{A}_0 x(k) \oplus \underline{A}_1 x(k-1) \leq x(k) \leq \overline{A}_0 x(k) \oplus \overline{A}_1 x(k-1),$$

then define $x = x(k)$, $\underline{v} = \underline{A}_1 x(k-1)$ and $\overline{v} = \overline{A}_1 x(k-1)$ thus

$$\begin{cases} \underline{A}_0 x \oplus \underline{v} \leq x, \\ \overline{A}_0 x \oplus \overline{v} \geq x. \end{cases}$$

Now, taking advantage of the partial order relation on this algebraic structure, the two-sided equation below is obtained

$$\overbrace{\begin{pmatrix} (\underline{A}_0 \oplus I_n) & \underline{v} \\ (\overline{A}_0 \oplus I_n) & \overline{v} \end{pmatrix}}^{:=LD(\underline{v},\overline{v})} \begin{pmatrix} x \\ e \end{pmatrix} = \overbrace{\begin{pmatrix} I_n & \mathcal{E}_{n\times 1} \\ \overline{A}_0 & \overline{v} \end{pmatrix}}^{:=UD(\overline{v})} \begin{pmatrix} x \\ e \end{pmatrix} \tag{5}$$

such that all $x \in \mathrm{Im}_{[A_0],[A_1]}\{x(k-1)\}$ of (2) satisfy the above equation. For the measurement function of (1), a similar procedure exists and was originally derived in Winck et al. (2022b). Briefly, $\underline{C}x(k) \leq z(k) \leq \overline{C}x(k)$ is written as

$$\overbrace{\begin{pmatrix} \underline{C} & z \\ \overline{C} & z \end{pmatrix}}^{:=LO} \begin{pmatrix} x \\ e \end{pmatrix} = \overbrace{\begin{pmatrix} \mathcal{E}_{p\times n} & z \\ \overline{C} & \mathcal{E}_{p\times 1} \end{pmatrix}}^{:=UO} \begin{pmatrix} x \\ e \end{pmatrix} \tag{6}$$

with $x = x(k)$ and $z = z(k)$, such that all $x \in \mathrm{Im}^{-1}_{[C]}\{z(k)\}$ of (3) satisfy this two-sided equation. It is evident that all $x(k) \in \mathcal{X}_k$, as defined in (4), satisfy both (5) and (6) simultaneously. By vertically concatenating the associated matrices, we obtain a single matrix equation that represents all $x(k) \in \mathcal{X}_k$.

4. FEASIBILITY GUARANTEES FOR SET-ESTIMATION

In a set-estimation scheme, we aim at computing a value for $x(k)$ within $\mathcal{X}_k$ of (4). Clearly (4) is useless, since $x(k-1)$ is unknown. Then, an estimate $\hat{x}(k)$ is computed such that

$$\hat{x}(k) \in \hat{\mathcal{X}}_k = \mathrm{Im}_{[A_0],[A_1]}\{\hat{x}(k-1)\} \cap \mathrm{Im}^{-1}_{[C]}\{z(k)\}$$

where $\hat{x}(k-1)$ is the estimate of $x(k)$ at $k-1$. Of course, the success of this approach is related to the distance between $x(k)$ and $\hat{x}(k)$. An estimate $\hat{x}(k) \in \mathrm{Im}_{[A_0],[A_1]}\{\hat{x}(k-1)\}$ may be not in $\mathrm{Im}^{-1}_{[C]}\{z(k) = C(k)x(k)\}$ since we cannot guarantee that $x(k-1)$ is equal to $\hat{x}(k-1)$ and thus $\hat{\mathcal{X}}_k$ may be empty. In this case, $\hat{x}(k)$ is said to be an *unfeasible* estimation. Based on Section 3, we derive a disjunctive and a concise tests to verify feasibility of $\hat{x}(k)$ at each k and, in the affirmative case, return an estimate.

4.1 Symbolic disjunctive method

In Mufid et al. (2022), the authors presented a numerical benchmark showing the efficiency of the SMT-based approach for reachability problems.

In a procedural way, consider the *symbolic* formula that represents $\mathcal{X}_k$ of (4). Let us replace $\mathtt{x}_1^{(k-1)}, \ldots, \mathtt{x}_n^{(k-1)}$ with $\hat{x}_1(k-1), \ldots, \hat{x}_n(k-1)$, hence defining $\mathtt{X}^{k|k}$ as the *prediction* formula. In the same way, we replace $\mathtt{z}_1^{(k)}, \ldots, \mathtt{z}_p^{(k)}$ with $z_1(k), \ldots, z_p(k)$, hence defining $\tilde{\mathtt{X}}_{k|k}$ as the *likelihood* formula. Thus

$$\mathtt{X}^{k|k} : \mathtt{X}^{k|k-1} \wedge \tilde{\mathtt{X}}^{k|k}$$

represents the *correction* formula, i.e., $\hat{\mathcal{X}}_k$. Using Z3 *solver* of De Moura and Bjørner (2008), we are able to verify if $\mathtt{X}^{k|k}$ is `SAT` and return a *solution* that makes each asserted constraint true, defining then a value for $x(k)$, i.e., an *arbitrary* estimate $\hat{x}(k)$. As part of a filtering algorithm, a recursion is defined, i.e., $\hat{x}(k-1) \leftarrow \hat{x}(k)$ and the solver is called once again. If the solver returns `UNSAT` for some k, then $\hat{x}(k)$ is unfeasible and we stop the filtering procedure.

4.2 Concise fixed-point method

In a procedural way, let us consider (5) with $\underline{v} = \underline{A}_1\hat{x}(k-1)$ and $\overline{v} = \overline{A}_1\hat{x}(k-1)$, hence defining $X_{L,k|k-1}(x^\intercal, e)^\intercal = X_{U,k|k-1}(x^\intercal, e)^\intercal$ as the *prediction* equation, with $X_{L,k|k-1} = LD(\underline{v}, \overline{v})$, $X_{U,k|k-1} = UD(\overline{v})$. In the same way, let us consider (6) with $z = z(k)$, hence defining $\tilde{X}_{L,k|k}(x^\intercal, e)^\intercal = \tilde{X}_{U,k|k}(x^\intercal, e)^\intercal$ as the *likelihood* equation with $\tilde{X}_{L,k|k} = LO$, $\tilde{X}_{U,k|k} = UO$. Thus,

$$\begin{pmatrix} X_{L,k|k-1} \\ \tilde{X}_{L,k|k} \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} X_{U,k|k-1} \\ \tilde{X}_{U,k|k} \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

represents the *correction* equation, i.e., $\hat{\mathcal{X}}_k$. Furthermore, by using the fixed-point iteration algorithm presented in Subsection 2.1, we are able to verify if the previous two-sided equation has solution and compute the *greatest* estimate $\hat{x}(k) \in \hat{\mathcal{X}}_k$. As part of a filtering algorithm, a recursion is defined, i.e., $\hat{x}(k-1) \leftarrow \hat{x}(k)$ and we repeat the procedure. If no solution exists for some k, then $\hat{x}(k)$ is unfeasible and we stop the filtering procedure.

4.3 Numerical simulations

For the numerical simulation's comparison [7], let us consider (1) with $A_0(k)$ in strictly lower form, i.e., $a_{0_{ij}}(k) \neq \varepsilon$ for all $i \leq j$, $i, j \in \{1, \ldots, n\}$ and $A_1(k), C(k)$ be full max-plus matrices, i.e., $a_{1_{ij}}(k) \neq \varepsilon$ for all $i, j \in \{1, \ldots, n\}$ and $c_{ij}(k) \neq \varepsilon$ for all $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, n\}$. Every element of these matrices are randomly chosen between the arbitrary bounds 0 and 10 at each k, i.e., the realizations $A_0(k), A_1(k), C(k)$. We suppose that $x(0) = (e, \ldots, e)^\intercal$ and then we obtain the following sequences $\{x(k) = A_0(k)x(k) \oplus A_1(k)x(k-1)\}_{k \in \mathbb{N}_{>0}}$ and $\{z(k) = C(k)x(k)\}_{k \in \mathbb{N}_{>0}}$. We compare in the sequel the previous approaches to compute feasible estimate $\hat{x}(k)$ for $x(k)$ at each k. If no feasible estimate can be guaranteed, then we stop the simulation.

Table 1 shows the minimum, average and maximum execution times for each call of the estimators for 20 experiments of the disjunctive method $T^{symb}(s)$ and the concise method $T^{mat}(s)$ for $k \in \{1, \ldots, N\}$, where N is the event-horizon. We analyze simulations that are not stopped, i.e., experiments that do not violate the feasibility guarantee of the set-estimation using either approach. Furthermore, we analyze the error-estimation of both approaches. We compute the mean-absolute-percentage-error (MAPE) between $x_i(k)$ and $\hat{x}_i(k)$ for $i \in \{1, \ldots, n\}$, precisely

$$error_i(x_i(k), \hat{x}_i(k)) = \frac{100\%}{N} \sum_{k=1}^{N} \left| \frac{x_i(k) - \hat{x}_i(k)}{x_i(k)} \right|, \; i \in \{1, \ldots, n\}$$

for all $i \in \{1, \ldots, n\}$ and then we take the average of the resulting vector, i.e.,

$$error_{avg} = \frac{1}{n} \sum_{i=1}^{n} error_i(x_i(k), \hat{x}_i(k)).$$

We show the minimum, average and maximum values of $error^{symb}_{avg}(\%), error^{mat}_{avg}(\%)$ for each experiment out of 20.

Table 1. Numerical analysis comparison.

n	p	N	$error^{symb}_{avg}(\%)$	$error^{mat}_{avg}(\%)$	$T^{symb}(s)$	$T^{mat}(s)$
5	3	500	{0.40; 0.46; 0.53}	{0.02; 0.03; 0.04}	{0.03; 0.04; 0.116}	{0.005; 0.009; 0.02}
10	5	20	{1.44; 1.62; 1.94}	{0.43; 0.51; 0.57}	{0.12; 0.21; 1.43}	{0.02; 0.03; 0.05}
10	8	20	{1.46; 1.58; 1.85}	{0.44; 0.51 0.58}	{0.14; 0.24; 1.52}	{0.02; 0.03; 0.05}
20	10	5	{2.27; 3.00; 3.84}	{1.54; 1.80; 2.02}	{0.57; 4.90; 28.20}	{0.07; 0.10; 0.12}
100	50	10	{-}	{0.96; 1.01; 1.05}	{-}	{1.99; 2.24; 2.65}

As it can be noted, the execution times of both approaches are related to n. However, the disjunctive approach is more affected by p because there are more symbolic constraints to be evaluated by the SMT solver, thus increasing the

[7] Running `Python` with C++ wrappers for Z3 SMT solver (De Moura and Bjørner (2008)) and Armadillo (Sanderson and Curtin (2016)) for fast (sparse) matrix operations in max-plus algebra on a Dell Precision 5530 - 2.6 GHz Intel(R) Core(TM) i7 processor.

execution time. In terms of error-estimation, these experiments suggest that the concise method leads to lower error-estimation values. For the last row of Table 1, we evaluate an example with a large n and we only present the results for the concise method because the running time of the disjunctive method exceeds a predefined threshold (timeout).

5. CONCLUSION

In this work, we have studied two approaches: one developed by the authors and another drawn from the existing literature to provide feasibility guarantees for set-estimation of MPL systems with bounded uncertainties. We *indirectly* characterize reachable sets from previous estimations that respect the measurement output and compute values within these sets. Firstly, we examine a disjunctive approach utilizing SMT techniques. Secondly, we propose a concise method based on solving two-sided equations in max-plus algebra with pseudo-polynomial complexity. The latter method outperforms the former in terms of speed and accuracy. Future works involve integrating probabilistic aspects for additional feasibility certificates and exploring the application of the concise method for *directly* characterizing the reachable sets.

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