# Weak dual residuations applied to tropical linear equations

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**Abstract**

An extension to an algorithm of R.A. Cuninghame-Green and K. Zimmermann for solving equations with residuated functions is presented. This extension relies on the concept of weak residuation and in the so called "strong property". It is shown that a contextualization of this method to tropical linear equations, which will be denoted as Primal Method (in constrast with the Dual Method, another algorithm described in literature), generates a non-decreasing sequence which converges to the smallest solution inside a special semimodule. It is also shown the connections of this method with previously published works.

*Keywords:* Tropical Algebra, Weak Residuation, Semimodule, Equations, Kleene Closure

## 1. Introduction

Tropical Algebra (also known as the Max-Plus Algebra), is the semiring[1]

$$\mathbb{T}_{max} = \{\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes\} \tag{1}$$

in which $\oplus$ is the maximum and $\otimes$ is the traditional sum. It is usual, as well, to denote the neutral element of the sum, $-\infty$, as $\varepsilon$. It is also usual to define the complete dioid $\overline{\mathbb{T}}_{max}$ augmented with the element $\infty$, here denoted as $\top$. It is also defined $\top \otimes \varepsilon = \varepsilon \otimes \top = \varepsilon$. As in the traditional algebra, the

[1]Usually, the name is used to denote the isomorphic dioid Min-Plus.

symbol $\otimes$ is usually omitted. The dual dioid $\mathbb{T}_{min}$ in which the maximum is swapped with the minimum is also defined (that is, replace $\wedge$ with $\oplus$, residuations with dual residuations and swap the roles of $\top$ and $\varepsilon$).

It is assumed from now on that the reader is familiar with this algebra basics and also with the concepts of residuation, dual residuation, Kleene Closure ( Baccelli et al. (1992)) and semimodules ( Cohen et al. (2004)). Results and identities which are common but not very straightforward will be given in footnotes. The tropical identity matrix of appropriate order is denoted by $I$, and $\diagdown\!\!\!\circ$ is used to denote the left residuation of the product. The pointwise infimum is denoted by $\wedge$. By analogy with the traditional algebra, if $A$ is a matrix and $\alpha$ a scalar, $A \not\circ \alpha$ will be the pointwise scalar residuation of the entries of $A$ by $\alpha$. $A^* = \bigoplus_{i=0}^{\infty} A^i$ and $\rho(A)$ are, respectively, the Kleene Closure and spectral radius of $A$. It is also defined a matrix composed entirely of $\varepsilon$ and $\top$, of convenient dimension, as $\overline{\varepsilon}$ and $\overline{\top}$, respectively. Finally, a matrix is said to be *upper bounded* (resp. *lower bounded*) if all the entries are different from $\top$ (resp. $\varepsilon$). The image of a matrix $M$ is the (tropical) linear span of the columns of $M$.

An important problem in the tropical algebra concerns the solution of two-sided linear equations

$$E\mathbf{x} = D\mathbf{x}. \tag{2}$$

Cuninghame-Green and Zimmermann (2001) introduced a general iterative algorithm for solving equations of the form

$$f(\mathbf{x}) = g(\mathbf{y}) \tag{3}$$

when $f$ and $g$ are residuated functions. A specialization of this algorithm to the linear tropical equation $A\mathbf{x} = B\mathbf{y}$ can be adapted to the (equivalent) equation $E\mathbf{x} = D\mathbf{x}$. Then, it has the important property of generating a non-increasing sequence which converges to the greatest solution $\mathbf{x}$ smaller or equal than the initial condition $\mathbf{x}_0$.

Algorithms for solving tropical linear equations can also solve their affine counterparts $R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}$, by introducing an auxiliary scalar variable $y$ (see Cuninghame-Green and Butkovic (2003))

$$R\mathbf{p} \oplus \mathbf{r}y = S\mathbf{p} \oplus \mathbf{s}y. \tag{4}$$

Equation (4) is linear in the extended vector $\mathbf{x} = (\mathbf{p}^T \mid y)^T$ if one sets

$E = (R \mid \mathbf{r})$ and $D = (S \mid \mathbf{s})$. If one employs Cuninghame-Green and Zimmerman method with the initial condition $\mathbf{p}[0] = \mathbf{p}_0$ and $y[0] = 0$, the $\mathbf{p}$ vector will converge to a solution - provided that one such that $\mathbf{p} \preceq \mathbf{p}_0$ exists - and $y$ will remain equal to 0. Due to the algorithm properties, the resulting solution will be the greatest one of the original affine equation which is smaller than $\mathbf{p}_0$. Thus - provided that the solution set is not empty - the greatest solution of an affine tropical equation exists and can be found using the greatest possible initial condition $\mathbf{p}[0] = \overline{\top}$.

However, in general, the smallest solution does not exist. This is a consequence to the fact that the product in $\overline{\mathbb{T}}_{max}^{n \times m}$ is not dually residuated (in general). Affine equations appear in some control applications (see Gonçalves et al. (2012)), and it may be desirable that these solutions be small and sparse (full of null $\varepsilon$ entries). This motivates the research for a method which return small solutions. Since seeking for the smallest solution is futile in general, one can weaken the problem asking for a solution inside a special set. As an example, a special *semimodule* $\mathcal{S}$ can be considered. Then, according to this constraint, the proposed problem may have a smallest solution.

To this end, the concept of *weak residuation* and *strong residuation for an element* will be introduced. Then, one can weaken the requirement of residuated functions $f$ and $g$ in Cuninghame-Green and Zimmermann (2001), and instead require that $f$ and $g$ has a weak residuation which have the strong property for a previously found solution. Thus, one can use this general algorithm in the dual dioid $\mathbb{T}_{min}$ (so the minimum becomes the maximum and $\preceq$ becomes $\succeq$) and obtain a method for generating other solutions, which are "small", to the tropical affine equation. In fact, the method can find the smallest inside a particular semimodule $\mathcal{S}$ using a special initial condition. So, the proposed method uses an already known solution for finding others with a special property.

The aforementioned method, which will be called *Primal Method* hereafter, is closely related to the specialization of the Cuninghame-Green method for equations of the form $E\mathbf{x} = D\mathbf{x}$, which will be called in this paper *Dual Method*. It is also closely related - and this will be explicitly addressed latter in Subsection 3.5- to the cellular decomposition of Develin and Sturmfels (2004), the mean payoff games and the algorithms presented in Truffet (2010), Lorenzo and de la Puente (2011) and Gaubert et al. (2012) .

Equation (2) has also been studied in several other works other than the previously mentioned ones.. Baccelli et al. (1992) provides a method for finding solutions using the symmetrized tropical algebra, which introduces a weak form of subtraction (in a weak inequality, the *balance*) and therefore allows analogous algorithms from traditional algebra to be adapted to the problem. Following this idea, many algorithms were also discussed in Gaubert (1992). Butkovic and Hegedüs (1984) provided a method, the *Elimination Method*, which can generate the entire set of solutions by solving the system of equations row-by-row. As a consequence of this method, it was proved that this set has a finite (albeit possibly very large) representation. Using concepts of residuation, Cuninghame-Green and Butkovic (2003) proposes the *Alternating Method*, which generates a non-increasing sequence (after the first step) which converges to a solution. Butkovic and Zimmermann (2006) provided an algorithm for finding a single solution, the *Stepping Stone Method*. It works by checking at each step which of the equalities holds and the ones which does not. Then, it decreases the values of the current vector in a way that the non-achieved equalities began to hold and the ones that already do continues holding. Akian et al. (2010) shows that the existence of a non-trivial solution is related to the problem of solving mean payoff games. The *Tropical Double-Description Method* in Allamigeon et al. (2010) is conceptually similar to the one proposed in Butkovic and Hegedüs (1984), being capable of generating the entire set of solutions by solving the system row-by-row. It uses, however, a more elaborated approach for solving each row equation, using the concept of *extreme rays*. This leads to a more compact representation of the intermediate solutions set and thus the method has a substantially better average complexity than the Elimination Method. Finally, the analogue of Equation (2) to the interval of dioids was established in Hardouin et al. (2009) and the related problem of solving inequations of the form $A\mathbf{x} \preceq \mathbf{x} \preceq B \backslash\!\!\circ\, \mathbf{x}$ in Brunsch et al. (2012).

In summary, the contributions of this paper are:

(i) An extension of the algorithm presented in Cuninghame-Green and Zimmermann (2001), considering the concepts of weak residuation and the strong property for it, which is presented in Section 2.

(ii) A method for generating solutions to tropical linear equations, the Primal Method, which is presented in Section 3. This method is a contextualization of the the aforementioned extension to tropical linear equations. It is also discussed the similarities of this method with previously published works.

## 2. Solving equations with weak residuated functions

In Cuninghame-Green and Zimmermann (2001), an algorithm was proposed for solving Equation (3), when $f$ and $g$ are residuated. It can be implemented by computing the sequences

$$\begin{aligned} \mathbf{x}[k+1] &= f^{\sharp}(g(\mathbf{y}[k])) \wedge \mathbf{x}[k]; \\ \mathbf{y}[k+1] &= g^{\sharp}(f(\mathbf{x}[k])) \wedge \mathbf{y}[k] \end{aligned} \tag{5}$$

(6)

for an initial $\mathbf{x}[0]$, $\mathbf{y}[0]$, in which $f^{\sharp}$ and $g^{\sharp}$ are the residuation for $f$ and for $g$, respectively.

Using a similar reasoning, the following sequence can be derived, which converges to a solution of $f(\mathbf{x}) = g(\mathbf{x})$:

$$\mathbf{x}[k+1] = f^{\sharp}(g(\mathbf{x}[k])) \wedge g^{\sharp}(f(\mathbf{x}[k])) \wedge \mathbf{x}[k]. \tag{7}$$

This algorithm can be extended if the residuation is relaxed to a weaker form.

**Definition 2.1.** (Weak residuation) *A non-decreasing function $f$ is said to have a weak residuation if there is a non-decreasing function $f^{\natural}$ such that*

$$f(f^{\natural}(\mathbf{x})) \preceq \mathbf{x} \quad \forall \mathbf{x}. \tag{8}$$

□

For a given function, there exists many weak residuations, and as the name implies, the requirement in Equation (8) by itself is not very useful. So, it is important to introduce another definition.

**Definition 2.2.** (Weak residuation with strong property) *For an element $\mathbf{z}$, a weak residuation $f^{\natural}_{\mathbf{z}}$ with the additional property*

$$f^{\natural}_{\mathbf{z}}(f(\mathbf{z})) \succeq \mathbf{z} \tag{9}$$

*is said to have the strong property for $\mathbf{z}$.*

□

Remember that the usual residuation is such that $f(f^\sharp(\mathbf{x})) \preceq \mathbf{x}$ and $f^\sharp(f(\mathbf{x})) \succeq \mathbf{x}$, *both* holding for all $\mathbf{x}$. Then, clearly, the usual residuation is a weak residuation which is strong for any element $\mathbf{z}$. It is also important to remark that one can, in analogy, define weak *dual* residuations with a strong property with an element. These kind of residuations will be used later on this paper.

Now, the *residuated* requirement in the algorithm presented in Cuninghame-Green and Zimmermann (2001) can be replaced by *weak residuated with a residuation which is strong for a solution* $\mathbf{z}$ *of the equation.* If $\mathbf{z}$ is lower bounded, the fact that $\mathbf{z}$ is a solution guarantees the convergence to a lower bounded solution.

**Proposition 2.1.** (Convergence with weak residuation with a strong property for a solution)*: The sequence generated by Equation* (7) *with initial condition* $\mathbf{x}[0] \succeq \mathbf{z}$ *converges to a lower bounded solution of Equation* (3) *if weak residuation functions with the strong property to a lower bounded solution* $\mathbf{z}$ *are used (that is, switching* $f^\sharp$, $g^\sharp$ *to* $f^\natural$, $g^\natural$*, respectively, in Equation* (7)*).*

**Proof 2.1.** *It is straightforward to see that the sequence generated by Equation* (7) *is non-increasing (due to the minimum with* $\mathbf{x}[k]$*). Therefore, it either degenerates to the trivial solution* $\overline{\varepsilon}$ *or stabilizes.*

*Suppose it stabilizes, thus*

$$\begin{aligned} \mathbf{x} &\preceq f^\natural_{\mathbf{z}}(g(\mathbf{x})); \\ \mathbf{x} &\preceq g^\natural_{\mathbf{z}}(f(\mathbf{x})). \end{aligned} \tag{10}$$

*Then, using Equation* (8) *(that is, after applying* $f$ *and* $g$ *in both sides of the top and bottom inequalities of Equation* (10)*, respectively)*

$$\begin{aligned} f(\mathbf{x}) &\preceq g(\mathbf{x}); \\ g(\mathbf{x}) &\preceq f(\mathbf{x}). \end{aligned} \tag{11}$$

*and thus it stabilizes to a solution.*

*The concern is that this solution can be the trivial one,* $\overline{\varepsilon}$*. This is addressed by the fact that* $\mathbf{z}$ *has the strong residuation property with a lower bounded solution* $\mathbf{z}$*. Thus, by Equation* (9)

$$f_{\mathbf{z}}^{\natural}(f(\mathbf{z})) \succeq \mathbf{z} \;\Rightarrow\; f_{\mathbf{z}}^{\natural}(g(\mathbf{z})) \succeq \mathbf{z} \tag{12}$$

*where the fact that* $f(\mathbf{z}) = g(\mathbf{z})$ *was used. Then, also*

$$g_{\mathbf{z}}^{\natural}(f(\mathbf{z})) \succeq \mathbf{z}. \tag{13}$$

*And thus, combining Equations* (12) *and* (13)

$$\mathbf{z} = f_{\mathbf{z}}^{\natural}(g(\mathbf{z})) \wedge g_{\mathbf{z}}^{\natural}(f(\mathbf{z})) \wedge \mathbf{z}. \tag{14}$$

*Then,* $\mathbf{z}$ *is a fixed point for the iteration map in Equation* (7)*. Then, the function*

$$h(\mathbf{x}) = f_{\mathbf{z}}^{\natural}(g(\mathbf{x})) \wedge g_{\mathbf{z}}^{\natural}(f(\mathbf{x})) \wedge \mathbf{x} \tag{15}$$

*is monotonic. As* $\mathbf{x}[0] \succeq \mathbf{z}$*, by induction and using Equation* (14)*, it can be shown that* $\mathbf{x}[k] \succeq \mathbf{z}$*. Thus, as* $\mathbf{z}$ *is lower bounded, the sequence will converge to a lower bounded solution.*

□

**Remark 2.1.** *When the residuated functions* $f(\mathbf{x}) = E\mathbf{x}$*, and* $g(\mathbf{x}) = D\mathbf{x}$ *(that is, one is dealing with Equation* (2)*) are used in the original algorithm of Cuninghame-Green and K. Zimmermann, the resulting algorithm is iterations of the function*

$$h(\mathbf{x}) = E \diamond\backslash (D\mathbf{x}) \wedge D \diamond\backslash (E\mathbf{x}) \wedge \mathbf{x} \tag{16}$$

*on an initial* $\mathbf{x}[0] = \mathbf{x}_0$*.*

*This algorithm is well known and has been exploited in literature. For example, Equation* (16) *appears in V.Dhingra and Gaubert (2006) and then in Gaubert and Sergeev (2010), in connection with mean payoff games (that will be discussed in Subsection 3.5). It is also related to the Alternating Method of Cuninghame-Green and Butkovic (2003).*

*This method enjoys many important properties, such as for example generating a non-increasing sequence* $\mathbf{x}[k]$ *which converges to the greatest solution of Equation* (2) *smaller or equal than* $\mathbf{x}_0$ *(see Cuninghame-Green and Zimmermann (2001)). This method will be denoted in this paper by* ***Dual Method****, in contrast with one that will be presented further that share many* *(almost) dualized properties with them (and also the same origin, as a particular case of the proposed extended algorithm in Proposition 2.1), the* ***Primal Method****.*

□

## 3. The Primal Method

As a contextualization of the proposed extended algorithm, the *Primal Method* will be presented. It concerns tropical linear equations of the form Equation (2). First, the method will be established with independent concepts. Then, the connection with the extended algorithm will be given.

It will be assumed from now on that the matrices $E$ and $D$ have their entries only in $\mathbb{T}_{max}$, so no $\top$ entries is allowed. This is a weak assumption that permits to avoid some technicalities concerning the expression $\top \otimes \varepsilon$ in the proposed results.

### *3.1. Introduction*

The Primal Method will be presented by a sequence of definitions and propositions.

**Definition 3.1.** (Dominance) *A* dominance *is a mapping* $\Upsilon : \{1, 2, ..., n\} \mapsto \{1, 2, .., m\}$. □

The reason behind this name will be clear later.

**Definition 3.2.** (Matrix generated by the dominance) *Let* $E \in \mathbb{T}_{max}^{n\times m}$ *and* $\Upsilon$ *be a dominance. The matrix* $\mathsf{W}(\Upsilon, E) \in \mathbb{T}_{max}^{n\times m}$ *is defined as the matrix constructed in the following way:*

$$\begin{aligned} \{\mathsf{W}(\Upsilon, E)\}_{ij} &\equiv E_{ij} \textit{ if } \Upsilon(i) = j; \\ \{\mathsf{W}(\Upsilon, E)\}_{ij} &\equiv \varepsilon \textit{ otherwise.} \end{aligned} \tag{17}$$

□

The matrix $\mathsf{W}(\Upsilon, E)$ generated by a dominance is simply a matrix constructed from $E$, such that all rows have at most one non-null entry, and the only (possible) non-null entry on row $i$ is exactly $j = \Upsilon(i)$.

**Property 3.1.** (Dual residuation) *The map* $\mathbf{x} \mapsto \mathsf{W}(\Upsilon, E)\mathbf{x}$ *is* dually residuated *if all rows of the matrix have exactly one non-null entry (by Definition 3.2, it can have one or zero non-null entries). This means that there exists a matrix, that will be denoted by* $\mathsf{W}^{\flat}(\Upsilon, E) \in \mathbb{T}_{max}^{m\times n}$, *such that for any* $\mathbf{x} \in \mathbb{T}_{max}^{m}$, $\mathbf{y} \in \mathbb{T}_{max}^{n}$

$$\mathsf{W}(\Upsilon, E)\mathbf{x} \succeq \mathbf{y} \Longleftrightarrow \mathbf{x} \succeq \mathsf{W}^{\flat}(\Upsilon, E)\mathbf{y}. \tag{18}$$

□

**Remark 3.1.** *Hereafter, it will be assumed, without loss of generality, that all rows of the matrix* $\mathsf{W}(\Upsilon, E)$ *have at least one non-null entry (it is* row G-astic*, see Butkovic and Hegedüs (1984)). The same must hold in regard to* $\mathsf{W}(\Upsilon, D)$. *It will be discussed later why this assumption is not restrictive.* □

It can be seen, by inspection, that this matrix $\mathsf{W}^{\flat}(\Upsilon, E)$ is simply obtained by switching the sign of all non-null entries and transposing the resulting matrix. Thus, one introduces the following definitions.

**Definition 3.3.** (Dual residuation matrix) *If* $\mathsf{W}(\Upsilon, E) \in \mathbb{T}_{max}^{n\times m}$ *has one non-null entry per row, then it is defined by* $\mathsf{W}^{\flat}(\Upsilon, E)$ *as the matrix obtained by* $\mathsf{W}(\Upsilon, E)$ *by switching the sign of all non-null entries and transposing the result.*□

**Definition 3.4.** (Induced dominance) *A dominance* $\Upsilon_E^{\mathbf{z}}$ *can be induced by a vector* $\mathbf{z}$ *in a matrix* $E$ *as follows*

$$\Upsilon_E^{\mathbf{z}}(i) \equiv \arg\max_j \{\bigoplus_{j=1}^{m} E_{ij} z_j\}. \tag{19}$$

*with the additional constraint that, for all* $i$, *if* $j = \Upsilon_E^{\mathbf{z}}(i)$ *then* $E_{ij} \neq \varepsilon$.

□

**Remark 3.2.** *Note that an induced dominance exists on a given matrix* $E$ *if and only if it is row G-astic. Otherwise, the additional constraint that* $E_{ij} \neq \varepsilon$ *cannot hold. This constraint guarantees that* $\mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)$ *is row G-astic (as assumed without loss of generality in Remark 3.1). Then, if one considers* only *induced dominances, the assumption that without loss of generality one can assume* $\mathsf{W}(\Upsilon, E)$ *as row G-astic can be transferred to the assumption that without loss of generality* $E$ *is row G-astic. The same must hold in regard to* $D$. *This will be addressed further.* □

A given vector $\mathbf{z}$ can induce multiple dominances, since two or more indexes can lead to the maximum values, as in the sum $2 \oplus 1 \oplus 2$ in which the first and third entry achieve the greatest value. Thus, in this case, $\Upsilon^{\mathbf{z}}$ is multiple-defined and can be any of those dominances.

**Property 3.2.** *Now the meaning of the label "dominance" can be made clear. If $\Upsilon^{\mathbf{z}}_E$ is an induced dominance from a given $\mathbf{z}$, then it can be shown by inspection that*

$$E\mathbf{z} = \mathsf{W}(\Upsilon^{\mathbf{z}}_E, E)\mathbf{z}. \tag{20}$$

*Thus, the mapping $\Upsilon^{\mathbf{z}}_E$ maps to each row the dominating index in the product $E\mathbf{z}$ in this row (see Equation* (19)*).*

□

In addition:

**Property 3.3.** *It is straightforward by the structure of the matrix $\mathsf{W}(\Upsilon, E)$ that, for any $\Upsilon$*

$$E \succeq \mathsf{W}(\Upsilon, E). \tag{21}$$

□

**Definition 3.5.** ($\mathsf{H}$ matrix) *Let $E, D \in \mathbb{T}^{n\times m}_{max}, \mathbf{z} \in \mathbb{T}^{m}_{max}$ and $\Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D$ be induced dominances. Then, the $\mathsf{H}$ matrix is defined as*

$$\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D) \equiv \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}_E, E)D \oplus \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}_D, D)E. \tag{22}$$

□

The $\mathsf{H}$ matrix has an important property.

**Proposition 3.1.** (Obtaining solutions) *Any linear combination $\mathbf{x}$ of columns of $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^* \in \overline{\mathbb{T}}^{m\times m}_{max}$ is a solution to the equation $E\mathbf{x} = D\mathbf{x}$.*

**Proof 3.1.** *Let $\mathbf{x} = \mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^*\mathbf{y}$. Due to Kleene Closure properties, this is equivalent to the following statement* [2]

[2] If $\mathbf{x} = H^*\mathbf{y}$, then $H^*\mathbf{x} = H^*H^*\mathbf{y} = H^*\mathbf{y}$ since $H^*H^* = H^*$ holds for Kleene Closures. Thus $\mathbf{x} = H^*\mathbf{x}$, and it can be stated that $\mathbf{x} \succeq H^*\mathbf{x} \succeq H\mathbf{x}$.

$$\mathbf{x} \succeq \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})\mathbf{x}. \tag{23}$$

*Which is equivalent to:*

$$\begin{aligned} \mathbf{x} &\succeq \mathsf{W}^{\flat}(\Upsilon_E^{\mathbf{z}}, E)D\mathbf{x}; \\ \mathbf{x} &\succeq \mathsf{W}^{\flat}(\Upsilon_D^{\mathbf{z}}, D)E\mathbf{x}. \end{aligned} \tag{24}$$

*Due to the fact that the maps are dually residuated (Property 3.1), this is equivalent to*

$$\begin{aligned} \mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{x} &\succeq D\mathbf{x}; \\ \mathsf{W}(\Upsilon_D^{\mathbf{z}}, D)\mathbf{x} &\succeq E\mathbf{x}. \end{aligned} \tag{25}$$

*By using Property 3.3, it can be deduced that $E\mathbf{x} \succeq D\mathbf{x}$ and $D\mathbf{x} \succeq E\mathbf{x}$. Then, the statement is proved.*

□

Proposition 3.1, however, does not guarantee that the Kleene Closure of $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})$ will be upper bounded (thats why the matrix is, generally, in the complete dioid $\in \overline{\mathbb{T}}_{max}^{m\times m}$). Indeed, dominances $\Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}}$ in which the closure is a matrix full of $\top$'s can be chosen. One must note that these are degenerate solutions, but solutions nonetheless.

The next Proposition ensures how dominances that guarantee at least partial upper boundedness of the Kleene Closure can be chosen, thus guaranteeing that non-trivial solutions are found.

**Proposition 3.2.** (Upper bounded Kleene Closure) *Let $\mathbf{z}$ be an upper bounded solution of $E\mathbf{x} = D\mathbf{x}$. Then, for all the rows $j$ in which $\mathbf{z}$ is non-null, the $j^{th}$ column of the matrix $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$ is upper bounded.*

**Proof 3.2.** *By hypothesis*

$$E\mathbf{z} = D\mathbf{z}. \tag{26}$$

*Using Property 3.2, on the left side*

$$\mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{z} = D\mathbf{z}. \tag{27}$$

*Then*

$$\mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{z} \succeq D\mathbf{z}. \tag{28}$$

*Using Property 3.1*

$$\mathbf{z} \succeq \mathsf{W}^\flat(\Upsilon_E^{\mathbf{z}}, E)D\mathbf{z}. \tag{29}$$

*Similarly*

$$\mathbf{z} \succeq \mathsf{W}^\flat(\Upsilon_D^{\mathbf{z}}, D)E\mathbf{z}. \tag{30}$$

*And then, by summing the statements in Equations* (29) *and* (30)

$$\mathbf{z} \succeq \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})\mathbf{z}. \tag{31}$$

*Using the property of Kleene Closures* [3]*, Equation* (31) *is equivalent to*

$$\mathbf{z} = \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*\mathbf{z}. \tag{32}$$

*And then the conclusion of the Proposition can be clear: if the $j^{th}$ entry of $\mathbf{z}$ is non-null and $\mathbf{z}$ is upper bounded, then necessarily the $j^{th}$ column of $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$ must be upper bounded. Otherwise, one cannot draw any conclusion.*

□

Propositions 3.1 and 3.2 together constitute a method for computing more solutions from the equation $E\mathbf{x} = D\mathbf{x}$ from a given known one $\mathbf{z}$. First, find $\mathbf{z}$ (using any method). Then, find the dominance induced by $\mathbf{z}$ and then compute $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$. This procedure is what is denoted by ***Primal Method***.

**Algorithm 3.1.** Primal Method for tropical linear equations

1. `Solve Equation` (2)`, using any method, obtaining a solution` $\mathbf{z}$`;`

[3] If $\mathbf{x} \succeq H\mathbf{x}$, by pre-multiplying by $H$ one concludes that $H\mathbf{x} \succeq H^2\mathbf{x}$ and thus $\mathbf{x} \succeq H^2\mathbf{x}$. By induction, $\mathbf{x} \succeq H^k\mathbf{x}$ for any natural $k$. By adding all these inequalities for all $k$, $\mathbf{x} \succeq H^*\mathbf{x}$. Since $H^*\mathbf{x} \succeq \mathbf{x}$, one can finally conclude that $\mathbf{x} = H^*\mathbf{x}$. Further, by Footnote 2, $\mathbf{x} = H^*\mathbf{y}$ for some $\mathbf{y}$.

2. Use this solution to induce dominances, $\Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}}$ (see Definition 3.4);

3. Construct the matrix $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})$, as in Definition 3.5;

4. Any linear combination of columns of $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$ is a solution.

□

**Remark 3.3.** *It was mentioned in Remark 3.2 that, without loss of generality, one can assume both $E$ and $D$ row G-astic . No generality is lost because, if this is not the case, it is possible to rewrite the equation removing these rows and appropriate columns/corresponding entries in $E, D$ and $\mathbf{x}$ (these will be fixed to $\varepsilon$) such that the new system has this property. If both rows of $E$ and $D$ are null, they can be removed without any problem. If it is only in $E$ or $D$, say the $i^{th}$ of $D$, there is a situation*

$$\bigoplus_{j=1}^{m} E_{ij} x_j = \varepsilon \tag{33}$$

*and then, for all $j$ such that $E_{ij} \neq \varepsilon$ necessarily $x_j = \varepsilon$. These variables can be set to $\varepsilon$, then they can be removed from the vector $\mathbf{x}$ along with the $i^{th}$ row and $j^{th}$ column of both $E$ and $D$. One can proceed in that way till there is nothing to remove and no row in $E$ or $D$ is null.*

□

*3.2. Connection with the extended Cuninghame-Green and Zimmerman algorithm*

The Dual Method (a specialization of the Cuninghame-Green and Zimmerman algorithm to linear equations, see Remark 2.1) is an iterative algorithm. Regardless of the method used to compute the Kleene Closure, the Primal Method *as presented* is not iterative. However, one can note that $\mathbf{z} = \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^* \mathbf{x}_0$ can be implemented by the sequence

$$\mathbf{x}[k+1] = (\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}}) \oplus I)\mathbf{x}[k], \tag{34}$$

for the initial $\mathbf{x}[0] = \mathbf{x}_0$. Then, the Primal Method can also be seen (implemented) as an iterative method. This is not the most computationally efficient way to implement it, since computing powers of $(A \oplus I)$ is not the best algorithm for computing $A^*$.

If the dioid is swapped from $\mathbb{T}_{max}$ to $\mathbb{T}_{min}$, the usual residuation in the dual dioid is just the dual residuation in the original dioid (to avoid confusion, everything will be nominated by the reference of $\mathbb{T}_{max}$). Then, it can be seen that the map $f(\mathbf{x}) = E\mathbf{x}$ has a weak dual residuation with a strong property for an element $\mathbf{z}$: the map $f^{\dagger}_{\mathbf{z}}(\mathbf{x}) = \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E)\mathbf{x}$ as defined in Definition 3.2 using an induced dominance.

**Proposition 3.3.** (Dominances induce a weak dual residuation with a strong property) *$f(\mathbf{x}) = E\mathbf{x}$ has as weak dual residuation the map $f^{\dagger}_{\mathbf{z}}(\mathbf{x}) = \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E)\mathbf{x}$ with strong property for $\mathbf{z}$.*

**Proof 3.3.** *It can be noted that*

$$E\mathbf{y} \succeq \mathsf{W}(\Upsilon^{\mathbf{z}}, E)\mathbf{y} \ \forall \mathbf{y} \tag{35}$$

*(see Property 3.3). Thus, using $\mathbf{y} = \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E)\mathbf{x}$ and the fact that*

$$\mathsf{W}(\Upsilon^{\mathbf{z}}, E)\mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E) \succeq I \tag{36}$$

*(since $\mathbf{x} \mapsto \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E)\mathbf{x}$ is a dual residuation for $\mathbf{x} \mapsto \mathsf{W}(\Upsilon^{\mathbf{z}}, E)\mathbf{x}$), it can be concluded that*

$$E\mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E)\mathbf{x} \succeq \mathbf{x} \ \forall \mathbf{x} \tag{37}$$

*which is exactly the requirement for a weak* dual *residuation, that is, Equation* (8) *with the order swapped. The strong property comes from Property 3.2:*

$$\mathsf{W}(\Upsilon^{\mathbf{z}}, E)\mathbf{z} = E\mathbf{z}. \tag{38}$$

*Thus*

$$\mathsf{W}(\Upsilon^{\mathbf{z}}, E)\mathbf{z} \succeq E\mathbf{z} \iff \mathbf{z} \succeq \mathsf{W}^{\flat}(\Upsilon^{\mathbf{z}}, E)E\mathbf{z} \tag{39}$$

*using the dual residuation of the map $\mathbf{x} \mapsto \mathsf{W}(\Upsilon^{\mathbf{z}}, E)\mathbf{x}$. Thus, the dual residuation version of Equation* (9) *arises and this weak dual residuation has the strong property for $\mathbf{z}$.*

□

Then, if Equation (7) is contextualized to $\mathbb{T}_{min}$ and also weak residuations who have a strong property for a solution are used, it can be concluded that this equation reduces to

$$\mathbf{x}[k+1] = \mathsf{W}^\flat(\Upsilon^{\mathbf{z}}, E)D\mathbf{x}[k] \oplus \mathsf{W}^\flat(\Upsilon^{\mathbf{z}}, D)E\mathbf{x}[k] \oplus \mathbf{x}[k]. \tag{40}$$

It is straightforward to see that the resulting Equation (40) is the iterative form of the Primal Method, as in Equation (34). Thus, as claimed, the Dual Method and Primal Method share the same origin.

It will now be proved that they also share (almost) dualized properties.

*3.3. Properties of the method*

Some results concerning the properties of the solutions found by the Primal Method are presented now. For this, it is useful to use the iterative form of the method, Equation (34). Thus, this form will be the one considered in this subsection.

First, a *dominance space* is defined.

**Definition 3.6.** (Dominance space) *Given a dominance $\Upsilon$ and a matrix $E$, it is called $\mathcal{D}(\Upsilon, E)$, the dominance space of $\Upsilon$ under $E$, the sets of all $\mathbf{x}$ such that*

$$E\mathbf{x} = \mathsf{W}(\Upsilon, E)\mathbf{x}. \tag{41}$$

$\square$

It is important to remark that $\mathcal{D}(\Upsilon, E)$ is a semimodule, since it is the solution set of a linear tropical equation which can be given as the image of a finite matrix (see Butkovic and Hegedüs (1984)). In fact, by Property 3.3, Equation (41) is equivalent to $\mathsf{W}(\Upsilon, E)\mathbf{x} \succeq E\mathbf{x}$. Then, by using Property 3.1, $\mathbf{x} \succeq \mathsf{W}^\flat(\Upsilon, E)E\mathbf{x}$. This fact has two interesting implications. The first one is that this implies that the semimodule is convex in the traditional sense. The second one is that this semimodule is generated by the matrix $(\mathsf{W}^\flat(\Upsilon, E)E)^*$ (see Footnote 3).

**Proposition 3.4.** (Exhaustion of dominance space) *Any solution $\mathbf{x}$ generated by the Primal Method using a matrix $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^*$ is such that $\mathbf{x} \in \mathcal{D}(\Upsilon^{\mathbf{z}}_E, E) \cap \mathcal{D}(\Upsilon^{\mathbf{z}}_D, D)$. Furthermore, $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^*$ is a generator matrix for all such $\mathbf{x}$'s.*

**Proof 3.4.** *Consider $\mathbf{x}$ a solution. For the first part, using Property 3.3 and post multiplying by $\mathbf{x}$*

$$E\mathbf{x} \succeq \mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{x}. \tag{42}$$

*Now, from the first Equation in* (25)*, and using the fact that* $E\mathbf{x} = D\mathbf{x}$

$$\mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{x} \succeq E\mathbf{x}. \tag{43}$$

*And thus* $E\mathbf{x} = \mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{x}$*, and then* $\mathbf{x} \in \mathcal{D}(\Upsilon_E^{\mathbf{z}}, E)$*. A similar results holds for* $D$ *and the first part is proved.*

*For the second part, suppose that* $\mathbf{x} \in \mathcal{D}(\Upsilon_E^{\mathbf{z}}, E) \cap \mathcal{D}(\Upsilon_D^{\mathbf{z}}, D)$*. Then*

$$\begin{aligned} \mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{x} &= E\mathbf{x}; \\ \mathsf{W}(\Upsilon_D^{\mathbf{z}}, D)\mathbf{x} &= D\mathbf{x}. \end{aligned} \tag{44}$$

*See Property 3.2. Using the fact that* $E\mathbf{x} = D\mathbf{x}$*, and also using the fact that the equality implies, in particular, the inequality.*

$$\begin{aligned} \mathsf{W}(\Upsilon_E^{\mathbf{z}}, E)\mathbf{x} &\succeq D\mathbf{x}; \\ \mathsf{W}(\Upsilon_D^{\mathbf{z}}, D)\mathbf{x} &\succeq E\mathbf{x}. \end{aligned} \tag{45}$$

*Then, using the same manipulations as in Proposition 3.1, it can be concluded that* $\mathbf{x} = \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*\mathbf{x}$*. Then, clearly* $\mathbf{x}$ *is a linear combination of columns of* $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$ *and the proof is completed.*

□

Using the iterative form of the Primal Method, one may show the following result concerning the evolution of the sequence and its final value.

**Proposition 3.5.** *(*Sequence characterization*) The sequence* $\mathbf{x}[k]$ *is non-decreasing, and converges to the smallest solution* $\mathbf{x}$ *such that* $\mathbf{x} \in \mathcal{D}(\Upsilon_E^{\mathbf{z}}, E) \cap \mathcal{D}(\Upsilon_D^{\mathbf{z}}, D)$ *and* $\mathbf{x} \succeq \mathbf{x}_0$*.*

**Proof 3.5.** *The fact that it is a non-decreasing sequence is straightforward by the sum of* $\mathbf{x}[k]$ *in Equation* (34)*.*

*For the second part, let* $\mathcal{X}$ *be the set of all solutions* $\mathbf{x}$ *(*$E\mathbf{x} = D\mathbf{x}$*) inside* $\mathcal{D}(\Upsilon_E^{\mathbf{z}}, E) \cap \mathcal{D}(\Upsilon_D^{\mathbf{z}}, D)$ *and* $\mathcal{X}^{\succeq}(\mathbf{x}_0)$ *all* $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{x} \succeq \mathbf{x}_0$*. Then, by the conclusions of Proposition 3.4, if* $\mathbf{x} \in \mathcal{X}$

$$\mathbf{x} = \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^* \mathbf{x} \;\Rightarrow\; \mathbf{x} \succeq \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^* \mathbf{x}. \tag{46}$$

*With this in regard, if additionally* $\mathbf{x} \succeq \mathbf{x}_0$ *one has* $\mathbf{x} \succeq \mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^* \mathbf{x}_0$. *Thus, any member of* $\mathcal{X}^{\succeq}(\mathbf{x}_0)$ *is minored by* $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^* \mathbf{x}_0$, *which is the value in which the sequence* $\mathbf{x}[k]$ *converges and also a member of* $\mathcal{X}^{\succeq}(\mathbf{x}_0)$. *Then, the proof is completed.*

□

$\mathcal{S} = \mathcal{D}(\Upsilon_E^{\mathbf{z}}, E) \cap \mathcal{D}(\Upsilon_D^{\mathbf{z}}, D)$ is a semimodule, being an intersection of two semimodules. This semimodule has the special property that the smallest solution of $E\mathbf{x} = D\mathbf{x}$ exists inside it.

**Property 3.4.** *It can be seen that the non-iterating Primal Method (computing* $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$*) and iterating Primal Method (iterating the map in Equation* (34)*) are related by using the initial conditions formed by the columns of the identity matrix in the iterating Primal Method .* □

Note that the Dual Method generates a non-increasing sequence which converges to the greatest solution smaller or equal than the initial $\mathbf{x}_0$. The Primal Method has an "almost" dual property in that regard: it generates a non-decreasing sequence which converges to the smallest solution greater or equal than the initial $\mathbf{x}_0$ which is inside the dominance spaces. The "...which is inside the dominance space" part, not present in the Dual, is which justifies the "almost".

*3.4. Numerical example*

Let

$$R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}; \tag{47}$$

be an equation in which $R, \mathbf{r}, S$ and $\mathbf{s}$ are given by

$$\begin{pmatrix} 2 & -3 & -2 \\ -4 & -3 & 4 \\ 0 & 3 & -3 \\ -3 & 1 & -2 \end{pmatrix} \mathbf{p} \oplus \begin{pmatrix} -2 \\ -3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & -3 \\ -3 & 2 & -3 \end{pmatrix} \mathbf{p} \oplus \begin{pmatrix} 1 \\ -3 \\ 2 \\ -5 \end{pmatrix} \tag{48}$$

Using the Dual Method, the augmented solution $\mathbf{z} = (3\ -2\ 0\ 0)^T$ can be found, thus $\mathbf{p}_{solD} = (3\ -2\ 0)^T$ is a solution.

By inspection of the products $E\mathbf{z}$ and $D\mathbf{z}$, it is possible to conclude that

$$\begin{gathered}\Upsilon_E^{\mathbf{z}}(1) = 1;\quad \Upsilon_E^{\mathbf{z}}(2) = 3;\\ \Upsilon_E^{\mathbf{z}}(3) = 1;\quad \Upsilon_E^{\mathbf{z}}(4) = 1.\end{gathered} \tag{49}$$

$$\begin{gathered}\Upsilon_D^{\mathbf{z}}(1) = 1;\quad \Upsilon_D^{\mathbf{z}}(2) = 1;\\ \Upsilon_D^{\mathbf{z}}(3) = 1;\quad \Upsilon_D^{\mathbf{z}}(4) = 1,\end{gathered} \tag{50}$$

and thus

$$\mathsf{W}(\Upsilon_E^{\mathbf{z}}, E) = \begin{pmatrix} 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix};\ \mathsf{W}(\Upsilon_D^{\mathbf{z}}, D) = \begin{pmatrix} 2 & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}; \tag{51}$$

$$\mathsf{W}^\flat(\Upsilon_E^{\mathbf{z}}, E) = \begin{pmatrix} -2 & \varepsilon & 0 & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix};\ \mathsf{W}^\flat(\Upsilon_D^{\mathbf{z}}, D) = \begin{pmatrix} -2 & -1 & 0 & 3 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}. \tag{52}$$

And also

$$\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^* = \begin{pmatrix} 0 & 5 & 3 & 2 \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ -3 & 2 & 0 & -1 \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}. \tag{53}$$

The first, second and fourth columns of $\mathsf{H}(E, D, \Upsilon_E^{\mathbf{z}}, \Upsilon_D^{\mathbf{z}})^*$ are linearly independent. The last column generates the solution $\mathbf{p}_{solP} = (2\ \varepsilon\ -1)^T$ , which is smaller or equal than $\mathbf{p}_{solD}$. It is also remarkable that this solution

Table 1: Shown in the table: the mean of number of linearly independent solutions found in the Primal Method, the standard deviation of $LI_p$, the mean of time taken to end the Primal Method (seconds, and including the time for solving the equation with the Dual Method), the standard deviation of $t_p$, the mean sparsity of matrix $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)$ (proportion of $\varepsilon$ entries), the standard deviation of $n_\varepsilon$.

| $n$ | $\overline{LI_p}$ | $\sigma(LI_p)$ | $\bar{t}_p$ | $\sigma(t_p)$ | $\overline{n}_\varepsilon$ | $\sigma(n_\varepsilon)$ |
|---|---|---|---|---|---|---|
| 50 | 35.45 | 2.03 | 0.04 | 0.005 | 71% | 3% |
| 100 | 77.65 | 2.49 | 0.15 | 0.005 | 76% | 3% |
| 200 | 166.25 | 2.75 | 0.74 | 0.082 | 82% | 2% |
| 300 | 254.15 | 3.16 | 1.88 | 0.060 | 85% | 1% |

$\mathbf{p}_{solP}$ is the smallest one in $\mathcal{D}(\Upsilon^{\mathbf{z}}_E, E) \cap \mathcal{D}(\Upsilon^{\mathbf{z}}_D, D)$. This is due to the fact that it was generated by the augmented initial condition $(\varepsilon\ \varepsilon\ \varepsilon\ 0)^T$ in the iterating Primal Method (see Property 3.4), and thus used as initial $\mathbf{p}$ the smallest initial condition possible: $\overline{\varepsilon}$.

**Remark 3.4.** *In order to find the smallest solution of Equation* (4) *inside a dominance space, it is sufficient to transform it into linear one and use the iterating Primal Method, in the augmented vector* $\mathbf{x}$*, with the initial condition* $\mathbf{p}[0] = \overline{\varepsilon}$ *and* $y = 0$*.*

□

As an illustration of the practical behavior of the method, an experiment (using the computer package ScicosLab 4.4.1) in which the Primal Method was applied to systems of the form of Equation (47) was done. The matrices $E, D \in \mathbb{T}^{n\times n}_{max}$ were square, with random entries between $-10$ and 10 or null, with 20% of entries equal to $\varepsilon$. The experiment was done with values of $n$ being 50, 100, 200 and 300. For each $n$ the experiment was repeated 50 times. The results are shown in Table 1. From this table, it is possible to infer that the Primal Method seems to yields a high number of linearly independent solutions and also with a high sparsity. Further, that it has a considerable speed even with a relatively large dimension as $n = 300$.

As mentioned in Section 1, the Primal Method generates solution with attractive properties in some applications. For example, when designing feedback controllers for guaranteeing that certain constraints holds in the firing dates of Timed Event Graphs, it may be also desirable that this control has the secondary property of not slowing down the system performance (for example, it may be a manufacturing system in which the firings are related to the production rate). Further, a tertiary desirable characteristic could be that this control rule is "simple" enough to implement.

This control can be designed to be completely characterized by a matrix $F$, which can be obtained by solving a tropical affine equation as Equation (47) (after vectorizing the matrix $F$ in a vector $\mathbf{x}$). Then, the requirement of performance can be transferred to the matrix $F$ having small entries, and the simplicity in the sparsity of it (because it would require less connections between the transitions of the Timed Event Graph). This is fitting for the Primal Method, which can generate small and sparse solutions. So, the affine equation can be solved (by any method, the Dual Method for example) and the resulting $F$ can be used in the Primal Method to generate a better solution, $F^{primal}$ by finding, for example, the smallest solution inside the dominance (see Remark 3.4). The reader is invited to read Gonçalves et al. (2012) in order to obtain some results concerning feedback synthesis and the utility of the Primal Method in that context.

### *3.5. Connection with other works*

The proposed method has similarities with some other previously published results.

**Connection to the Cellular Decomposition of Develin and Sturmfels (2004)**: The Develin-Sturmfels cellular decomposition decomposes a tropically convex polytope $tconv(V)$ , considered as the image of a finite matrix $V \in \mathbb{T}_{max}^{n\times m}$, in a finite number of convex (in the traditional sense) polytopes. It does so using the concept of *type*: a type of a vector $\mathbf{x}$ relative to $V$, $type_V^{\mathbf{x}}$, is a set of $n$ subsets of $\{1, 2, ..., m\}$ defined such that [4]:

$$type_V^{\mathbf{x}}(i) = \{j \in \{1, 2, ..., m\} \text{ such that } \bigoplus_{k=1}^{n}(-x_k)V_{kj} = (-x_i)V_{ij}\}. \quad (54)$$

A type and an induced dominance are closely related. In fact, if $\Upsilon_M^{\mathbf{x}}(i) = j$ then $i \in type_{M^T}^{-\mathbf{x}}(j)$ (note that the type must act on $-\mathbf{x}$ and the matrix $M$ must be transposed).

[4] The original work of Develin and Sturmfels (2004) uses $\mathbb{T}_{min}$ instead of $\mathbb{T}_{max}$. Further, it assumes that the linear span of the rows of $V$ generates the tropically convex polytope. The definition was adapted to the settings of this paper: $\mathbb{T}_{max}$ and column linear span.

If one defines for a type $S$ the set $\mathcal{X}_S$ of all points $\mathbf{x} \in \mathbb{T}^n_{max}$ such that their type $type^{\mathbf{x}}_V$ contains $S$, then (i) $\mathcal{X}_S$ is convex in the traditional sense (ii) $\mathcal{X}_S$ is bounded if and only if $S(j) \neq \emptyset$ for all $j$ and (iii) $tconv(V)$ is the union of all bounded $\mathcal{X}_S$. Further, $\mathcal{X}_S$ can be completely characterized as the image of a Kleene Closure matrix $C(S)^*$.

The solution set of Equation (2) -an *implicit* characterization of the semimodule of solutions- is a tropically convex polytope, that is, there is a finite matrix $G$ (see Butkovic and Hegedüs (1984)) such that all solutions can be written as $\mathbf{x} = G\mathbf{y}$ for a vector $\mathbf{y}$ - an *explicit* characterization of the semimodule of solutions. Thus, in possession of a solution $\mathbf{z}$, one can compute its type $type^{\mathbf{z}}_G$ and with it characterize a convex set of solutions as the image of the matrix $C(type^{\mathbf{z}}_G)^*$. The Primal Method works similarly, but uses an implicit characterization (instead of the explicit), that is Equation (2), to compute a convex set of solutions as the image of the matrix $\mathsf{H}(E, D, \Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D)^*$. Further, the Primal Method also induces a decomposition by convex cells of the semimodule of solutions if one enumerates all dominances $\Upsilon$ and considers all the sets generated by the image of the upper bounded matrices $\mathsf{H}(E, D, \Upsilon_E, \Upsilon_D)^*$. This approach was the one taken in the work described below.

**Connections with the Algorithm of Truffet (2010)**: That work deals with equations of the form $A\mathbf{x} \preceq B\mathbf{x}$, $A, B \in \mathbb{T}^{n' \times m}_{max}$. Null row considerations aside (the author considers explicitly the null rows of $B$ when constructing the solution, as opposed to this paper in which the null rows are considered to be, without loss of generality, non existent by Remark 3.1), the author enumerates a set of $n'$-tuples with values ranging from 1 to $m$, that is, a set of functions $\underline{j} : \{1, 2, ..., n'\} \mapsto \{1, 2, ..., m\}$ . Then, they are used to construct the entire semimodule of solutions to the tropical linear equation as an image of a matrix $G$ by the augmentation of individual matrices

$$G^{\underline{j}} = \left( \bigoplus_{i=1}^{n} Q^{i\underline{j}(i)} \right)^* \tag{55}$$

for all $n'$-tuples $\underline{j}$ in the set.

These results can be interpreted by using the notations of this paper. But since in this paper equations of the form of Equation (2) are considered, it is necessary to use $A = (E^T \,\vdots\, D^T)^T$, $B = (D^T \,\vdots\, E^T)^T$ so $A\mathbf{x} \preceq B\mathbf{x} \Rightarrow E\mathbf{x} = D\mathbf{x}$ (so, $n' = 2n$ since $E, D \in \mathbb{T}^{n \times m}_{max}$). Then, in the notation of this paper, a $\underline{j}$ is a dominance on the matrix $B$ and thus a dominance on $E$ (denoted by $\overline{\Upsilon}_E$)

augmented with a dominance in $D$ (denoted by $\Upsilon_D$). Then, the matrix inside the Kleene Closure of Equation (55) can be written as $\mathsf{H}(E, D, \Upsilon_E, \Upsilon_D) \oplus I$ and, by consequence, $G^{\underline{j}} = \mathsf{H}(E, D, \Upsilon_E, \Upsilon_D)^*$.

The main difference of the approach of this paper and that one is that the former presents a guidance for choosing dominances in which the "usefulness" is guaranteed (that is, those induced by solutions with appropriate properties). The latter work generates all solutions by enumerating a set of "promising " dominances and computing the $G^{\underline{j}}$ for all of them. These promising dominances are obtained by discarding the $n'$-tuples $\underline{j}$ that would surely generate an unbounded (and therefore useless) $G^{\underline{j}}$. However, (in general) it is possible that event the promising dominances generates unbounded solutions, and these must be removed from the matrix $G$ later.

**Connection with Mean Payoff Games**: Consider a directed bipartite graph with two disjoint sets of nodes, say "CIRCLE" nodes ($n$ nodes $i = 1, 2, .., n$) and "SQUARE" nodes ($m$ nodes $j = 1, 2, .., m$). A game is played in which, initially, a pawn is in one SQUARE node $j$. A player, MIN, plays by moving the pawn to a CIRCLE node $i$ and receives from the other player, MAX, an integer amount $A_{ij}$. Then, it is time to the player MAX to move the pawn to a SQUARE node $j'$ and then receiving from MIN an integer amount $B_{ij'}$. Then, a *turn* ends and this zero-sum game proceeds again with a move from MIN player, and so on. The player MAX aims to maximize his receiving while the player MIN aims to minimize this same amount. Then, one considers the *mean payoff at the turn* $k$ as the cumulative reward of the player MAX at turn $k$, divided by $k$. With these definitions, it is of interest the vector $\chi_j$ which gives the value [5] of this game when the starting SQUARE node is $j$ and $k$ goes to infinity. This is a *mean payoff game*.

There is a close connection between tropical linear equations, written on the form $A\mathbf{x} \preceq B\mathbf{x}$ $A, B \in \mathbb{T}_{max}^{n\times m}$ (which can be formulated as Equation (2) and *vice-versa*, and thus are equivalent in terms of what they can describe) and mean payoff games described above. In fact, the map $f(\mathbf{x}) = A \backslash\!\!\!\circ (B\mathbf{x})$ can be seen as a dynamic programming operator of the described game. Then, solving the tropical linear equation equation can be reduced to the problem of finding an *invariant half-line* to this map. These ideas have been developed in V.Dhingra and Gaubert (2006); Akian et al. (2010); Gaubert et al. (2012); Gaubert and Sergeev (2010) and in the references therein.

[5] In the sense of the MINIMAX theorem, see Osborne and Rubinstein (1994).

The Primal Method can be also interpreted by what is denoted in the literature as *one player mean payoff game* (V.Dhingra and Gaubert (2006)), which establishes mean payoff games as another common ground to both Primal and Dual, other than the Extended Cuninghame-Green and K. Zimmermann algorithm presented in Section 2. In an (MAX) one player mean payoff game, the MAX player uses a *positional strategy* $\sigma : \{1, 2, .., n\} \mapsto \{1, 2, ..., m\}$, that is, he chooses an *a priori* strategy that he will chose SQUARE node $j = \sigma(i)$ when it is at the CIRCLE node $i$. The player MIN then aims to minimize the rewards of player MAX based on a more general strategy. This positional strategy is simply what is called *dominance* is this paper. With this strategy, the equation $B\mathbf{x} \succeq A\mathbf{x}$ is reduced to an Equation $B^{\sigma}\mathbf{x} \succeq A\mathbf{x}$, in which $B^{\sigma}$ is dually residuated, that is $B^{\sigma}\mathbf{x} \succeq A\mathbf{x} \iff \mathbf{x} \succeq (B^{\sigma})^{\flat}A\mathbf{x}$. Then, a standard Kleene Closure can be used. This step is essentially an application of the Primal Method, but adapted to Equation $A\mathbf{x} \preceq B\mathbf{x}$ instead of Equation (47). However, not all strategies $\sigma$ generate "useful" dominances (upper bounded solutions): one needs that all $\chi_j \preceq 0$ (or equivalently, $\rho((B^{\sigma})^{\flat}A) \preceq 0$) to this to happen. The fact that the Primal Method presented in this paper uses another solution (with special characteristics) to generate dominances addresses this problem.

**Connections with the Algorithm of Lorenzo and de la Puente (2011)**: The concept of dominance is closely related to the concept of *Winning sequences* in Lorenzo and de la Puente (2011). Following this work, given a system as Equation (2), with $E, D \in \mathbb{T}^{n\times m}_{max}$, a *winning pair* is a pair $(i, j) \in \{1, 2, ..., m\} \times \{1, 2, ..., m\}$ of indexes. A winning sequence is a set of $n$ winning pairs such that a compatibility requirement for the matrices $E$, $D$ holds. This compatibility is a necessary condition for the proposed algorithm in Lorenzo and de la Puente (2011) to be successful in returning solutions when this winning sequence is used.

The winning sequences are replaced in this work by a pair of induced dominances, $\{\Upsilon^{\mathbf{z}}_E, \Upsilon^{\mathbf{z}}_D\}$. The compatibility requirement then comes naturally from the fact that $\mathbf{z}$ is a solution, by hypothesis. Using a winning sequence, the authors derive from Equation (2) a set of bivariate equalities and inequalities. Then, a specialized gaussian elimination is used to generate the entire set of solutions that are induced by that winning sequence. The reasoning here is similar, but a previously found solution is used for finding adequate dominances and Kleene Closures are used instead of the gaussian elimination. As it was proved in Proposition 3.4, it also generates the entire set of solutions inside that particular dominance.

**Connections with the Algorithm of Gaubert et al. (2012)**: In Gaubert et al. (2012), algorithms for solving tropical fractional linear programs (tropical analogues of fractional linear programs) are presented. One of the algorithms (Algorithm 2) concerns minimization problems, and an idea closely related to the Primal Method was used for solving them. At each step, the current suboptimal solution $\mathbf{x}[k]$ is used to transform the constraint equation (which determines the feasible set of the optimization problem)

$$R\mathbf{x} \oplus \mathbf{r} \succeq S\mathbf{x} \oplus \mathbf{s} \tag{56}$$

(which is equivalent to equations of the form Equation (47) discussed in this paper) in a "simplified form" $R^{\sigma}\mathbf{x} \succeq S\mathbf{x} \oplus \mathbf{s}$. The matrix $R^{\sigma}$ is, in the notations of this paper, the matrix generated by the dominance of $\mathbf{x}[k]$ in $R$, that is, $R^{\sigma} = \mathsf{W}(\Upsilon_R^{\mathbf{x}[k]}, R)$. Then, the equation is reduced to the form $\mathbf{x} \succeq \mathsf{W}^{\flat}(\Upsilon_R^{\mathbf{x}[k]}, R)(S\mathbf{x} \oplus \mathbf{s})$ in which, as discussed in this paper, the smallest solution $\mathbf{x}[k+1] \preceq \mathbf{x}[k]$ exists and can be computed using Kleene Closures. This step is essentially an application of the Primal Method, but adapted to Equation (56) instead of Equation (47) (see the connection with mean payoff games above). Then, after this procedure one has either a smaller (therefore better) objective function - and thus the iteration must continue - or the algorithm converged to (one possible) optimal solution.

Among the works cited, this one is probably the closest one to the proposed Primal Method. This is due to the fact that it contains implicitly an important feature of the algorithm: the idea that a solution is useful to guarantee the convergence (upper boundedness) of the Kleene Closure of a specially constructed matrix.

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