Optimal control for (max,+)-linear systems in the presence of disturbances

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Abstract. This paper deals with control of (max,+)-linear systems when a disturbance acts on system state. In a first part we synthesize the greatest control which allows to match the disturbance action. Then, we look for an output feedback which makes the disturbance matching. Formally, this problem is very close to the disturbance decoupling problem for continuous linear systems.

1 Introduction

The (max,+) working group [1] has developed a linear theory for discrete event systems which are characterized by synchronization phenomena and time-delays. They have also proposed an optimal control law in regards of just in time criterion. Roughly speaking, it consists in computing the latest date of input events (which are controllable) in order to obtain output events before given desired output dates. This control synthesis needs a complete knowledge of the desired output. Since it is an open loop control, it is not robust when disturbances act on the system. In [5] we have proposed a closed loop control approach where the control objective is expressed as a reference model. The controller design is based on the residuation theory applied to particular mappings. Residuation theory makes possible to consider a kind of mapping inversion defined on ordered sets, and then plays naturally a significant role in controller synthesis.
This presentation deals with controller design when disturbances act on the system. As in conventional linear systems theory [10], the control is synthesized in order to keep the system state x in the kernel of the output matrix C. Section 2 recalls some algebraic tools and in particular that the kernel of a linear mapping defined on dioids (or lattices) is an equivalence relation. In Section 3 it is shown that our objective is equivalent to match the output due to the disturbance. Then we show that the optimal control is the greatest (in the dioid sense) which keeps the system state in the equivalence class gener-

ated by the disturbance. This means that the inputs are delayed as much as possible in order to match the output due to the disturbance.

2 Elements of dioid and residuation theories

2.1 Dioid Theory

We first recall in this section some notions from the dioid theory. A general introduction is given in [4], and a detailed introduction can be found[1] in [1].

Definition 1 (Dioid). *A dioid is a set $\mathcal{D}$ endowed with two inner operations denoted $\oplus$ and $\otimes$. The sum is associative, commutative, idempotent ($\forall a \in \mathcal{D}$, $a \oplus a = a$) and admits a neutral element denoted ε. The product is associative, distributes over the sum and admits a neutral element denoted e. The element ε is absorbing for the product.*

Definition 2 (Order relation). *An order relation can be associated with a dioid $\mathcal{D}$ by the following equivalence : $\forall a, b \in \mathcal{D}$, $a \succeq b \Leftrightarrow a = a \oplus b$.*

Definition 3 (Complete dioid). *A dioid $\mathcal{D}$ is complete if it is closed for infinite sums and if the product distributes over infinite sums too.*

Theorem 1. *Over a complete dioid $\mathcal{D}$, the implicit equation $a = ax \oplus b$ admits $x = a^*b$ as least solution, where $a^* = \bigoplus_{i \in \mathbb{N}}$ (Kleene star operator) with $a^0 = e$.*

The Kleene star operator, over a complete dioid $\mathcal{D}$, will be represented by the following mapping $\mathcal{K} : \mathcal{D} \to \mathcal{D}, x \mapsto x^*$.

Definition 4 (Kernel [4],[3]). *Let $C : \mathcal{X} \to \mathcal{Y}$ be a mapping. We call kernel of C (denoted by $\ker C$), the equivalence relation over $\mathcal{X}$:*

$$x \overset{\ker C}{\equiv} y \Leftrightarrow C(x) = C(y). \tag{1}$$

The set of equivalence classes is denoted by $\mathcal{X}_{/\ker C}$ and $[x]_C$ denotes the equivalence class of x.

Remark 1. The usual kernel definition $\{x \in \mathcal{X} \mid C(x) = \varepsilon\}$ becomes meaningless in dioid algebra. Each equivalence class contains all the elements which map to the same image, in [4], the term "fibration" is used. Relation (1) corresponds to the kernel definition of a mapping defined on lattices [6].

[1] An electronic version is available on http://maxplus.org.

2.2 Residuation theory

The residuation theory provides, under some assumptions, *optimal* solutions to inequalities such as $f(x) \preceq b$ where f is an isotone mapping (f s.t. $a \preceq b \Rightarrow f(a) \preceq f(b)$) defined over ordered sets. Some theoretical results are summarized below. Complete presentations are given in [2] and [1].

Definition 5 (Residual and residuated mapping). *An isotone mapping $f : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are ordered sets, is a residuated mapping if for all $y \in \mathcal{E}$, the least upper bound of the subset $\{x | f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted $f^\sharp(y)$. Mapping $f^\sharp$ is called the residual of f. When f is residuated, $f^\sharp$ is the unique isotone mapping such that $f \circ f^\sharp \preceq \mathsf{Id}_{\mathcal{E}}$, and $f^\sharp \circ f \succeq \mathsf{Id}_{\mathcal{D}}$ where Id is the identity mapping respectively on $\mathcal{E}$ and $\mathcal{D}$.*

Theorem 2 ([1]). *Consider the mapping $f : \mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ and $\mathcal{F}$ are complete dioids of which the bottom elements are, respectively, denoted by $\varepsilon_{\mathcal{E}}$ and $\varepsilon_{\mathcal{F}}$. Then, f is residuated iff $f(\varepsilon_{\mathcal{E}}) = \varepsilon_{\mathcal{F}}$ and $f(\bigoplus_{x \in \mathcal{G}} x) = \bigoplus_{x \in \mathcal{G}} f(x)$ for each $\mathcal{G} \subseteq \mathcal{E}$.*

Corollary 1. *The mappings $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ defined over a complete dioid $\mathcal{D}$ are both residuated [2]. Their residuals are usually denoted, respectively, $L_a^\sharp(x) = a \backslash\!\!\circ x$ and $R_a^\sharp(x) = x \phi a$ in $(\max, +)$ literature.*[3]

Theorem 3 ([1]). *Let $\mathcal{D}$ be a complete dioid and $A \in \mathcal{D}^{q \times m}$ be a matrix with entries in $\mathcal{D}$. Then, $A \phi A$ is a matrix in $\mathcal{D}^{q \times q}$ which verifies*

$$A \phi A = (A \phi A)^*. \tag{2}$$

2.3 Mapping restriction

In this subsection, the problem of mapping restriction and its connection with the residuation theory is addressed. In particular the Kleene star mapping, becomes residuated as soon as its codomain is restricted to its image.

Definition 6 (Restricted mapping). *Let $f : \mathcal{E} \to \mathcal{F}$ be a mapping and $\mathcal{A} \subseteq \mathcal{E}$. We will denote*[4] *$f_{|\mathcal{A}} : \mathcal{A} \to \mathcal{F}$ the mapping defined by $f_{|\mathcal{A}} = f \circ \mathsf{Id}_{|\mathcal{A}}$ where $\mathsf{Id}_{|\mathcal{A}} : \mathcal{A} \to \mathcal{E}$, $x \mapsto x$ is the canonical injection. Identically, let $\mathcal{B} \subseteq \mathcal{F}$ with $\mathsf{Im} f \subseteq \mathcal{B}$. Mapping ${}_{\mathcal{B}|}f : \mathcal{E} \to \mathcal{B}$ is defined by $f = \mathsf{Id}_{|\mathcal{B}} \circ {}_{\mathcal{B}|}f$, where $\mathsf{Id}_{|\mathcal{B}} : \mathcal{B} \to \mathcal{F}$, $x \mapsto x$ is the canonical injection.*

Definition 7 (Closure mapping). *An isotone mapping $f : \mathcal{E} \to \mathcal{E}$ defined on an ordered set $\mathcal{E}$ is a closure mapping if $f \succeq \mathsf{Id}_{\mathcal{E}}$ and $f \circ f = f$.*

[2] This property concerns as well a matrix dioid product, for instance $X \mapsto AX$ where $A, X \in \mathcal{D}^{n \times n}$. See [1] for the computation of $A \backslash\!\!\circ B$ and $B \phi A$.

[3] $a \backslash\!\!\circ b$ is the greatest solution of $ax \preceq b$.

[4] These notations are borrowed from classical linear system theory see [10].

Proposition 1 ([5]). *Let $f : \mathcal{E} \to \mathcal{E}$ be a closure mapping. A closure mapping restricted to its image* ${}_{\mathsf{Im}f|}f$ *is a residuated mapping whose residual is the canonical injection* $\mathsf{Id}_{|\mathsf{Im}f} : \mathsf{Im}f \to \mathcal{E}$, $x \mapsto x$.

Corollary 2. *The mapping* ${}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}$ *is a residuated mapping whose residual is* $\left({}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}\right)^\sharp = \mathsf{Id}_{|\mathsf{Im}\mathcal{K}}$.
This means that $x = a^*$ is the greatest solution to inequality $x^* \preceq a^*$. Actually, the greatest solution achieves equality.

2.4 Projectors [4, 3]

Lemma 1. *Let $C : \mathcal{X} \to \mathcal{Y}$ be a residuated mapping and let*

$$\Pi^C = C^\sharp \circ C. \tag{3}$$

Π^C is a projector, i.e. $\Pi^C \circ \Pi^C = \Pi^C$ and $C \circ \Pi^C = C$.

Lemma 2. *Let $B : \mathcal{U} \to \mathcal{X}$ be a residuated mapping and let*

$$\Pi_B = B \circ B^\sharp. \tag{4}$$

Π_B is a projector, i.e. $\Pi_B \circ \Pi_B = \Pi_B$ and $\Pi_B \circ B = B$.

2.5 Projections on the Image of a Mapping Parallel to the Kernel of Another Mapping

We consider $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$, the *projection* of $x \in \mathcal{X}$ *on* $\mathsf{Im}B$ *parallel to* $\ker C$ is any x' which belongs to $\mathsf{Im}B$ and is equivalent to x modulo $\ker C$, that is,

$$\text{find } x' \in \mathcal{X}, \text{ s.t. } \exists u \in \mathcal{U} : C(x') = C(x) \text{ and } B(u) = x'.$$

From (3)-(4), it comes that $z = \Pi^C(x) = C^\sharp \circ C(x)$ is the greatest element in the equivalence class of x modulo $\ker C$, and $\xi = \Pi_B(z) = B \circ B^\sharp(z)$ is the greatest element in $\mathsf{Im}B$ which is less than z. Then z is 'subequivalent' (see [4]) to x modulo $\ker C$, *i.e.* $C \circ \Pi_B \circ \Pi^C(x) = C(\xi) \preceq C(x)$. If equality holds true (*i.e.* $C(\xi) = C(x)$), $\Pi_B \circ \Pi^C$ will be denoted by Π_B^C, which is a projector (*i.e.* $\Pi_B^C = \Pi_B^C \circ \Pi_B^C$). The question of existence and uniqueness of projections for given operators B and C are studied in [4, 3]. We summarize the results

- *Existence* of projections for all x is equivalent to the condition $C = C \circ \Pi_B^C$ (*i.e.* $\xi = \Pi_B^C(x) \in [x]_C$).
- *Uniqueness* is equivalent to the condition $B = \Pi_B^C \circ B$ (*i.e.* any $x \in \mathsf{Im}B$ remains invariant by Π_B^C).

3 Control in the presence of disturbances

$$\begin{cases} x = Ax \oplus Bu \oplus Sq \\ y = Cx \end{cases} \Rightarrow \begin{cases} x = A^*Bu \oplus A^*Sq \\ y = Cx \end{cases} \tag{5}$$

where $u \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^p$ is the control vector, $x \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^n$ the state vector, $y \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^q$ the output vector, $q \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^m$ the disturbance (uncontrollable input) vector. Matrices of proper size A, B, C, S have entries in dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ with only non-negative exponents integer values. In the conventional linear system theory [10], the disturbance decoupling problem consists in finding a control u such that the disturbance q has no influence on the controlled output y (*i.e.* $y = 0$, $\forall q \in \mathcal{Q}$, the control keeps system state x in the kernel of C). Our problem must be stated in a different way since trajectories u, x, y and q are monotonous and no decreasing. The output cancellation is consequently meaningless in this context. Here we seek for a control u which keeps the system state x in the equivalence class of A^*Sq modulo $\ker C$. We say that such a control u ensures the disturbance matching, if u is such that

$$A^*Sq \overset{\ker C}{\equiv} x \Leftrightarrow A^*Sq \overset{\ker C}{\equiv} A^*Bu \oplus A^*Sq \Leftrightarrow CA^*Sq = CA^*Bu \oplus CA^*Sq. \tag{6}$$

The right statement shows that the objective will be achieved iff $CA^*Sq \succeq CA^*Bu$. Obviously the set of controls verifying (6) may contain many elements, hence we are interested in computing the greatest one, formally

$$u_{opt} = \bigoplus_{\{u \mid A^*Bu \oplus A^*Sq \overset{\ker C}{\equiv} A^*Sq\}} u. \tag{7}$$

The greatest element in $\left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^n$ such that $y = CA^*Sq$ is by definition the greatest element in $[A^*Sq]_C$, *i.e.*

$$\Pi^C(A^*Sq) = C^\sharp \circ C(A^*Sq).$$

We denote with the same symbol the matrix C and the linear mapping $x \mapsto Cx$. However, since this greatest state is not necessarily reachable, we seek for the greatest reachable state x ensuring the disturbance matching. This state is the projection of $\Pi^C(A^*Sq)$ in $\mathsf{Im}A^*B$, *i.e.*,

$$\xi = \Pi_{A^*B} \circ \Pi^C(A^*Sq) = A^*B \circ (A^*B)^\sharp \circ C^\sharp \circ C(A^*Sq). \tag{8}$$

It is the greatest element in $\mathsf{Im}A^*B$ which is 'subequivalent' to $\Pi^C(A^*Sq)$, *i.e.* such that $C(\xi) \preceq C(A^*Sq)$. If $C \circ \Pi_{A^*B} \circ \Pi^C = C$ then $\Pi_{A^*B} \circ \Pi^C = \Pi^C_{A^*B}$ is a projector in $\mathsf{Im}A^*B$ parallel to $\ker C$ and ξ is the greatest element in $[A^*Sq]_C$.

Remark 2. System (5) can represent a Timed Event Graph (TEG), where u represents controllable transitions, x internal transitions and q represents uncontrollable transitions which delay the firing of internal transitions. In this context, it is useless that tokens enter too soon into the system. Then the control objective is to delay maximally tokens input by taking the disturbance into account. The control u_{opt} achieves optimally the just-in-time criterion when some disturbances q acts on the system.

The greatest control u_{opt} allowing to reach this greatest state x (for a given q), is $u_{opt} = (A^*B)^\sharp \circ C^\sharp \circ C(A^*Sq) = (CA^*B) \backslash (CA^*Sq)$.
Practically, this control computation requires the disturbance[5] knowledge. Our problem is then to find a feedback F which allows to avoid this assumption.

3.1 Output feedback

We discuss the existence and the computation of an output feedback controller which leads to a closed-loop system making the disturbance matching. The objective of the control (denoted by $u = Fy$ with $F \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{p \times q}$) is to keep the transfer relation between y and q unchanged. System (5) becomes

$$\begin{cases} x = A^*BFy \oplus A^*Sq \\ y = Cx = CA^*BFy \oplus CA^*Sq \end{cases} \tag{9}$$

The output equation ($y = CA^*BFy \oplus CA^*Sq$) can be solved by considering Theorem 1. We obtain[6] $y = (CA^*BF)^*CA^*Sq = CA^*(BFCA^*)^*Sq$. According to the previous section, the disturbance matching problem with output feedback can be stated as follows : find the greatest output feedback (denoted $\hat{F}$) such that the transfer function between y and q remains unchanged, *i.e.*

$$\hat{F} = \{\bigoplus F | M(F) \preceq CA^*S\}, \tag{10}$$

where mapping $M : X \mapsto (CA^*BX)^*CA^*S$ is not residuated since $M(\varepsilon) = CA^*S \neq \varepsilon$. Nevertheless the following result shows that it is possible to compute the greatest output feedback $\hat{F}$.

Proposition 2. *The greatest solution of (10) is* $\hat{F} = CA^*B \backslash CA^*S / CA^*S$.

Proof. $F = \varepsilon$ is a solution of (10) (since $M(\varepsilon) = CA^*S$), hence the greatest solution, if it exists, also achieves equality. From (10), we seek for the

[5] In a manufacturing system, q may represent the supply of raw material which is a priori known. The problem is then very similar to the problem introduced in [9] which establishes an optimal open-loop control in presence of known uncontrollable inputs.
[6] We recall that $(ab)^*a = a(ba)^*$ (see [5]).

greatest feedback verifying $(CA^*BF)^*CA^*S \preceq CA^*S$. Since $R_{CA^*S} : x \mapsto xCA^*S$ is residuated (cf. Corollary 1), we have $(CA^*BF)^*CA^*S \preceq CA^*S \Leftrightarrow (CA^*BF)^* \preceq CA^*S \phi CA^*S$. According to (2), the last expression shows that $CA^*S \phi CA^*S$ belongs to the image of $\mathcal{K}$. Since ${}_{\mathsf{Im}\mathcal{K}|}\mathcal{K}$ is residuated (cf. Corollary 2), there is also the following equivalence $(CA^*BF)^* \preceq CA^*S \phi CA^*S \Leftrightarrow CA^*BF \preceq CA^*S \phi CA^*S$. Finally, since mapping $L_{CA^*B} : x \mapsto CA^*Bx$ is residuated too (see Corollary 1), we verify that $\hat{F} = CA^*B \backslash CA^*S \phi CA^*S$ is the greatest solution of $M(\hat{F}) = (CA^*B\hat{F})^*CA^*S \preceq CA^*S$.

Property 1. This feedback is the greatest such that $x = A^*(BFCA^*)^*Sq \in [A^*Sq]_C$ and obviously the resulting state x is lower than $\xi = \Pi^C(A^*Sq)$. Furthermore $x = A^*(BFCA^*)^*Sq \succeq A^*Sq$. Therefore, if $x \in [\xi]_C$ it exists a control $u = Fy$ such that $x = Ax \oplus Bu \oplus Sq \in [\xi]_C$. It seems interesting to characterize under which conditions if $x \in [\xi]_C$ it exists a control $u = Fy$ such that $x = Ax \oplus Bu \in [\xi]_C$ and to exhibit the links with the $(A, B)-$invariant definition given in [7].

4 Conclusion

The objective is to synthesize a control law keeping state x in the kernel of C. It presents a strong analogy with the disturbance decoupling of the traditional control systems. However it must be noted that the reached objective does not lead to an output cancellation. Indeed the specific kernel definition of a mapping on a lattice and the nature of the considered systems allow to obtain the greatest control such that the output remains unchanged for any disturbance. The problem is obviously linked with the problem of characterization of $(A, B)-$invariant and future works will discuss this point [8].

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