On the Regulation Problem for Tropical Linear Event-Invariant Dynamical Systems

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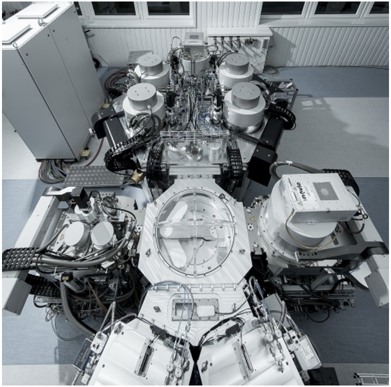

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- $Im\{M\}$ is the *right image* of M, that is, the set $\{x | \exists y \ , \ \ x = My\}$;

Problem statement

- Consider a *Tropical Linear Event-Invariant System*

$$x[k+1] = Ax[k] \oplus Bu[k] \tag{1}$$

in which $x[k] \in \mathcal{X}$ is the *state vectors* and $u[k] \in \mathcal{U}$ is the *control vectors*;

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- $\mathcal{S}$ is the set of *desirable specifications*;
- **Tropical regulation problem** $\mathcal{R}(A, B, E, D)$: find a control action $u[k]$ such that *for all* initial condition $x[0]$ there exists a natural number K such that for all $k \geq K$, $x[k] \in \mathcal{S}$;

(A,B) geometrical invariance and coupled problems

- [Katz, 2007]: A semimodule $\mathcal{K} \subseteq \mathcal{X}$ is said to be *(A,B) geometrical invariant* if for any $x \in \mathcal{K}$ there exists $u \in \mathcal{U}$ such that $Ax \oplus Bu \in \mathcal{K}$;

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- [Katz, 2007]: Given a specification semimodule $\mathcal{S}$ of a problem $\mathcal{R}(A, B, E, D)$, there exists a maximal (A, B) geometrical invariant semimodule inside $\mathcal{S}$. It will be denoted by $\mathcal{K}_{max}(\mathcal{R})$;

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- **Definition**: a problem $\mathcal{R}$ is said to be *coupled* if any member x of $\mathcal{K}_{max}(\mathcal{R})$, except the null vector itself, has only finite entries;

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- Very practical sufficient condition: suppose the constraints $Ex = Dx$ can be written in the special form $x = M^*x$ (often the case);
- M^* only having finite entries implies *finite volume* [Katz, 2007], which in turn implies the coupled property;

Control characteristic equation

- **Definition**: the *control characteristic equation* $\mathcal{C}(\mathcal{R})$ associated to the problem $\mathcal{R}(A, B, E, D)$ is the following equation for the unknowns $\chi \in \mathcal{X}$, $\mu \in \mathcal{U}$ and $\lambda \in \mathbb{R}$

$$\begin{aligned} \lambda\chi &= A\chi \oplus B\mu; \\ E\chi &= D\chi. \end{aligned} \tag{3}$$

  Furthermore, a solution $\{\lambda, \chi, \mu\}$ is *proper* if no entry of χ is the null element $\perp$;

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- **Definition**: the *control characteristic spectrum* of a problem, $\Lambda(\mathcal{R})$, is the set of λ such that $\{\lambda, \chi, \mu\}$ is a proper solution;

Non-critical problems

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- **Definition**: a problem $\mathcal{R}$ is said to be *critical* if the control characteristic spectrum $\Lambda(\mathcal{R})$ is the singleton $\{\rho(A)\}$. Otherwise, it is said to be *non-critical*;

Convergence number

- **Definition**: Given a square matrix M with $\rho(M) \leq 0$, the *convergence number* $\kappa(M)$ is the smallest number k such that

$$M^* = I \oplus M \oplus M^2 \oplus M^3 \oplus ... \oplus M^k. \tag{4}$$

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- If M has n rows (and hence n columns), then $\kappa(M) \leq n$.

Main results

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- **Theorem 2**: a coupled and non-critical problem $\mathcal{R}$ is solvable *if and only if* its control characteristic equation $\mathcal{C}(\mathcal{R})$ has a proper solution $\{\lambda, \chi, \mu\}$. The control action is a simple state feedback of the form

$$u[k] = Fx[k] \tag{5}$$

in which $F = \mu(-\chi)^T$. Furthermore, the closed loop system will have eigenvalue equal to λ and convergence to $\mathcal{S}$ is achieved in at most $\kappa(\lambda^{-1}A)$ events.

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- Let $\mathcal{K}$ be one of these sets. The fact that the problem is coupled, by hypothesis, implies that it is finitely generated. Then it can be written as $\mathcal{K} = Im\{X\}$ for a matrix X;

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- Since $\mathcal{K}$ is (A, B) geometrical invariant, there exist matrices U and V such that
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- Finally, since the problem is coupled, all the entries of X are not $\perp$, and therefore Xv has no $\perp$ entry. By this and the Equation (8), clearly $\chi = Xv$, $\mu = Uv$ and λ compose a proper solution of the control characteristic equation.

Solving the control characteristic equation

- The control characteristic equation $\mathcal{C}(\mathcal{R})$ can be written conveniently as

$$\begin{pmatrix} A & B \\ EA & EB \end{pmatrix} \begin{pmatrix} \chi \\ \mu \end{pmatrix} = \lambda \begin{pmatrix} I & \perp \\ D & \perp \end{pmatrix} \begin{pmatrix} \chi \\ \mu \end{pmatrix}; \qquad (9)$$

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- *Pseudopolynomial algorithms*: not very difficult to solve currently for medium-sized systems [S. Gaubert and S. Sergeev, 2013];

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- If the problem is coupled, however, any solution to the two-sided eigenproblem generates a proper solution to the control characteristic equation $\mathcal{C}(\mathcal{R})$, that is, $y \neq \perp$ implies that χ does not have $\perp$ entries;

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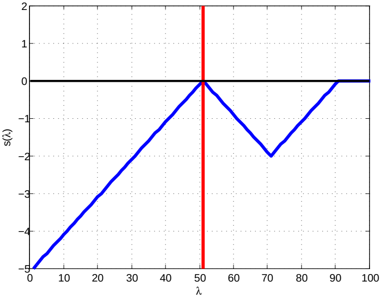

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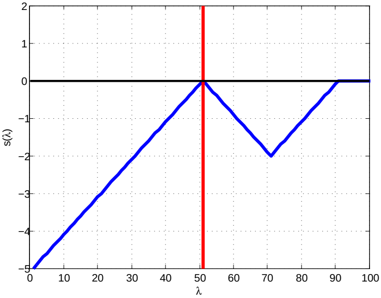

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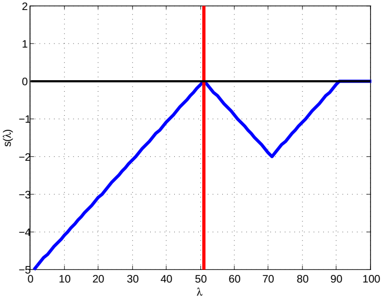

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- In the given example, $\rho(A) = 50$ (red line). The control characteristic spectra is $\Lambda = \{50\} \cup [90, 100]$, which is not the singleton $\{\rho(A)\} = \{50\}$, so the problem is non-critical;

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- More results/details in V. M. Gonçalves's thesis [V. M. Gonçalves, 2014].

References

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