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Modelling and Control of Periodic Time-Variant Event Graphs in Dioids

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Abstract Timed Event Graphs (TEGs) (max,+) linear systems. This formalism has been studied for modelling, analysis and control synthesis for decision-free timed Discrete Event Systems (DESs), for instance specific manufacturing processes or transportation networks operating under a given logical schedule. However, many applications exhibit time-variant behaviour, which cannot be modelled in a standard TEG framework. In this paper we extend the class of TEGs in order to include certain periodic time-variant behaviours. This ex-

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tended class of TEGs is called Periodic Time-variant Event Graphs (PTEGs). It is shown that the input-output behaviour of these systems can be described by means of ultimately periodic series in a dioid of formal power series. These series represent transfer functions of PTEGs and are a convenient basis for performance analysis and controller synthesis.

Keywords Dioids; controller synthesis; timed event graph; discrete-event systems; residuation; time-variant behaviour.

1 Introduction and Motivation

The class of Discrete Event Systems (DESs) studied in this paper are persistent timed DESs. A DES is called persistent if the occurrence of an event never disables another event. In other words an enabled event remains enabled until it occurs. Persistent DESs are often obtained from non-persistent ones by solving the underlying conflicts, i.e., by determining the logical order in which events can occur. For many applications, such as manufacturing systems, these logic schedules can often be computed offline. The resulting system is a persistent DES which describes the timed dynamics of the original non-persistent DES with respect to the predefined logic schedule. A well studied class of persistent timed DESs are Timed Event Graphs (TEGs), which are a subclass of timed Petri nets and suitable to describe synchronization phenomena arising in DESs. Over the last decades, TEGs have been extensively studied because they admit linear representations in particular algebraic structures called dioids (Baccelli et al (1992); Heidergott et al (2005)). Based on dioids, many concepts of standard control theory have been adapted to TEGs. In the particular dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, the input-output behaviour of TEGs can be described by transfer functions defined of a set of formal power series in two variables γ and δ (Baccelli et al (1992)) [1]. These transfer functions represent the main properties such as latency and throughput of a system in a compact form. Moreover, based on these transfer functions, several model matching control problems have been solved for TEGs. This includes state or output feedback design as well as observer design (Libeaut and Loiseau (1996); Maia et al (2003); Hardouin et al (2017, 2018)). Usually the objective of the control strategy is to modify the system behaviour such that the resulting closed-loop is bounded by the reference model. For instance, we can specify a desired throughput (resp. latency) behaviour of a production line in such a reference model. The resulting controller optimizes the production process under the "just-in-time" criterion while the specified throughput is guaranteed. Thus, materials spend the minimal required time in the production line, which leads to a reduction of internal stocks. In (Hardouin et al (2009)), software tools are presented, for evaluation and controller synthesis of TEGs based on the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Model predictive control for (max,+)-linear systems was

[1] In Bouillard and Thierry (2008) a similar approach, the so called network calculus, was presented to analyze communication networks.

studied in (Schutter and van den Boom (2001)). Moreover, in Declerck (2013) and Amari et al (2012) the control of TEGs under additional time window constraints was addressed. For these TEGs, sojourn times of tokens in some places have to respect an upper bound. The control problem is then to find an admissible trajectory such that these upper bounds are satisfied.

An important property of TEGs is that they are time-invariant. From an operator point of view, for a transfer function $H \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, we have $H\delta^1 = \delta^1 H$. Here δ represents the time-shift operator.

In this paper we study time-variant DESs, which cannot be described by ordinary TEGs. To consider time-variant behaviour is motivated by several applications. For example, time-variant behaviour can be found in transportation networks, with traffic light control or communication networks with time-division-multiplexing. A simple example in the field of manufacturing is a resource which is shared by several processes on the basis of a periodic schedule, e.g., the resource is available for process 1 at times $2n$ and for process 2 at times $1 + 2n$, with $n \in \mathbb{N}_0$.

First results for time-variant (max,+)-systems have been obtained in (David-Henriet et al (2015, 2014)). There, TEGs are extended by allowing a weaker form of synchronization, called partial synchronization (PS). PS of a transition means that the transition can only fire when it is enabled by an external signal $\mathcal{S} : \mathbb{N}_0 \to \{0, 1\}$. $\mathcal{S}$ enables the firing of the transition at times $\xi \in \mathbb{N}_0$ where $\mathcal{S}(\xi) = 1$. For instance, such a signal can represent a traffic light, and a vehicle can cross a crossroad only when the traffic light is green. In the case where such signals are predefined and ultimately periodic, it is possible to obtain a transfer function of a TEG under partial synchronization (David-Henriet et al (2015)).

Standard TEGs are also event-invariant. From an operator point of view, for a transfer function $H \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ we have $H\gamma^1 = \gamma^1 H$, where γ is the event shift operator. Another extension of standard TEGs refers to event-variant timed DESs, e.g., Lahaye et al (2008); Cottenceau et al (2014b); Cofer and Garg (1993); Brat and Garg (1998)). In (Lahaye et al (2008)), the authors introduce first in first out (FIFO) TEGs in which holding times of places change periodically based on event-sequences. Therefore, these systems can describe event-variant time behaviours. In FIFO TEGs, places must respect a FIFO behaviour, in other words tokens must not overtake each other. In (Cottenceau et al (2014b)), it is shown that the input-output behaviour of these systems can be represented as formal power series in the 3-dimensional dioid $\mathcal{E}^*[\![\delta]\!]$. The studied system class is an extension of TEGs which is called Weight-Balanced Timed Event Graph (WBTEG).

In this paper, we suggest a new approach to model time-variant behaviours. First, we introduce the class of Periodic Time-variant Event Graphs, in which the holding times of places depend on times when tokens enter the place. More precisely, the holding time $\mathcal{H}(\xi)$ of a place at time $\xi \in \mathbb{Z}$ is time-variant and immediately periodic, i.e., $\mathcal{H}(\xi + \omega) = \mathcal{H}(\xi)$. The main contribution of this paper is to show that the input-output behaviour (transfer function) of PTEGs can be described by ultimately periodic series in a new dioid denoted $\mathcal{T}^*[\![\gamma]\!]$. As

PTEGs are time-variant, implying that for a transfer function $H \in \mathcal{T}^*[\![\gamma]\!]$ of a PTEG $H\delta^1 \neq \delta^1 H$. This means, the response of a PTEG to an input trajectory varies over time. In addition to the synchronization and time delay phenomena already described by standard TEGs, PTEG can describe phenomena that can only occur during certain time windows. The operational representation of PTEGs allows us to extend methods for performance evaluation and controller synthesis for TEGs to the more general class of PTEGs. Furthermore, we elaborate the relation between the impulse response of a PTEG and its transfer behaviour. First results on the dioid $\mathcal{T}^*[\![\gamma]\!]$ were obtained in (Trunk et al (2018)).

This paper is organized as follows: Section 2 summarizes the necessary facts on TEGs and dioids. In Section 3, we present PTEGs as suitable models for some time-variant discrete event systems. In Section 4, we introduce a new periodic timing operator Δ_ω and define the dioid $\mathcal{T}^*[\![\gamma]\!]$. In Section 5, the dioid $\mathcal{T}^*[\![\gamma]\!]$ is used to model the input-output behaviour of PTEGs. Furthermore, the relation between impulse response and transfer function is investigated. Finally, Section 6 illustrates the controller design process for PTEGs.

2 Timed Event Graphs and Dioids

2.1 Timed Event Graphs

In the following, we briefly recall the necessary facts on TEGs (see, e.g., Baccelli et al (1992); Heidergott et al (2005) for a more thorough discussion). TEGs are a subclass of timed Petri nets, with $P = \{p_1, \cdots, p_n\}$ the set of places, $T = \{t_1, \cdots, t_m\}$ the set of transitions and, $A \subseteq (P \times T) \cup (T \times P)$ the set of arcs connecting places with transitions and transitions with places. p_i is an upstream place of transition t_j (and t_j is a downstream transition of place p_i), if $(p_i, t_j) \in A$. Conversely, p_i is a downstream place of transition t_j (and t_j is an upstream transition of place p_i), if $(t_j, p_i) \in A$. For TEGs, each place p_i has exactly one upstream transition and exactly one downstream transition. Note that in TEGs, each arc has weight 1. Moreover, each place p_i exhibits an initial marking $M_i^0 \in \mathbb{N}_0$ and a nonnegative holding time $\phi_i \in \mathbb{N}_0$. A transition t_j can fire if the marking in every upstream place is at least 1. If t_j fires, the marking M_i in every upstream place p_i is reduced by 1 and the marking M_o in every downstream place p_o is increased by 1. The holding time ϕ_i is the time a token must remain in place p_i before it contributes to the firing of the downstream transition of p_i. We can partition the set of transitions of a TEG into input, output and internal transitions. Input transitions are transitions without upstream places. Output transitions are transitions without downstream places, and all other transitions are called internal transitions.

Definition 1 (Earliest Functioning Rule) A TEG is operating under the earliest functioning rule if all internal and output transitions are fired as soon as they are enabled.

For the purpose of modelling a TEG, a dater function $x : \mathbb{Z} \to \mathbb{Z}_{max}$ ($\mathbb{Z}_{max} := \mathbb{Z} \cup \{\pm\infty\}$) is associated to each transition. $x(k)$ gives the time (or date) when the transition fires the $(k+1)^{st}$ time. Note that dater functions are nondecreasing (Baccelli et al (1992)), i.e. $x(k+1) \geq x(k)$. Note that we assume that time is discrete and takes values in $\mathbb{Z}_{max}$.

Example 1 Consider the TEG of Figure 1. By assigning u_1 (resp. u_2) to the input transition t_1 (resp. t_2), x_1 (resp. x_2) to internal transition t_3 (resp. t_4) and y to the output transition t_5, the behaviour of the TEG can be described by the following inequalities

$$x_1(k) \geq \max(x_2(k-2), u_1(k)+1, u_2(k-1)+3),$$
$$y(k) \geq x_2(k) \geq x_1(k)+2.$$

If the TEG operates under the earliest functioning rule, its behaviour is described by equations instead of inequalities,

$$\begin{aligned} x_1(k) &= \max(x_2(k-2), u_1(k)+1, u_2(k-1)+3), \\ y(k) &= x_2(k) = x_1(k)+2. \end{aligned} \tag{1}$$

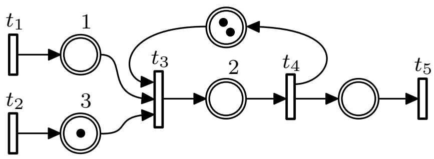


Figure 1: A simple TEG.

2.2 Dioid Theory

In this section we briefly recall some basic facts on dioids and discuss (max,+)-algebra as a specific case. Formally, a dioid is an algebraic structure that consists of a set $\mathcal{D}$ equipped with two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication). Addition is commutative, associative and idempotent (i.e. $\forall a \in \mathcal{D},\ a \oplus a = a$). The neutral element for addition (or zero element), denoted by ε, is absorbing for multiplication (i.e., $\forall a \in \mathcal{D},\ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). Multiplication is associative, distributive over addition and has a neutral element (or unit element) denoted by e. Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted. Both operations can be extended to the matrix case. For matrices $A, B \in \mathcal{D}^{m \times n}$ and $C \in \mathcal{D}^{n \times q}$, matrix addition and multiplication are defined by

$$(A \oplus B)_{ij} := A_{ij} \oplus B_{ij}, \quad (A \otimes C)_{ij} := \bigoplus_{k=1}^{n} (A_{ik} \otimes C_{kj}).$$

Moreover, the dioid structure carries over to the case of matrices, if nonsquare matrices are suitably extended by zero rows or columns (for details see Baccelli et al (1992)). A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. On a complete dioid, the Kleene star of an element $a \in \mathcal{D}$, denoted a^*, is defined by $a^* = \bigoplus_{i=0}^{\infty} a^i$ with $a^0 = \mathrm{e}$ and $a^{i+1} = a \otimes a^i$. In any dioid, there is an order naturally defined by $a \preceq b \Leftrightarrow a \oplus b = b$.

Theorem 1 (Baccelli et al (1992)) *On a complete dioid $\mathcal{D}$, $x = a^*b$ is the least (in the sense of $\preceq$) solution of the implicit equation $x = ax \oplus b$.*

A TEG can be conveniently modelled as a linear system in a particular dioid called (max,+)-algebra. The (max,+)-algebra is the set $\mathbb{Z}_{max}$ endowed with max as addition $\oplus$ and + as multiplication $\otimes$, e.g., $5 \otimes 4 \oplus 7 = \max(5+4, 7) = 9$. Moreover, the zero element is $\varepsilon = -\infty$ and the unit element is $\mathrm{e} = 0$, respectively. By convention $(\infty) \otimes (-\infty) = -\infty = \varepsilon$.

Example 2 In the (max,+)-algebra, the system (1) is expressed as

$$\begin{aligned} x_1(k) &= x_2(k-2) \oplus 1u_1(k) \oplus 3u_2(k-1), \\ y(k) &= x_2(k) = 2x_1(k). \end{aligned} \tag{2}$$

2.3 Dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

Using the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, it is straightforward to obtain transfer functions for TEGs. It was formally introduced in (Baccelli et al (1992); Gaubert and Klimann (1991)), and is based on the event-shift operator γ^ν and time-shift operator δ^τ with $\tau, \nu \in \mathbb{Z}$. These operators map dater functions to dater functions in the following way:

$$(\gamma^\nu x)(k) = x(k-\nu) \text{ and } (\delta^\tau x)(k) = x(k) + \tau. \tag{3}$$

For both operators, addition is defined as follows

$$\begin{aligned} ((\gamma^\nu \oplus \gamma^{\nu'})x)(k) &:= (\gamma^\nu x \oplus \gamma^{\nu'} x)(k) = (\gamma^\nu x)(k) \oplus (\gamma^{\nu'} x)(k), \\ ((\delta^\tau \oplus \delta^{\tau'})x)(k) &:= (\delta^\tau x \oplus \delta^{\tau'} x)(k) = (\delta^\tau x)(k) \oplus (\delta^{\tau'} x)(k). \end{aligned}$$

Furthermore, the operators γ^ν and δ^τ commute, i.e. $\gamma^\nu \delta^\tau = \delta^\tau \gamma^\nu$, and obey the following simplification rules,

$$\gamma^\nu \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')}, \quad \delta^\tau \oplus \delta^{\tau'} = \delta^{\max(\tau,\tau')}. \tag{4}$$

$\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is then the dioid of power series in γ and δ with Boolean coefficients $\tilde{\mathrm{e}}$, $\tilde{\varepsilon}$ and exponents in $\mathbb{Z}$, with a quotient structure induced by the simplification rules (4). A series $s \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is written as $s = \bigoplus_{\nu,\tau \in \mathbb{Z}} s(\nu, \tau)\gamma^\nu \delta^\tau$

with $s(\nu,\tau) \in \{\tilde{e}, \tilde{\varepsilon}\}$. Furthermore, for $s_1, s_2 \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ addition and multiplication is defined as

$$s_1 \oplus s_2 = \bigoplus_{\nu,\tau\in\mathbb{Z}} \big(s_1(\nu,\tau) \oplus s_2(\nu,\tau)\big)\gamma^\nu\delta^\tau,$$
$$s_1 \otimes s_2 = \bigoplus_{\nu,\tau\in\mathbb{Z}} \Bigg(\bigoplus_{\substack{n+n'=\nu \\ t+t'=\tau}} (s_1(n,t) \otimes s_2(n',t')) \Bigg)\gamma^\nu\delta^\tau.$$

The unit element is denoted by $\mathrm{e} = \tilde{e}\gamma^0\delta^0$ and the zero element is denoted by $\varepsilon = \bigoplus_{\nu,\tau\in\mathbb{Z}} \tilde{\varepsilon}\gamma^\nu\delta^\tau$.

Example 3 With the γ and δ operators, system (2) can be expressed by $x_1 = \gamma^2 x_2 \oplus \delta^1 u_1 \oplus \gamma^1\delta^3 u_2$, $y = x_2 = \delta^2 x_1$. Or, equivalently, with $x = \begin{bmatrix} x_1 \; x_2 \end{bmatrix}^T$, $u = \begin{bmatrix} u_1 \; u_2 \end{bmatrix}^T$, in matrix form $x = Ax \oplus Bu$; $y = Cx$, where

$$A = \begin{bmatrix} \varepsilon & \gamma^2 \\ \delta^2 & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} \delta^1 & \gamma^1\delta^3 \\ \varepsilon & \varepsilon \end{bmatrix}, \quad C = \begin{bmatrix} \varepsilon \; \mathrm{e} \end{bmatrix}.$$

Due to Theorem 1, the least solution for the output y is given by $y = Hu$, with transfer function matrix

$$H = CA^*B = \begin{bmatrix} \delta^3(\gamma^2\delta^2)^* \; \gamma^1\delta^5(\gamma^2\delta^2)^* \end{bmatrix}.$$

A dater function $u : \mathbb{Z} \to \mathbb{Z}_{max}$ can be expressed as a series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, such that

$$s_u = \bigoplus_{\{k|-\infty<u(k)<\infty\}} \gamma^k\delta^{u(k)} \oplus \bigoplus_{\{k|u(k)=\infty\}} \gamma^k\delta^*,$$

see Baccelli et al (1992); Cohen et al (1991). By expressing an input u as a series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, the least output y of a single-input and single-output (SISO) system can be obtained as the product of the transfer function h and the input series s_u, i.e., $s_y = (h \otimes s_u) \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, where s_y is the series associated to the output counter y (Cohen et al (1991)). As in conventional systems theory, there is a link between the impulse response and the transfer function of a system. An impulse is a specific dater function $\mathcal{I}$ such that:

$$\mathcal{I}(k) = \begin{cases} -\infty, & \text{for } k < 0, \\ 0, & \text{for } k \geq 0. \end{cases} \tag{5}$$

Choosing an impulse as the input of a SISO TEG means that its input transition fires infinitely often at time 0. This input can be expressed as a series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, $s = \bigoplus_{k\geq 0} \gamma^k\delta^0 = \gamma^0\delta^0 = \mathrm{e}$ and we have $h = h\mathrm{e}$, i.e., the transfer function is the impulse response in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ (Baccelli et al (1992); Cohen et al (1991)). In (Hardouin et al (2009)), software tools are introduced for the computation of rational expressions of periodic series (matrices) in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

3 Periodic Time-variant Event Graphs

In this section we discuss time-variant DES where the time behaviour changes in a periodic form. Such periodic timing phenomena occur for instance in traffic networks. As an example, let us consider a crossroad which is controlled by a traffic light. A vehicle can only cross during the green phase. If it reaches the crossing during this phase, it can immediately proceed. But if it reaches the crossing during the red phase, it has to wait for the next green phase. The vehicle is delayed by a time that depends on its time of arrival. Under the assumption that the behaviour of the traffic light is periodic, the crossroad can be modelled as a nonstandard TEG where the timing behaviour of the traffic light is described by a periodic mapping associated with a place. This periodic mapping $\mathcal{H} : \mathbb{Z}_{max} \to \mathbb{Z}_{max}$ describes the holding time of the place at each time instant $\xi \in \mathbb{Z}_{max}$. We call such a mapping holding-time-function, and it is defined as follows.

Definition 2 (holding-time-function $\mathcal{H}$) A holding-time-function $\mathcal{H} : \mathbb{Z}_{max} \to \mathbb{Z}_{max}$ is an ω-periodic function, i.e., $\exists \omega \in \mathbb{N},\ \forall \xi \in \mathbb{Z}_{max} :\ \mathcal{H}(\xi) = \mathcal{H}(\xi + \omega)$.

Hence, $\forall j \in \mathbb{Z}_{max}$

$$\mathcal{H}(\xi) = \begin{cases} \overline{n}_0 & \text{if } \ \xi = 0 + \omega j, \\ \overline{n}_1 & \text{if } \ \xi = 1 + \omega j, \\ \ \vdots & \\ \overline{n}_{\omega-1} & \text{if } \ \xi = (\omega - 1) + \omega j, \end{cases} \tag{6}$$

where for $i \in \{0, \cdots, \omega - 1\}$, $\overline{n}_i \in \mathbb{Z}$ are the holding times in each period.

The short form of a holding-time-function is defined as a string $\langle \overline{n}_0\ \overline{n}_1 \cdots \overline{n}_{\omega-1} \rangle$. The period ω is implicitly given by the number of elements in the string. For the modelling process of TEGs in the (max,+)-algebra, it is necessary that tokens must enter and leave each place in the same order (Baccelli et al (1992))[Section 2.5.2]. In other words, a place must respect a FIFO behaviour. This property leads to the following constraint on holding-time-functions

$$\forall \xi \in \mathbb{Z}_{max},\ \ \mathcal{H}(\xi + 1) + 1 \geq \mathcal{H}(\xi). \tag{7}$$

A holding-time-function which respects (7) is called FIFO holding-time-function. Moreover, a holding-time-function is called causal if all holding times are non-negative, i.e., $\forall i \in \{0, \cdots, \omega - 1\}$, $\overline{n}_i \in \mathbb{N}_0$.

Definition 3 (Periodic Time-variant Event Graph) A PTEG is a TEG where the holding times of places are given by causal FIFO holding-time-functions.

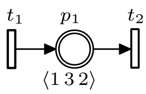
Example 4 Consider the PTEG in Figure 2a where the holding time of p_1 is changing according to, $\forall j \in \mathbb{Z}_{max}$

$$\mathcal{H}_1(\xi) = \langle 0\ 0\ 2\ 1 \rangle = \begin{cases} 0 & \text{if } \ \xi = 0 + 4j, \\ 0 & \text{if } \ \xi = 1 + 4j, \\ 2 & \text{if } \ \xi = 2 + 4j, \\ 1 & \text{if } \ \xi = 3 + 4j. \end{cases}$$

The holding time is such that tokens enter and leave place p_1 in the same order, hence the function satisfies (7). In contrast, let us consider the TEG in Figure 2b, where the holding time of place p_2 is changing according to $\mathcal{H}_2(\xi) = \langle 3\ 0\ 2\ 1 \rangle$. In this case tokens which enter the place p_2 at time instant $\xi = 0$ enable the firing of transition t_4 at time instant $0 + \mathcal{H}_2(0) = 3$. Tokens which enter the place p_2 at time instant $\xi = 1$ immediately enable the firing of t_4, since $\mathcal{H}_2(1) = 0$. The function $\mathcal{H}_2$ violates the FIFO condition of p_2, and therefore the TEG in Figure 2b is not in the class of PTEGs.

Figure 2: In (a) $\mathcal{H}_1 = \langle 0\ 0\ 2\ 1 \rangle$ satisfies the FIFO condition. In (b) $\mathcal{H}_2 = \langle 3\ 0\ 2\ 1 \rangle$ violates the FIFO condition.

Example 5 Consider the following simple PTEG. By associating a dater func-



tion x_1 with transition t_1 and a dater function x_2 with transition t_2, the behaviour of this PTEG is described by

$$x_2(k) \geq \left\lceil \frac{x_1(k)}{3} \right\rceil 3 + 1, \tag{8}$$

where $\lceil a \lceil$ is the smallest integer greater than or equal to a. In standard TEGs, the effect of constant holding times τ are expressed by inequalities of the form $x_2(k) \geq x_1(k) + \tau$. This corresponds to a specific PTEG with $\mathcal{H}_1 =< \tau >$. Hence, PTEG can describe a broader class of behaviours. Moreover, when considering equality for (8), i.e., the earliest functioning of the system, it is easy to see that this cannot be written as a (max,+)-linear equation. In contrast standard TEGs, with constant holding times, have a linear representation in the (max,+)-algebra, e.g., see Example 2.

Definition 4 (Release-time-function $\mathcal{R}$) A release-time-function $\mathcal{R} : \mathbb{Z}_{max} \to \mathbb{Z}_{max}$ is defined as $\mathcal{R}(\xi) = \mathcal{H}(\xi)+\xi$, where $\mathcal{H}(\xi)$ is a FIFO holding-time-function. A release-time-function is called causal if $\mathcal{R}(\xi) \geq \xi,\ \forall \xi \in \mathbb{Z}_{max}$.

As $\mathcal{H}(\xi+1)+1 \geq \mathcal{H}(\xi)$, it follows that $\mathcal{R}(\xi+1) = \mathcal{H}(\xi+1)+\xi+1 \geq \mathcal{H}(\xi)+\xi = \mathcal{R}(\xi)$, i.e. $\mathcal{R}$ is nondecreasing. The release-time-function can be seen as an alternative representation of the time-variant behaviour of a place in a PTEG. This function describes the time when a token in a place is available to contribute to the firing of the downstream transition of the place. The argument of this function is the time ξ when the token enters the place and its value is the time when the token is available to leave the place. By defining $n_i = \bar{n}_i + i$, we can express a release-time-function as, $\forall j \in \mathbb{Z}_{max}$

$$\mathcal{R}(\xi) = \mathcal{H}(\xi) + \xi = \begin{cases} n_0 + \omega j & \text{if } \ \xi = 0 + \omega j, \\ n_1 + \omega j & \text{if } \ \xi = 1 + \omega j, \\ \ \vdots & \\ n_{\omega-1} + \omega j & \text{if } \ \xi = (\omega - 1) + \omega j. \end{cases} \tag{9}$$

Clearly, nonnegative holding-times $\overline{n}_i$ (causal holding-time-functions) lead to causality of $\mathcal{R}$.

Example 6 (PTEG) Figure 3 shows a PTEG with holding-time-functions of places p_1, p_2, p_3 given by

$$\mathcal{H}_1(\xi) = \langle 0\ 0\ 2\ 1\rangle,\ \mathcal{H}_2(\xi) = \langle 1\rangle,\ \mathcal{H}_3(\xi) = \langle 1\ 3\ 2\ 1\rangle.$$

The corresponding release-time-functions are, $\forall j \in \mathbb{Z}_{max}$,

$$\begin{aligned} \mathcal{R}_1(\xi) &= \begin{cases} 0+4j & \text{if } \ \xi = 0+4j, \\ 1+4j & \text{if } \ \xi = 1+4j, \\ 4+4j & \text{if } \ \xi = 2+4j, \\ 4+4j & \text{if } \ \xi = 3+4j, \end{cases} \\ \mathcal{R}_2(\xi) &= 1+\xi, \\ \mathcal{R}_3(\xi) &= \begin{cases} 1+4j & \text{if } \ \xi = 0+4j, \\ 4+4j & \text{if } \ \xi = 1+4j, \\ 4+4j & \text{if } \ \xi = 2+4j, \\ 4+4j & \text{if } \ \xi = 3+4j. \end{cases} \end{aligned}$$

In this example, place p_2 has a constant holding time of 1 time unit, whereas the holding times of places p_1 and p_3 are changing periodically with period

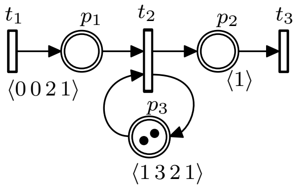
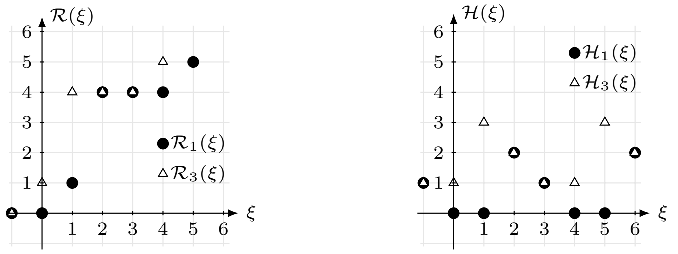


Figure 3: PTEG with holding-time-functions of places p_1, p_2, p_3 expressed in the short form at each place.



(a) Release-time-functions

(b) Holding-time-functions

Figure 4: Release-time-function $\mathcal{R}_1, \mathcal{R}_3$ and holding-time-functions $\mathcal{H}_1, \mathcal{H}_3$ of places p_1, p_3.

4. $\mathcal{R}_1, \mathcal{R}_3$, respectively $\mathcal{H}_1, \mathcal{H}_3$, are illustrated in Figure 4a, respectively, Figure 4b. The place p_1 can be interpreted as the model of a traffic light which is green for time instants $\{0, 1, 4, 5, \cdots\}$ and red for time instants $\{2, 3, 6, 7, \cdots\}$. Therefore, if a car arrives at times $2, 6, \cdots$ it has to wait for 2 time instants, if it arrives at times $3, 7, \cdots$, it has to wait for 1 time instant.

4 Introduction of Timing Operators

As in Baccelli et al (1992), where TEGs are described by rational compositions of operators, we introduce a family of specific timing operators to handle time variation. Similar to TEGs, for the modelling process of PTEGs, a dater function $x_i : \mathbb{Z} \to \mathbb{Z}_{max}$ is associated to each transition t_i. Recall that $x_i(k)$ gives the date when the transition fires the $(k+1)^{st}$ time and that dater functions are nondecreasing functions, i.e., $x_i(k+1) \geq x_i(k)$. The set of dater functions is denoted by Σ, and on Σ addition, $\oplus$, and multiplication by constants, $\otimes$, are defined as follows:

$$x, y \in \Sigma,\ (x \oplus y)(k) := \max(x(k), y(k)),$$
$$\lambda \in \mathbb{Z}_{max},\ (\lambda \otimes x)(k) := \lambda + x(k).$$

The $\oplus$ operation induces an order relation on Σ, i.e., $\forall x, y \in \Sigma,\ x \preceq y \Leftrightarrow x \oplus y = y$. An operator $\rho : \Sigma \to \Sigma$ is linear if (a) $\forall x, y \in \Sigma : \rho(x \oplus y) = \rho(x) \oplus \rho(y)$ and (b) $\lambda \otimes \rho(x) = \rho(\lambda \otimes x)$. An operator is additive if (a) is satisfied.

Definition 5 (Cottenceau et al (2014a)) The set of additive operators on Σ is denoted $\mathcal{O}$. On the set $\mathcal{O}$, addition and multiplication is defined as follows: $x \in \Sigma, \forall \rho_1, \rho_2 \in \mathcal{O}$,

$$(\rho_1 \oplus \rho_2)(x) = \rho_1(x) \oplus \rho_2(x),\ (\rho_1 \otimes \rho_2)(x) = \rho_1(\rho_2(x)).$$

Multiplication is not commutative, and the set $\mathcal{O}$ equipped with $\otimes$ and $\oplus$ is a noncommutative complete dioid. The identity operator (unit element) is denoted by $\mathrm{e} : \forall x \in \Sigma,\ (\mathrm{e}(x))(k) = x(k)$, and the zero operator (zero element) is denoted by $\varepsilon : \forall x \in \Sigma,\ (\varepsilon(x))(k) = -\infty$. To simplify notation, we usually write ρx instead of $\rho(x)$.

Definition 6 (Basic operators in PTEGs) Dynamic phenomena arising in PTEGs can be described by the following basic additive operators in $\mathcal{O}$:

$$\varsigma \in \mathbb{Z},\ \delta^{\varsigma} : \forall x \in \Sigma,\ (\delta^{\varsigma} x)(k) = x(k) + \varsigma, \tag{10}$$

$$\nu \in \mathbb{Z},\ \gamma^{\nu} : \forall x \in \Sigma,\ (\gamma^{\nu} x)(k) = x(k - \nu), \tag{11}$$

$$\omega \in \mathbb{N},\ \Delta_{\omega} : \forall x \in \Sigma,\ (\Delta_{\omega} x)(k) = \lceil x(k)/\omega \rceil \omega, \tag{12}$$

where $\lceil a \rceil$ is the smallest integer greater than or equal to a.

The identity operator can be expressed as: $\mathrm{e} = \gamma^0 = \delta^0 = \Delta_1$. In particular, the Δ_{ω} operator models a time-variant delay behaviour. For example, consider transitions t_1 and t_2 with associated dater functions x_1 and x_2. Then, $x_2 = \Delta_4 x_1$ implies $x_2(k) = \lceil x_1(k)/4 \rceil 4, \forall k \in \mathbb{Z}$. Hence, if the $(k+1)^{st}$ firing of t_1 is at time instant $x_1(k) = 5$, the $(k+1)^{st}$ firing of t_2 is at $x_2(k) = 8$, and the delay is 3. If the $(k+1)^{st}$ firing t_1 is at time instant $x_1(k) = 8$, the $(k+1)^{st}$ firing time of t_2 is at $x_2(k) = 8$, and the delay is 0. Clearly, this operator is nonlinear as $\Delta_4(\lambda \otimes x) \neq \lambda \otimes \Delta_4(x)$. E.g., for $\lambda = 1$ and $x(k) = 1$ $(\Delta_4(\lambda \otimes x))(k) = \lceil(\lambda + x(k))/4\rceil 4 = 4 \neq \lambda + \lceil x(k)/4 \rceil 4 = 5$.

Proposition 1 *The basic operators* (10) - (12) *satisfy the following relations*

$$\delta^{\varsigma}\delta^{\varsigma'} = \delta^{\varsigma+\varsigma'}, \qquad \gamma^{\nu}\gamma^{\nu'} = \gamma^{\nu+\nu'}, \tag{13}$$

$$\delta^{\varsigma} \oplus \delta^{\varsigma'} = \delta^{\max(\varsigma,\varsigma')}, \qquad \gamma^{\nu} \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')}, \tag{14}$$

$$\delta^1\gamma^1 = \gamma^1\delta^1, \qquad \Delta_{\omega}\gamma^1 = \gamma^1\Delta_{\omega}, \tag{15}$$

$$\Delta_{\omega}\delta^{\varsigma} = \delta^{\lceil \frac{\varsigma}{\omega} \rceil \omega}\Delta_{\omega}\delta^{\varsigma - \lceil \frac{\varsigma}{\omega} \rceil \omega}, \qquad \delta^{\varsigma}\Delta_{\omega} = \delta^{\varsigma - \lceil \frac{\varsigma}{\omega} \rceil \omega}\Delta_{\omega}\delta^{\lceil \frac{\varsigma}{\omega} \rceil \omega}, \tag{16}$$

$$\Delta_{\omega}\delta^{\varsigma}\Delta_{\omega} = \delta^{\lceil \frac{\varsigma}{\omega} \rceil \omega}\Delta_{\omega}. \tag{17}$$

Proof See (Baccelli et al (1992)) for (13), (14), (15) and Appendix C.1 for (16), (17).

4.1 A dioid of time operators

Definition 7 (Dioid of T-operators $\mathcal{T}$) We denote by $\mathcal{T}$ the dioid of operators obtained by addition and composition of operators in $\{\varepsilon, \mathrm{e}, \delta^{\varsigma}, \Delta_{\omega}\}$, with $\varsigma \in \mathbb{Z}$, and $\omega \in \mathbb{N}$. The elements of $\mathcal{T}$ are called T-operators (T is for time).

For example, $\delta^3 \Delta_4 \delta^1 \oplus \delta^2 \Delta_3 \in \mathcal{T}$. Note that the operator γ^ν is not in $\mathcal{T}$. A T-operator v describes the input-output delay occurring in a system and can be represented by a release-time-function $\mathcal{R}_v$. The release-time-function associated with v is obtained by replacing $x(k)$ by ξ in the expression of $v(x)(k)$, e.g., $((\delta^3\Delta_4\delta^1 \oplus \delta^2\Delta_3)x)(k) = \max(3+\lceil (x(k)+1)/4\rceil 4, 2+\lceil x(k)/3\rceil 3)$ and therefore $\mathcal{R}_{\delta^3\Delta_4\delta^1\oplus\delta^2\Delta_3}(\xi) = \max(3 + \lceil(\xi + 1)/4\rceil 4, 2 + \lceil\xi/3\rceil 3)$. A T-operator v is said to be causal if its corresponding release-time-function $\mathcal{R}_v$ is causal. Then $\mathcal{R}_v$ can be realized as a causal holding-time-function associated with a place in a PTEG. Furthermore we define periodicity for a T-operator as follows.

Definition 8 A T-operator $v \in \mathcal{T}$ is called ω-periodic if its corresponding release-time-function $\mathcal{R}_v$ satisfies, $\forall \xi \in \mathbb{Z}_{max}$,

$$\mathcal{R}_v(\xi + \omega) = \omega + \mathcal{R}_v(\xi).$$

For instance, the Δ_4 operator is 4-periodic and the $\delta^2\Delta_3$ operator is 3-periodic, but of course $\delta^2\Delta_3$ is not 4-periodic. In general all operators $v \in \mathcal{T}$ are ω-periodic for some ω. There is an isomorphism between the set of T-operators and the set of release-time-functions. The order relation over the dioid $\mathcal{T}$ corresponds to the order induced by the max operation on the release-time-functions. For $v_1, v_2 \in \mathcal{T}$,

$$\begin{aligned} & v_1 \succeq v_2 \Leftrightarrow v_1 \oplus v_2 = v_1 \Leftrightarrow v_1 x \oplus v_2 x = v_1 x \quad \forall x \in \Sigma, \\ & \Leftrightarrow (v_1 x)(k) \oplus (v_2 x)(k) = (v_1 x)(k) \quad \forall x \in \Sigma,\ \forall k \in \mathbb{Z}, \\ & \Leftrightarrow \mathcal{R}_{v_1}(\xi) \geq \mathcal{R}_{v_2}(\xi) \quad \forall \xi \in \mathbb{Z}_{max}. \end{aligned} \tag{18}$$

Clearly, $\forall \xi \in \mathbb{Z}_{max},\ \mathcal{R}_v(\xi) \geq \mathcal{R}_v(\xi) - 1 = \mathcal{R}_{\delta^{-1}v}(\xi)$, furthermore nondecreasingness of $\mathcal{R}_v$ implies that: $\forall \xi \in \mathbb{Z}_{max},\ \mathcal{R}_v(\xi) \geq \mathcal{R}_v(\xi - 1) = \mathcal{R}_{v\delta^{-1}}(\xi)$, therefore $v \succeq \delta^{-1}v$ and $v \succeq v\delta^{-1}$. This leads to the following equalities for $v \in \mathcal{T}$,

$$v = v(\delta^{-1})^* = (\delta^{-1})^* v. \tag{19}$$

A simple element in $\mathcal{T}$ is defined as: $\delta^\varsigma \Delta_\omega \delta^{\varsigma'}$. A polynomial in $\mathcal{T}$ is a finite sum of simple elements, i.e. $\bigoplus_{i=0}^{I} \delta^{\varsigma_i} \Delta_{\omega_i} \delta^{\varsigma'_i}$. A simple element $\delta^\varsigma \Delta_\omega \delta^{\varsigma'}$ corresponds to a release-time-function: $\mathcal{R}(\xi) = \varsigma + \lceil(\xi + \varsigma')/\omega\rceil\omega$. Figure 5a illustrates the release-time-function $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}$ of simple element $\delta^2\Delta_4\delta^{-1}$. Because of (18), the shaded area corresponds to the domain of T-operators less than or equal to $\delta^2\Delta_4\delta^{-1}$. Consider now the release-time-function $\mathcal{R}_{\delta^1\Delta_4\delta^{-2}}$ associated with the operator $\delta^1\Delta_4\delta^{-2}$. $\mathcal{R}_{\delta^1\Delta_4\delta^{-2}}$ is completely covered by $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}$ ($\mathcal{R}_{\delta^1\Delta_4\delta^{-2}}$ is beneath "in the shade of" $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}$) and therefore $\delta^1\Delta_4\delta^{-2} \preceq \delta^2\Delta_4\delta^{-1}$. However, two operators can also be incomparable, e.g., $\delta^{-3}\Delta_4\delta^0 \not\preceq \delta^0\Delta_4\delta^{-1}$ and $\delta^{-3}\Delta_4\delta^0 \not\succeq \delta^0\Delta_4\delta^{-1}$. Therefore we cannot simplify the expression $\delta^{-3}\Delta_4\delta^0 \oplus \delta^0\Delta_4\delta^{-1}$, see Figure 5b. Note that the representation of a simple element is not unique since, because of (16), δ^ω commutes with the Δ_ω operator, i.e., $\delta^\varsigma\Delta_\omega\delta^{\varsigma'} = \delta^{\varsigma+\omega}\Delta_\omega\delta^{\varsigma'-\omega}$. To simplify calculations we define a canonical form for simple elements. A simple element $\delta^\varsigma\Delta_\omega\delta^{\varsigma'}$

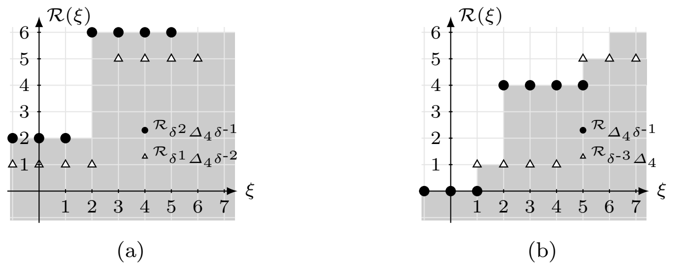


Figure 5: (a) $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}(\xi) > \mathcal{R}_{\delta^1\Delta_4\delta^{-2}}(\xi)\ \forall\xi$, i.e., $\delta^2\Delta_4\delta^{-1} \succeq \delta^1\Delta_4\delta^{-2}$. (b) $\delta^{-3}\Delta_4\delta^0$ and $\delta^0\Delta_4\delta^{-1}$ are not comparable.

can always be written in canonical form such that $-\omega < \varsigma' \leq 0$. This follows from (16). We choose this particular form, since, for $-\omega < \varsigma' \leq 0$, $\mathcal{R}_{\delta^{\varsigma}\Delta_\omega\delta^{\varsigma'}}(0) = \varsigma + \left\lceil \frac{0+\varsigma'}{\omega} \right\rceil \omega = \varsigma$. As, in general $\mathcal{R}_{\delta^{\varsigma}\Delta_\omega\delta^{\varsigma'}}(\xi) = \varsigma + i\omega$ for $-\varsigma' + (i-1)\omega < \xi \leq -\varsigma' + i\omega$, the ordering of two simple elements $\delta^{\varsigma_1}\Delta_\omega\delta^{\varsigma_1'}$ and $\delta^{\varsigma_2}\Delta_\omega\delta^{\varsigma_2'}$ in canonical form can be checked by

$$\delta^{\varsigma_1}\Delta_\omega\delta^{\varsigma_1'} \succeq \delta^{\varsigma_2}\Delta_\omega\delta^{\varsigma_2'} \Leftrightarrow \begin{cases} \varsigma_1 \geq \varsigma_2 \text{ and } \varsigma_1' \geq \varsigma_2', \\ \text{or } \varsigma_1 - \omega \geq \varsigma_2. \end{cases} \tag{20}$$

Proposition 2 *A release-time-function $\mathcal{R}(\xi)$, as given in (9), can be expressed by an operator $p \in \mathcal{T}$ in the following form:*

$$\begin{aligned} p =& \delta^{n_0}\Delta_\omega\delta^{1-\omega} \oplus \delta^{n_1-\omega}\Delta_\omega \oplus \delta^{n_2-\omega}\Delta_\omega\delta^{-1} \oplus \ldots \oplus \delta^{n_{\omega-1}-\omega}\Delta_\omega\delta^{2-\omega}. \\ =& \delta^{n_0}\Delta_\omega\delta^{1-\omega} \oplus \bigoplus_{i=1}^{\omega-1} \delta^{n_i-\omega}\Delta_\omega\delta^{1-i} \end{aligned} \tag{21}$$

Proof See Appendix C.2.

Corollary 1 *Since $\mathcal{H}(\xi) = \mathcal{R}(\xi) - \xi$, the T-operator associated with a holding-time-function $\langle \overline{n}_0\, \overline{n}_1 \cdots \overline{n}_{\omega-1} \rangle$ can be obtained by*

$$p = \delta^{\overline{n}_0}\Delta_\omega\delta^{1-\omega} \oplus \bigoplus_{i=1}^{\omega-1} \delta^{\overline{n}_i+(i-\omega)}\Delta_\omega\delta^{1-i}.$$

Proof This follows immediately from $n_i = \bar{n}_i + i$.

Recall that the operator $p \in \mathcal{T}$ associated with a causal release-time-function $\mathcal{R}(\xi)$ is causal.

Example 7 Consider $\mathcal{H}_1(\xi) = \langle 0\,0\,2\,1 \rangle$ given in Example 6. This holding-time-function corresponds to an operator given by

$$\begin{aligned} p =& \delta^0\Delta_4\delta^{-3} \oplus \delta^{-3}\Delta_4\delta^0 \oplus \delta^0\Delta_4\delta^{-1} \oplus \delta^0\Delta_4\delta^{-2}, \\ =& \delta^{-3}\Delta_4\delta^0 \oplus \delta^0\Delta_4\delta^{-1} \oplus \delta^0\Delta_4\delta^{-2} \oplus \delta^0\Delta_4\delta^{-3}, \\ =& \delta^{-3}\Delta_4 \oplus \Delta_4(\delta^{-1} \oplus \delta^{-2} \oplus \delta^{-3}) = \delta^{-3}\Delta_4 \oplus \Delta_4\delta^{-1}, \end{aligned}$$

because of (14): $\delta^{-1} \oplus \delta^{-2} \oplus \delta^{-3} = \delta^{-1}$. Respectively, $\mathcal{H}_3(\xi) = \langle 1\,3\,2\,1 \rangle$ corresponds to the operator $\Delta_4 \oplus \delta^1 \Delta_4 \delta^{-3}$.

Proposition 3 (Canonical form of an ω-periodic T-operator) *An ω-periodic operator $v \in \mathcal{T}$ has a canonical form given by a finite sum: $v = \bigoplus_{i=1}^{I} \delta^{\varsigma_i} \Delta_\omega \delta^{\varsigma_i'}$ of canonical simple elements where I is minimal. Furthermore, the simple elements are strictly ordered such that $\forall i \in \{1, \cdots, I-1\}$, $\varsigma_i < \varsigma_{i+1}$.*

Proof Recall the isomorphism between T-operators and release-time-functions. Because of Proposition 2 the release-time-function $\mathcal{R}_v$ of operator $v \in \mathcal{T}$ can be represented by a finite sum of simple elements in $\mathcal{T}$. The canonical expression can then be obtained by removing dominated elements according to the order relation in (20).

Remark 1 For a canonical T-operator $(\bigoplus_{i=1}^{I} \delta^{\varsigma_i} \Delta_\omega \delta^{\varsigma_i'})$, $I \leq \omega$.

Remark 2 Note that in the canonical form of v, every basic Δ operator has the same period ω and therefore $v\delta^\omega = \delta^\omega v$.

Remark 3 Clearly an ω-periodic T-operator is also $n\omega$-periodic, with $n \in \mathbb{N}$. Thus an ω-periodic T-operator v can be represented as an $n\omega$-periodic T-operator. This form can be obtained by expressing the release-time-function $\mathcal{R}_v$ of v with a multiple period and then applying Proposition 2.

4.2 Dioid $\mathcal{T}^*[\![\gamma]\!]$

Since the γ operator commutes with all T-operators, see (15), we can define a dioid of formal power series in the variable γ with coefficients in $\mathcal{T}$ and exponents in $\mathbb{Z}$. All elements of this dioid can be written as $\bigoplus_i v_i \gamma^i$, with $v_i \in \mathcal{T}$.

Definition 9 (Dioid $\mathcal{T}^*[\![\gamma]\!]$) We denote by $\mathcal{T}^*[\![\gamma]\!]$ the quotient dioid in the set of formal power series in one variable γ with exponents in $\mathbb{Z}$ and coefficients in the noncommutative complete dioid $\mathcal{T}$ induced by the equivalence relation, $\forall s \in \mathcal{T}^*[\![\gamma]\!]$,

$$s = (\gamma^1)^* s = s(\gamma^1)^*. \tag{22}$$

A monomial in $\mathcal{T}^*[\![\gamma]\!]$ is defined by $v\gamma^\nu$, where $v \in \mathcal{T}$. A polynomial is a finite sum of monomials, i.e., $\bigoplus_i v_i \gamma_i^\nu$. Moreover, we call a monomial in $\mathcal{T}^*[\![\gamma]\!]$ simple, if it can be written as $\delta^\varsigma \Delta_\omega \delta^{\varsigma'} \gamma^\nu$, i.e., if v is a simple element in $\mathcal{T}$. A series $s \in \mathcal{T}^*[\![\gamma]\!]$ can be written as $s = \bigoplus_{\nu \in \mathbb{Z}} s(\nu)\gamma^\nu$, where $s(\nu) \in \mathcal{T}$.

Definition 10 Let $s_1, s_2 \in \mathcal{T}^*[\![\gamma]\!]$, then addition and multiplication are defined by

$$s_1 \oplus s_2 = \bigoplus_{\nu \in \mathbb{Z}} \big(s_1(\nu) \oplus s_2(\nu)\big)\gamma^\nu,$$

$$s_1 \otimes s_2 = \bigoplus_{\nu \in \mathbb{Z}} \left(\bigoplus_{n+n'=\nu} (s_1(n) \otimes s_2(n')) \right) \gamma^\nu.$$

As before, $\oplus$ defines an order on $\mathcal{T}^*[\![\gamma]\!]$, i.e., for $a, b \in \mathcal{T}^*[\![\gamma]\!]$, $a \oplus b = b \Leftrightarrow a \preceq b$. The quotient structure in $\mathcal{T}^*[\![\gamma]\!]$, given by (22), is interpreted as a simplification rule on $\mathcal{T}^*[\![\gamma]\!]$. Given two monomials $m_1 = v_1\gamma^{\nu_1}$, $m_2 = v_2\gamma^{\nu_2}$ with $v_1, v_2 \in \mathcal{T}$ then $m_1 \succeq m_2$, iff $v_1 \succeq v_2$ and $\nu_1 \leq \nu_2$. Consider, for example, the polynomial $\delta^2\Delta_4\delta^{-1}\gamma^1 \oplus \delta^1\Delta_4\delta^{-3}\gamma^7$. Because of (20), $\delta^2\Delta_4\delta^{-1} \succeq \delta^1\Delta_4\delta^{-3}$ (in the dioid $\mathcal{T}$), the second monomial, $\delta^1\Delta_4\delta^{-3}\gamma^7$, is dominated by $\delta^2\Delta_4\delta^{-1}\gamma^1$, therefore $\delta^2\Delta_4\delta^{-1}\gamma^1 \oplus \delta^1\Delta_4\delta^{-3}\gamma^7 = \delta^2\Delta_4\delta^{-1}\gamma^1$.

A series $s = \bigoplus_i v_i\gamma^{\nu_i} \in \mathcal{T}^*[\![\gamma]\!]$ has a graphical representation in $\mathbb{Z}^2_{max} \times \mathbb{Z}$. For every exponent $\nu_i \in \mathbb{Z}$ the coefficient v_i is represented by its release-time-function $\mathcal{R}_{v_i}$ in the (input-time $\times$ output-time) plane. For instance, recalling that the release-time-function $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}$ of $\delta^2\Delta_4\delta^{-1}$ is given in Figure 5a, Figure 6a illustrates the graphical representation of $\delta^2\Delta_4\delta^{-1}\gamma^1 = \delta^2\Delta_4\delta^{-1}\gamma^1 \oplus \delta^2\Delta_4\delta^{-1}\gamma^2 \oplus \delta^2\Delta_4\delta^{-1}\gamma^3 \oplus \cdots$. For every event-shift value $k \geq 1$ the (input-time $\times$ output-time) plane in Figure 6a shows the release-time-function $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}$. Figure 6b shows the graphical representation of

$$\begin{aligned}\delta^2\Delta_4\delta^{-1}\gamma^1 \oplus \delta^3\Delta_4\delta^{-2}\gamma^4 &= (\delta^2\Delta_4\delta^{-1})\gamma^1\gamma^* \oplus (\delta^3\Delta_4\delta^{-2})\gamma^4\gamma^* \\ &= \bigoplus_i v_i\gamma^i,\end{aligned}$$

with $v_i = \delta^2\Delta_4\delta^{-1}$, for $i = 1, 2, 3$ and $v_i = \delta^2\Delta_4\delta^{-1} \oplus \delta^3\Delta_4\delta^{-2}$ for $i \geq 4$. Here for the event shift values $k = 1, 2, 3$ the release-time-function $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}}$ is depicted in the (input-time $\times$ output-time) plane and respectively for event shift values $k > 3$ the release-time-function $\mathcal{R}_{\delta^2\Delta_4\delta^{-1}\oplus\delta^3\Delta_4\delta^{-2}}$.

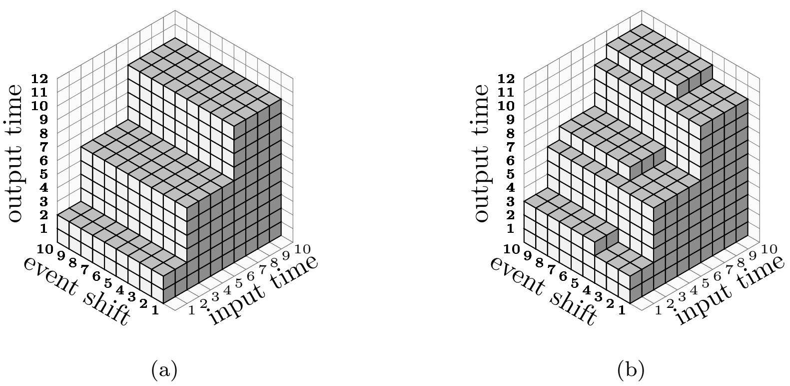


Figure 6: (a) graphical representation of $\delta^2\Delta_4\delta^{-1}\gamma^1$ and (b) graphical representation of $\delta^2\Delta_4\delta^{-1}\gamma^1 \oplus \delta^3\Delta_4\delta^{-2}\gamma^4$. To improve the readability the 3D representations have been truncated to positive values.

Remark 4 Let us note that, due to Proposition 3 and Remark 3 a polynomial $p = \bigoplus_{i=1}^{I} v_i \gamma^{n_i} \in \mathcal{T}^*[\![\gamma]\!]$ can always be represented as

$$p = \bigoplus_{i=1}^{I} \left(\bigoplus_{j=1}^{J_i} \delta^{\varsigma_{ij}} \Delta_\omega \delta^{\varsigma'_{ij}} \right) \gamma^{n_i}. \tag{23}$$

In this form, all monomials of the polynomial p have the same period ω, $J_i \leq \omega$.

Definition 11 *(Ultimately Periodic Series in $\mathcal{T}^*[\![\gamma]\!]$):* A series $s \in \mathcal{T}^*[\![\gamma]\!]$ is said to be ultimately periodic if it can be written as $s = p \oplus q(\gamma^\nu \delta^\tau)^*$, where $\nu, \tau \in \mathbb{N}_0$ and p, q are polynomials in $\mathcal{T}^*[\![\gamma]\!]$. Moreover, the asymptotic slope of s is defined by $\sigma(s) := \nu/\tau$.

Remark 5 Note that a polynomial $p = \bigoplus_{i=1}^{I} v_i \gamma^{n_i}$ can be considered as a specific ultimately periodic series $s = \varepsilon \oplus p(\gamma^0 \delta^0)^*$ where $\nu = 0$ and $\tau = 0$.

Proposition 4 *An ultimately periodic series $s \in \mathcal{T}^*[\![\gamma]\!]$, $s = p \oplus q'(\gamma^{\nu'} \delta^{\tau'})^*$ has a specific form $s = p \oplus q(\gamma^\nu \delta^\tau)^*$ in which $(\gamma^\nu \delta^\tau)^*$ commutes with the polynomial q, i.e., $s = p \oplus q(\gamma^\nu \delta^\tau)^* = p \oplus (\gamma^\nu \delta^\tau)^* q$. We call this form commute form.*

Proof For $s = p \oplus q'(\gamma^{\nu'} \delta^{\tau'})^*$, the polynomial q' can be represented with a common period ω, see (23). Then we can choose τ such that it is a multiple of ω, i.e., $\tau = l\tau' = lcm(\tau', \omega)$, thus the monomial $\delta^\tau \gamma^\nu$ commutes with q. Now we rewrite $(\gamma^{\nu'} \delta^{\tau'})^*$ as $\bar{q}(\gamma^\nu \delta^\tau)^* = (\mathrm{e} \oplus \gamma^\nu \delta^\tau \oplus \cdots \oplus \gamma^{(l-1)\nu} \delta^{(l-1)\tau})(\gamma^\nu \delta^\tau)^*$. Finally $q = \bar{q} \otimes q'$.

4.3 Operations in the Dioid $\mathcal{T}^*[\![\gamma]\!]$

When we want to compute the transfer function of a given PTEG, we have to perform addition, multiplication and the Kleene star operation on series $s \in \mathcal{T}^*[\![\gamma]\!]$. We investigate these calculations in this section. The product of two simple monomials in $\mathcal{T}^*[\![\gamma]\!]$ with the same period ω is a simple monomial in $\mathcal{T}^*[\![\gamma]\!]$. Because of (17),

$$\delta^{\varsigma_1} \Delta_\omega \delta^{\varsigma'_1} \gamma^{\nu_1} \otimes \delta^{\varsigma_2} \Delta_\omega \delta^{\varsigma'_2} \gamma^{\nu_2} = \delta^{\varsigma_1 + \lceil (\varsigma'_1 + \varsigma_2)/\omega \rceil \omega} \Delta_\omega \delta^{\varsigma'_2} \gamma^{\nu_1 + \nu_2}.$$

The Kleene star of a simple monomial $m = \delta^\varsigma \Delta_\omega \delta^{\varsigma'} \gamma^\nu$ is an ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$ and can be obtained by

$$\begin{aligned} m^* &= \mathrm{e} \oplus \delta^\varsigma \Delta_\omega \delta^{\varsigma'} \gamma^\nu \oplus \delta^\varsigma \Delta_\omega \delta^{\varsigma'} \gamma^\nu \delta^\varsigma \Delta_\omega \delta^{\varsigma'} \gamma^\nu \oplus \cdots \\ &= \mathrm{e} \oplus \left(\delta^{\lceil (\varsigma + \varsigma')/\omega \rceil \omega} \gamma^\nu \right)^* \delta^\varsigma \Delta_\omega \delta^{\varsigma'} \gamma^\nu. \end{aligned} \tag{24}$$

Hence, the Kleene star of a simple monomial in $\mathcal{T}^*[\![\gamma]\!]$ can be calculated based on the Kleene star of a monomial in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, as $\delta^{\lceil(\varsigma+\varsigma')/\omega\rceil\omega}\gamma^{\nu}$ is a monomial in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Clearly $\lceil(\varsigma+\varsigma')/\omega\rceil\omega$ is a multiple of ω, therefore

$$m^* = \mathrm{e} \oplus \delta^{\varsigma}\Delta_{\omega}\delta^{\varsigma'}\gamma^{\nu}\left(\delta^{\lceil(\varsigma+\varsigma')/\omega\rceil\omega}\gamma^{\nu}\right)^* .$$

In the following we extend the basic operations ($\oplus$, $\otimes$ and *) for simple monomials to polynomials and ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$. The sum of polynomial $p_1 \in \mathcal{T}^*[\![\gamma]\!]$ with period ω_1 and $p_2 \in \mathcal{T}^*[\![\gamma]\!]$ with period ω_2 can be obtained by expressing both polynomials with common period $\omega = lcm(\omega_1, \omega_2)$ see Remark 4. Then,

$$p_1 \oplus p_2 = \bigoplus_{i=1}^{I}\left(\bigoplus_{j=1}^{J_i}\delta^{\varsigma_{i_j}}\Delta_{\omega}\delta^{\varsigma'_{i_j}}\right)\gamma^{n_i} \oplus \bigoplus_{l=1}^{L}\left(\bigoplus_{k=1}^{K_l}\delta^{\tau_{l_k}}\Delta_{\omega}\delta^{\tau'_{l_k}}\right)\gamma^{\nu_l}, \tag{25}$$

where $J_i \leq \omega$, $K_l \leq \omega$. The complexity of this operation is $\mathcal{O}(\omega(I+L))$.

Proposition 5 (Product of polynomials) *Let* $p_1 = \bigoplus_{i=1}^{I} v_i\gamma^{n_i}$ *with period* ω_1 *and* $p_2 = \bigoplus_{l=1}^{L} \bar{v}_l\gamma^{\nu_l}$ *with period* ω_2 *be two polynomials in* $\mathcal{T}^*[\![\gamma]\!]$*, then the product* $p_1 \otimes p_2$ *is again a polynomial in* $\mathcal{T}^*[\![\gamma]\!]$ *with period* $\omega = lcm(\omega_1, \omega_2)$*. The complexity of the operation is* $\mathcal{O}(2\omega IL)$*.*

Proof See Appendix C.3.

The domination lemma given in Gaubert (1992) for series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ can be adapted to series in $\mathcal{T}^*[\![\gamma]\!]$ as follows.

Lemma 1 (Ultimate domination) *Let* $s_1 = \delta^{\varsigma_1}\Delta_{\omega}\delta^{\varsigma'_1}\gamma^{n_1}(\gamma^{\nu_1}\delta^{\tau_1})^* \in \mathcal{T}^*[\![\gamma]\!]$ *and* $s_2 = \delta^{\varsigma_2}\Delta_{\omega}\delta^{\varsigma'_2}\gamma^{n_2}(\gamma^{\nu_2}\delta^{\tau_2})^* \in \mathcal{T}^*[\![\gamma]\!]$ *be two series in the commute form (Proposition 4) with asymptotic slopes* $\sigma(s_1) = \tau_1/\nu_1 > \sigma(s_2) = \tau_2/\nu_2$ *then there exists a nonnegative integer* $K \in \mathbb{N}$ *such that,*

$$\delta^{\varsigma_2}\Delta_{\omega}\delta^{\varsigma'_2}\gamma^{n_2}(\gamma^{K\nu_2}\delta^{K\tau_2})(\gamma^{\nu_2}\delta^{\tau_2})^* \preceq s_1. \tag{26}$$

Therefore, s_1 *ultimately dominates* s_2*.*

Proof See Appendix C.4.

Proposition 6 (Sum of series) *The sum of two ultimately periodic series* $s_1, s_2 \in \mathcal{T}^*[\![\gamma]\!]$ *is an ultimately periodic series with an asymptotic slope given by* $\sigma(s_1 \oplus s_2) = \max(\sigma(s_1), \sigma(s_2))$*.*

Proof See Appendix C.5.

Proposition 7 (Product of series) *Let* $s_1, s_2 \in \mathcal{T}^*[\![\gamma]\!]$ *be two ultimately periodic series, then the product* $s_1 \otimes s_2$ *is again an ultimately periodic series in* $\mathcal{T}^*[\![\gamma]\!]$ *with an asymptotic slope* $\sigma(s_1 \otimes s_2) = \max(\sigma(s_1), \sigma(s_2))$*.*

Proof See Appendix C.6.

Proposition 8 (Kleene star of a polynomial) *The Kleene star of a polynomial* $p \in \mathcal{T}^*[\![\gamma]\!]$ *(*$p = \bigoplus_{i=1}^{I} \delta^{\varsigma_i} \Delta_\omega \delta^{\varsigma_i'} \gamma^{n_i}$*) is an ultimately periodic series in* $\mathcal{T}^*[\![\gamma]\!]$.

Proof See Appendix C.7.

Proposition 9 (Kleene star of a series) *The Kleene star of an ultimately periodic series* $s \in \mathcal{T}^*[\![\gamma]\!]$ *is again an ultimately periodic series in* $\mathcal{T}^*[\![\gamma]\!]$.

Proof See Appendix C.8.

Moreover, in Appendix C.5, C.6, C.8 it is shown that operations between ultimately periodic series can be reduced to operations between polynomials. The size of those polynomials (i.e., the number of their constituent monomials) depend of the point K of ultimate domination of the positive integer introduced in Lemma 1. Hence, complexity of operations between series also critically depends on this point.

Let us note that the dioid $\mathcal{T}^*[\![\gamma]\!]$ with periodic time operators is the counterpart to the dioid $\mathcal{E}^*[\![\delta]\!]$ with periodic event operators introduced in (Cottenceau et al (2014a)). The dioid $\mathcal{E}^*[\![\delta]\!]$ is used to model dynamic phenomena arising in Weight-Balanced Timed Event Graphs (Cottenceau et al (2014a, 2017)).

5 Modelling of PTEGs

We can use T-operators and the event shift operator γ to describe the transfer behaviour of PTEGs. The firing-relation between the two transitions t_i, t_j in Figure 7 is represented by $x_j = v_k \gamma^{M_k^0} x_i$, where M_k^0 is the initial marking in

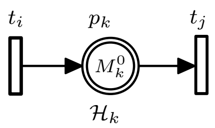


Figure 7

place p_k, v_k is the T-operator associated with the holding-time-function $\mathcal{H}_k$ of place p_k and x_i, x_j are the dater functions associated with t_i, t_j. Thus, the relation between input, output and internal transitions of a general PTEG can be modelled by

$$x = Ax \oplus Bu, \qquad y = Cx, \tag{27}$$

where x (resp. u, y) refers to vector of dater functions of the n internal (resp. m input, p output) transitions of the PTEG. The relations between internal transitions are modelled by the system matrix $A \in \mathcal{T}^*[\![\gamma]\!]^{n \times n}$, the relation between input and internal transitions by the input matrix $B \in \mathcal{T}^*[\![\gamma]\!]^{n\times m}$, and the relation between internal and output transitions by the output matrix $C \in \mathcal{T}^*[\![\gamma]\!]^{p\times n}$. This modelling procedure is similar to the modelling procedure of TEGs in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, see Example 3.

Example 8 Consider the PTEG in Figure 3 of Example 6. The firing relation between its transitions can be modelled by the following representation

$$\begin{aligned} x &= \left[(\Delta_4 \oplus \delta^1\Delta_4\delta^{-3})\gamma^2\right] x \oplus \left[\delta^{-3}\Delta_4 \oplus \Delta_4\delta^{-1}\right] u, \\ y &= \left[\delta^1\right] x, \end{aligned}$$

where $\Delta_4 \oplus \delta^1\Delta_4\delta^{-3}$ and $\delta^{-3}\Delta_4 \oplus \Delta_4\delta^{-1}$ are the T-operators corresponding to $\mathcal{H}_3 = \langle 1\,3\,2\,1\rangle$ and $\mathcal{H}_1 = \langle 0\,0\,2\,1\rangle$, see Example 7.

Theorem 2 (Transfer function matrix of PTEG) *The input-output behaviour of an m-input and p-output PTEG, defined by* (27)*, can be represented by a transfer function matrix $H \in \mathcal{T}^*[\![\gamma]\!]^{p\times m}$ of ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$. This transfer function matrix is obtained by $H = CA^*B$.*

Proof Holding-time-functions in PTEGs correspond to causal periodic T-operators (Proposition 2). Note that because of Remark 5 a monomial (resp. polynomial) in $\mathcal{T}^*[\![\gamma]\!]$ can be expressed as an ultimately periodic series. Hence, the entries of the A, B, C matrices are composed of ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$. Due to Proposition 6, Proposition 7 and Proposition 9 the sum, product and Kleene star of ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$ are again ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$. Therefore the matrix CA^*B is composed of ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$.

As indicated above, the entries of the A^* matrix are ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$. The domination point between series depend on their asymtotic slope and therefore on the circuits of the PTEG, in particular on their marking and time configuration. As argued in Section 4, complexity of operations on series depend critically on the point of domination between these series. Hence the complexity of forming an transfer function matrix is critically affected by the circuits of the PTEG. In $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, we have a similar situation, but without the dependence on a time-variant holding times. Hence, the complexity difference between the class of TEGs and PTEGs is the multiplication factor related to the periodicity of time operators Δ. Finally let us note that in Bouillard and Thierry (2008) more detailed results on operational complexity are given for the class of network calculus, which is similar to operations in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

Example 9 (Transfer function) Consider the PTEG in Figure 3 of Example 6. We can describe the firing relation between input transition t_1 and output transition t_3 by a transfer function h in $\mathcal{T}^*[\![\gamma]\!]$, i.e., $y = hu$, where

$$\begin{aligned} h &= \delta^1[(\delta^1\Delta_4\delta^{-3} \oplus \Delta_4)\gamma^2]^*(\delta^{-3}\Delta_4 \oplus \Delta_4\delta^{-1}) \\ &= (\gamma^4\delta^4)^* \left((\delta^1\Delta_4\delta^{-1} \oplus \delta^{-2}\Delta_4) \oplus (\delta^1\Delta_4 \oplus \delta^2\Delta_4\delta^{-1})\gamma^2\right) \\ &= \left(\delta^1\Delta_4\delta^{-1} \oplus \delta^{-2}\Delta_4\right)\gamma^0 \oplus \left(\delta^1\Delta_4 \oplus \delta^2\Delta_4\delta^{-1}\right)\gamma^2 \oplus \\ &\quad \left(\delta^5\Delta_4\delta^{-1} \oplus \delta^2\Delta_4\right)\gamma^4 \oplus \left(\delta^5\Delta_4 \oplus \delta^6\Delta_4\delta^{-1}\right)\gamma^6 \oplus \cdots \end{aligned}$$

This transfer function has a graphical representation, see Figure 8a.

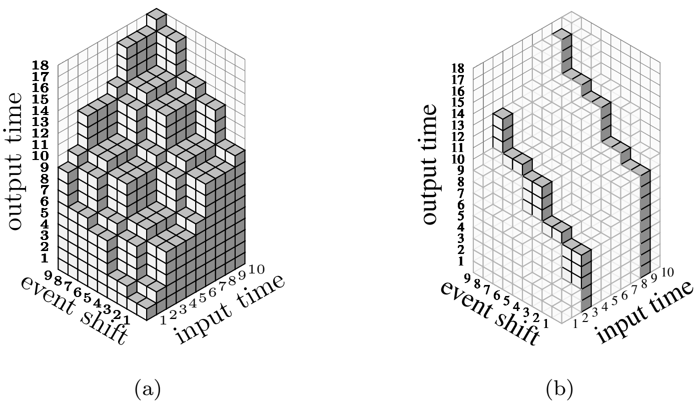


Figure 8: (a) transfer function h of Example 9. (b) the gray slice at input time 2 (resp. time 8) (event-shift/output-time plane) corresponds to the response to an impulse at time 2: $\delta^2\mathcal{I}$ (resp. time 8: $\delta^8\mathcal{I}$) of the system (Example 10).

5.1 Impulse response of a SISO PTEG

An impulse is a specific dater function $\mathcal{I}(k)$, see (5). As in conventional linear systems theory, the impulse response of a $(max, +)$ linear system provides complete knowledge of the input-output behaviour, see Baccelli et al (1992) and the paragraph immediately following (5). More precisely, the system's impulse response equals its transfer function. In contrast, the impulse response of a PTEG is not sufficient to describe its complete behaviour, because a PTEG is a time-variant system. Hence, the moment when the impulse is applied matters. One single impulse gives only partial information. In order to obtain complete knowledge, we need the system responses of ω consecutive time shifted impulses, i.e. $\delta^\varsigma\mathcal{I},\ \varsigma \in \{0, \cdots, \omega-1\}$. Each single response corresponds then to one slice in the 3D representation of the transfer function. The impulse response of a simple monomial $\delta^\varsigma\Delta_\omega\delta^{\varsigma'}\gamma^\nu$ is given by

$$\left(\delta^\varsigma\Delta_\omega\delta^{\varsigma'}\gamma^\nu\mathcal{I}\right)(k) = \left\lceil\frac{\mathcal{I}(k-\nu)+\varsigma'}{\omega}\right\rceil\omega + \varsigma = \mathcal{I}(k-\nu) + \left\lceil\frac{\varsigma'}{\omega}\right\rceil\omega + \varsigma,$$
$$= \mathcal{I}(k-\nu) + \mathcal{R}_{\delta^\varsigma\Delta_\omega\delta^{\varsigma'}}(0).$$

As $\mathcal{I}(k-\nu) = 0$ for $k-\nu \geq 0$ and $-\infty$ otherwise, the impulse response of a simple monomial is again an impulse which is event-shifted by ν units and time-shifted by $\mathcal{R}_{\delta^\varsigma\Delta_\omega\delta^{\varsigma'}}(0) = \varsigma + \lceil\frac{\varsigma'}{\omega}\rceil\omega$ units, i.e., $\delta^\varsigma\Delta_\omega\delta^{\varsigma'}\gamma^\nu\mathcal{I} = \delta^{(\varsigma+\lceil\frac{\varsigma'}{\omega}\rceil\omega)}\gamma^\nu\mathcal{I}$. For a simple canonical monomial $\delta^\varsigma\Delta_\omega\delta^{\varsigma'}$, with $-\omega < \varsigma' \leq 0$, this reduces to

$\delta^{\varsigma}\Delta_{\omega}\delta^{\varsigma'}\gamma^{\nu}\mathcal{I} = \delta^{\varsigma}\gamma^{\nu}\mathcal{I}$. The impulse response of a series $s = p \oplus qr^* \in \mathcal{T}^*[\![\gamma]\!]$ can be obtained by applying the above rule to every simple monomial in the p (resp. q) polynomial of s.

Example 10 The response of an impulse at time 2 of the system in Example 6 - with a transfer function given in Example 9 - is $(\delta^5 \oplus \delta^6\gamma^2)(\gamma^4\delta^4)^*\mathcal{I}$. This response corresponds to the slice at input-time 2 (event-shift/output-time plane) in Figure 8b. The system response of an impulse at time 8 is $(\delta^9 \oplus \delta^{10}\gamma^2)(\gamma^4\delta^4)^*\mathcal{I}$. We can interpret the 3D representation of a transfer function in $\mathcal{T}^*[\![\gamma]\!]$ as the juxtaposition of its time-shifted impulse responses.

6 Control of PTEGs

In general, the product in a dioid is not invertible. However, with residuation theory it is possible to find a greatest solution of inequality $A \otimes X \preceq B$. Therefore, this theory is suitable to solve some model matching control problems for PTEGs. This approach is well known for TEGs, see e.g., Baccelli et al (1992).

6.1 Complete Dioids and Residuation Theory

Residuation theory is a formalism to address the problem of approximate mapping inversion over ordered sets, see Baccelli et al (1992). Recall that a complete dioid is a partially ordered set, with a canonical order $\succeq$ defined by $a \oplus b = a \Leftrightarrow a \succeq b$. The infimum, or greatest lower bound, operator can then be defined by $a, b \in \mathcal{D}$, $a \wedge b = \bigoplus\{x \in \mathcal{D} \mid x \oplus a = a, x \oplus b = b\}$.

Definition 12 (Residuation) Let $\mathcal{F}$ and $\mathcal{L}$ be partially ordered sets and $f : \mathcal{F} \to \mathcal{L}$ a nondecreasing mapping. The mapping f is said to be residuated if for all $y \in \mathcal{L}$, the least upper bound of the subset $\{x \in \mathcal{F} | f(x) \preceq y\}$ exists and lies in this subset. It is denoted $f^{\sharp}(y)$, and mapping $f^{\sharp}$ is called the residual of f.

It can be shown (e.g. Baccelli et al (1992)) that, on a complete dioid, the mappings $R_a : x \mapsto xa$, (right multiplication) resp. $L_a : x \mapsto ax$ (left multiplication) are residuated. The residual mappings are denoted $R_a^{\sharp}(b) = b \phi a = \bigoplus\{x | xa \preceq b\}$ (right division by a) resp. $L_a^{\sharp}(b) = a \backslash\!\!\circ b = \bigoplus\{x | ax \preceq b\}$ (left division by a). In analogy to the extension of the product to the matrix case, we can extend left and right division to matrices with entries in a complete dioid. Since $\mathcal{T}$ and $\mathcal{T}^*[\![\gamma]\!]$ are complete dioids, left and right multiplication in these dioids are residuated.

Lemma 2 *Let $v \in \mathcal{T}$, then:*

$$\Delta_{\omega} \backslash\!\!\circ v = \Delta_{\omega}\delta^{1-\omega}v, \qquad v \phi \Delta_{\omega} = v\delta^{1-\omega}\Delta_{\omega}. \tag{28}$$

Proof To prove (28), recall that by definition of the residuated mapping, $\Delta_\omega \backslash v$ is the greatest solution of the inequality $v \succeq \Delta_\omega x$. This greatest solution is given by

$$\Delta_\omega \backslash v = \bigoplus \{u \in \mathcal{T} | \Delta_\omega u \preceq v\} = \bigoplus \{u \in \mathcal{T} | \mathcal{R}_{\Delta_\omega u}(\xi) \leq \mathcal{R}_v(\xi)\ \forall, \xi \in \mathbb{Z}_{max}\}.$$

Therefore, $\forall \xi \in \mathbb{Z}_{max}$

$$\mathcal{R}_{\Delta_\omega \backslash v}(\xi) = \max\{\mathcal{R}_u(\xi) |\ \lceil \mathcal{R}_u(\xi)/\omega \rceil \omega \leq \mathcal{R}_v(\xi)\}$$

Observe that,

$$\begin{aligned}
&\left\lceil \frac{\mathcal{R}_u(\xi)}{\omega} \right\rceil \omega \leq \mathcal{R}_v(\xi) \\
\Leftrightarrow &\left\lceil \frac{\mathcal{R}_u(\xi)}{\omega} \right\rceil \leq \frac{\mathcal{R}_v(\xi)}{\omega} \\
\Leftrightarrow &\frac{\mathcal{R}_u(\xi)}{\omega} \leq \left\lfloor \frac{\mathcal{R}_v(\xi)}{\omega} \right\rfloor = \left\lceil \frac{\mathcal{R}_v(\xi) - \omega + 1}{\omega} \right\rceil \\
\Leftrightarrow &\mathcal{R}_u(\xi) \leq \left\lceil \frac{\mathcal{R}_v(\xi) - \omega + 1}{\omega} \right\rceil \omega
\end{aligned}$$

where the equality above chain of equivalence follows from the basic properties of the "floor" and "ceil" operations listed in Appendix B. Consequently

$$\begin{aligned}
&\mathcal{R}_{\Delta_\omega \backslash v} \leq \left\lceil \frac{\mathcal{R}_v(\xi) - \omega + 1}{\omega} \right\rceil \omega, \quad \forall \xi \in \mathbb{Z}_{max} \\
\Leftrightarrow \quad &\omega \backslash v = \Delta_\omega \delta^{1-\omega} v.
\end{aligned}$$

The proof for $v \phi \Delta_\omega = v\delta^{1-\omega}\Delta_\omega$ is analogous.

Proposition 10 *Let $s \in \mathcal{T}^*[\![\gamma]\!]$, then:*

$$\gamma^\nu \backslash s = \gamma^{-\nu} s, \qquad s \phi \gamma^\nu = s\gamma^{-\nu}, \tag{29}$$

$$\delta^\varsigma \backslash s = \delta^{-\varsigma} s, \qquad s \phi \delta^\varsigma = s\delta^{-\varsigma}, \tag{30}$$

$$\Delta_\omega \backslash s = \Delta_\omega \delta^{1-\omega} s, \qquad s \phi \Delta_\omega = s\delta^{1-\omega}\Delta_\omega. \tag{31}$$

Proof For the proof of (29) and (30), note that the operators δ^ς and γ^ν are invertible, since $\delta^\varsigma\delta^{-\varsigma} = \gamma^\nu\gamma^{-\nu} = e$. Moreover, for the proof of (31), recall $\Delta_\omega \backslash v = \Delta_\omega \delta^{1-\omega} v$ with $v \in \mathcal{T}$ (Lemma 2) and $w\gamma^\eta \backslash (\bigoplus_i v_i \gamma^{n_i}) = (\bigoplus_i w \backslash v_i)\gamma^{n_i - \eta}$ with $v_i, w \in \mathcal{T}$, see Baccelli et al (1992) Remark 4.96. Therefore, for a series $s = \bigoplus_i v_i \gamma^{n_i} \in \mathcal{T}^*[\![\gamma]\!]$, one has

$$\begin{aligned}
\Delta_\omega \backslash s &= \Delta_\omega \gamma^0 \backslash \Big(\bigoplus_i v_i \gamma^{n_i} \Big) = \bigoplus_i \Big(\Delta_\omega \backslash v_i \Big) \gamma^{n_i - 0} = \bigoplus_i \Delta_\omega \delta^{1-\omega} v_i \gamma^{n_i}, \\
&= \Delta_\omega \delta^{1-\omega} s.
\end{aligned}$$

The proof of the second expression in (31) is analogous.

Left and right division of a series in $\mathcal{T}^*[\![\gamma]\!]$ by a T-operator can be generalized to left and right division by polynomials and series in $\mathcal{T}^*[\![\gamma]\!]$.

Proposition 11 (Infimum of series) *Let $s_1, s_2 \in \mathcal{T}^*[\![\gamma]\!]$ be two ultimately periodic series, then the infimum $s_1 \wedge s_2$ is an ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$.*

Proof The proof is similar to the sum of two series, therefore we only give a brief sketch. If $\sigma(s_1) = \sigma(s_2)$, then the asymptotic slope of the result is $\sigma(s_1 \wedge s_2) = \sigma(s_1) = \sigma(s_2)$. If $\sigma(s_1) > \sigma(s_2)$, then the result is a series with asymptotic slope given by the slope of s_2, i.e. $\sigma(s_1 \wedge s_2) = \sigma(s_2)$.

Proposition 12 *Let p_1 and p_2 be two polynomials in $\mathcal{T}^*[\![\gamma]\!]$, then $p_2 \circ\!\!\!\!\backslash\, p_1$ and $p_1 \circ\!\!\!\!/\, p_2$ are polynomials in $\mathcal{T}^*[\![\gamma]\!]$.*

Proof

$$
\begin{aligned}
p_2 \circ\!\!\!\!\backslash\, p_1 &= \left(\bigoplus_{i=1}^{I} \delta^{\varsigma_{1i}} \Delta_\omega \delta^{\varsigma'_{1i}} \gamma^{n_{1i}}\right) \circ\!\!\!\!\backslash \left(\bigoplus_{j=1}^{J} \delta^{\varsigma_{2j}} \Delta_\omega \delta^{\varsigma'_{2j}} \gamma^{n_{2j}}\right), \\
&\quad \text{with (34) :} (a \oplus b) \circ\!\!\!\!\backslash\, x = a \circ\!\!\!\!\backslash\, x \wedge b \circ\!\!\!\!\backslash\, x \\
&= \bigwedge_{i=1}^{I} \left(\left(\delta^{\varsigma_{1i}} \Delta_\omega \delta^{\varsigma'_{1i}} \gamma^{n_{1i}}\right) \circ\!\!\!\!\backslash \bigoplus_{j=1}^{J} \delta^{\varsigma_{2j}} \Delta_\omega \delta^{\varsigma'_{2j}} \gamma^{n_{2j}} \right), \\
&\quad \text{because of Proposition 10, (32): } (ab) \circ\!\!\!\!\backslash\, x = b \circ\!\!\!\!\backslash (a \circ\!\!\!\!\backslash\, x) \text{ and (17)} \\
&= \bigwedge_{i=1}^{I} \left(\bigoplus_{j=1}^{J} \delta^{-\varsigma'_{1i} + \lceil (1-\omega-\varsigma_{1i}+\varsigma_{2j})/\omega \rceil \omega} \Delta_\omega \delta^{\varsigma'_{2j}} \gamma^{n_{2j}-n_{1i}} \right).
\end{aligned}
$$

The proof for $p_1 \circ\!\!\!\!/\, p_2$ is analogous.

Lemma 3 (Baccelli et al (1992)) *The greatest fixed-point of $\Pi_l(x) = a \circ\!\!\!\!\backslash\, x \wedge b$ (resp. $\Pi_r(x) = x \circ\!\!\!\!/\, a \wedge b$) is $a^* \circ\!\!\!\!\backslash\, b$ (resp. $b \circ\!\!\!\!/\, a^*$).*

Proposition 13 (Left and Right Residuation of Product) *Let $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$, $s_2 = p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})^* \in \mathcal{T}^*[\![\gamma]\!]$ be two ultimately periodic series, then $s_2 \circ\!\!\!\!\backslash\, s_1$ (resp. $s_1 \circ\!\!\!\!/\, s_2$) is a series in $\mathcal{T}^*[\![\gamma]\!]$, if the mapping $(\gamma^{\nu_2}\delta^{\tau_2}) \circ\!\!\!\!\backslash\, x \wedge (q_2 \circ\!\!\!\!\backslash\, s_1)$ (resp. $x \circ\!\!\!\!/ (\gamma^{\nu_2}\delta^{\tau_2}) \wedge s_1 \circ\!\!\!\!/\, q_2$))has a fixed point.*

Proof The proof is similar to the proof for the division of series in $\mathcal{E}^*[\![\delta]\!]$, see Cottenceau et al (2014a). Because of (34), $s_2 \circ\!\!\!\!\backslash\, s_1$ can be written as

$$
(p_2 \oplus q_2(\gamma^{\nu_2}\delta^{\tau_2})) \circ\!\!\!\!\backslash\, s_1 = p_2 \circ\!\!\!\!\backslash\, s_1 \wedge (\gamma^{\nu_2}\delta^{\tau_2})^* \circ\!\!\!\!\backslash (q_2 \circ\!\!\!\!\backslash\, s_1).
$$

If $(\gamma^{\nu_2}\delta^{\tau_2}) \circ\!\!\!\!\backslash\, x \wedge (q_2 \circ\!\!\!\!\backslash\, s_1)$ has a fixed point then $s_2 \circ\!\!\!\!\backslash\, s_1$ can be expressed as a infimum of a finite set of periodic series with the same slope, see Proposition 11.

To obtain the fix point of $(\gamma^{\nu_2}\delta^{\tau_2}) \circ\!\!\!\!\backslash\, x \wedge (q_2 \circ\!\!\!\!\backslash\, s_1)$ is a particular method to compute the residuation of the product of two ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$, for more detail see also (Cottenceau et al (2014a)).

Model Reference Control

Model reference control for the case of TEGs was discussed in (Libeaut and Loiseau (1996); Maia et al (2003); Hardouin et al (2018)). For the class of PTEG the model reference control problem is as follows: Given a transfer matrix H describing the input-output relation of a PTEG and a reference transfer matrix G with entries in $\mathcal{T}^*[\![\gamma]\!]$. Find the greatest feedback matrix F with entries in $\mathcal{T}^*[\![\gamma]\!]$ such that the closed loop transfer matrix in Figure 9 $H_{cl} = (HF)^*H \preceq G$. In particular we are interested in the case $G = H$. This implies that we seek feedback that delays the firing of plant input transitions as much as possible without "slowing down" the transfer behaviour. This is often called a neutral "just in time" policy. As $\mathcal{T}^*[\![\gamma]\!]$ is a complete dioid, the maximal solution of $(HF)^*H \preceq H$ is given by $F_{opt} = H \circ\!\!\!\backslash H \circ\!\!\!/ H$, i.e., it has the same structure as for ordinary TEGs.

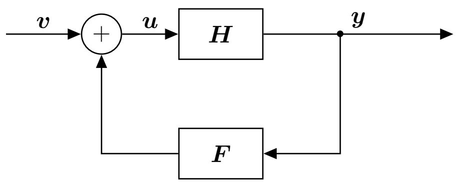


Figure 9: Closed loop structure with an output feedback F.

Example 11 The following example illustrates model reference control for the simple PTEG of Example 6, with transfer function given in Example 9. For this system, the neutral "just-in-time" feedback is: $f_{opt} = h \circ\!\!\!\backslash h \circ\!\!\!/ h = (\gamma^4\delta^4)^*((\Delta_4\delta^{-1} \oplus \delta^1\Delta_4\delta^{-2})\gamma^2 \oplus (\delta^1\Delta_4\delta^{-1} \oplus \delta^4\Delta_4\delta^{-2})\gamma^4)$. Recall the control law $u = f_{opt}y \oplus v$. To realize the feedback f_{opt} we rewrite $f_{opt}y$ as

$$\begin{aligned}\rho &= f_{opt}y \\ &= (\gamma^4\delta^4)^* \left[(\Delta_4\delta^{-1} \oplus \delta^1\Delta_4\delta^{-2})\gamma^2 \oplus (\delta^1\Delta_4\delta^{-1} \oplus \delta^4\Delta_4\delta^{-2})\gamma^4\right] y.\end{aligned}$$

The former expression is the solution of the following implicit equation

$$\rho = \left[\gamma^4\delta^4\right]\rho \oplus \left[(\Delta_4\delta^{-1} \oplus \delta^1\Delta_4\delta^{-2})\gamma^2 \oplus (\delta^1\Delta_4\delta^{-1} \oplus \delta^4\Delta_4\delta^{-2})\gamma^4\right] y.$$

From this expression we can implement the feedback f_{opt} by a PTEG as follows: The feedback has one transition, denoted by t_c, associate with the dater-function ρ. Because of operator $\gamma^4\delta^4$ transition t_c is attached with a self loop, constituted by place p_{c1} with 4 initial tokens and a constant holding time of 4 time units. The polynomial $(\Delta_4\delta^{-1} \oplus \delta^1\Delta_4\delta^{-2})\gamma^2 \oplus (\delta^1\Delta_4\delta^{-1} \oplus \delta^4\Delta_4\delta^{-2})\gamma^4$ describes the influence of the plant output transition t_3 onto the transition t_c of the feedback. Observe that we have two monomial, therefore we obtain two parallel path between t_3 and t_c, each with one place. First $(\Delta_4\delta^{-1} \oplus \delta^1\Delta_4\delta^{-2})\gamma^2$ is described by the place p_{c2} and the arcs (t_3, p_{c2}) and (p_{c2}, t_c). Because of the exponent of γ^2 the place p_{c2} contains 2 initial tokens. The holding-time-function of p_{c2} is determined by the T-operator $\Delta_4\delta^{-1} \oplus \delta^1\Delta_4\delta^{-2}$ as follows:

$$\begin{aligned}\mathcal{H}_{p_{c_2}}(\xi) &= \max\left(\mathcal{R}_{\Delta_4\delta^{-1}}(\xi), \mathcal{R}_{\delta^1\Delta_4\delta^{-2}}(\xi)\right) - \xi,\\ &= \max\left(\left\lceil\frac{\xi-1}{4}\right\rceil 4,\ 1+\left\lceil\frac{\xi-2}{4}\right\rceil 4\right) - \xi,\\ &= \langle 1\ 0\ 2\ 2\rangle\end{aligned}$$

Respectively, $(\delta^1\Delta_4\delta^{-1} \oplus \delta^4\Delta_4\delta^{-2})\gamma^4$ is described by the place p_{c3} and the arcs (t_3, p_{c3}) and (p_{c3}, t_c). Because of the exponent of γ^4 the place p_{c3} contains 4 initial tokens. Moreover, the holding-time-function of p_{c3} is

$$\begin{aligned}\mathcal{H}_{p_{c3}}(\xi) &= \max\left(\mathcal{R}_{\delta^1\Delta_4\delta^{-1}}(\xi), \mathcal{R}_{\delta^4\Delta_4\delta^{-2}}(\xi)\right) - \xi,\\ &= \max\left(1+\left\lceil\frac{\xi-1}{4}\right\rceil 4,\ 4+\left\lceil\frac{\xi-2}{4}\right\rceil 4\right) - \xi,\\ &= \langle 4\ 3\ 3\ 5\rangle\end{aligned}$$

The controller is is connected to the plant input transition t_1 via the arcs (t_c, p_{c4}) and (p_{c4}, t_1). Finally, transition t_v is associated with the new input v and is connected to the plant input transition t_1 via the arcs (t_v, p_v) and (p_v, t_1). Figure 10 illustrates the closed loop system. The feedback keeps the number of tokens in places p_1, p_2 as small as possible, while the throughput of the system is preserved.

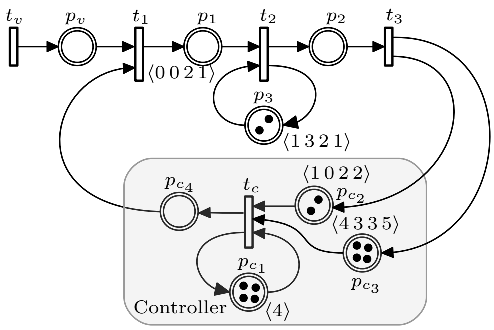


Figure 10: Closed loop system.

7 Conclusion

In this paper, we have introduced an extension of TEGs called Periodic Time-variant Event Graphs, where the holding times vary periodically over time. These time-variant systems allow to model particular time phenomena such as traffic light control, for which we need to describe varying waiting times. We show that the transfer behaviour of these systems can be modelled by ultimately periodic series in a dioid denoted $\mathcal{T}^*[\![\gamma]\!]$. These transfer functions are useful for performance evaluation and controller synthesis of PTEGs. In this paper, we have focused on fundamental results and simple examples that illustrates our theoretical results. In future work, we aim at applying the obtained results to more complex systems. For this purpose, the software tools ETVO has been developed Cottenceau et al (2019).

The class of PTEGs can be seen as the counterpart of Weight-Balanced Timed Event Graphs (Cottenceau et al (2014a)). In future work, we also aim at combining the results for PTEGs with the results for WBTEGs in a comprehensive modelling formalism. This would allow to describe a class of periodic time- and event-variant discrete event systems with a common set of algebraic tools.

A Formula of Residuation

In a complete dioid, the following formula hold for the residuation of left and right multiplication see (Baccelli et al 1992, Chap.4).

$$(ab) \backslash\!\!\circ\, x = b \backslash\!\!\circ (a \backslash\!\!\circ\, x) \qquad x \,/\!\!\circ (ba) = (x /\!\!\circ\, a) /\!\!\circ (b) \tag{32}$$

$$(a \backslash\!\!\circ\, x) /\!\!\circ\, b = a \backslash\!\!\circ (x /\!\!\circ\, b) \qquad a \backslash\!\!\circ (x /\!\!\circ\, b) = (a \backslash\!\!\circ\, x) /\!\!\circ\, b \tag{33}$$

$$(a \oplus b) \backslash\!\!\circ\, x = (a \backslash\!\!\circ\, x) \wedge (b \backslash\!\!\circ\, x) \qquad x /\!\!\circ (a \oplus b) = (x /\!\!\circ\, a) \wedge (x /\!\!\circ\, b) \tag{34}$$

B Formula for Floor and Ceil Operations (Graham et al (1989))

For $x \in \mathbb{R}$,

$$\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor, \qquad \lceil \lceil x \rceil \rceil = \lceil x \rceil.$$

For $x \in \mathbb{R}$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\left\lfloor \frac{x+m}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor, \qquad \left\lceil \frac{x+m}{n} \right\rceil = \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil.$$

For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil, \qquad \left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor.$$

C Proofs

C.1 Proof of Proposition 1 (Relations between T-operators):

Let us recall that $y \in \mathbb{R},\ \forall n \in \mathbb{Z}, \lceil y+n \rceil = \lceil y \rceil + n$. To prove (16), because of Definition 6, $\forall x \in \Sigma$,

$$\begin{aligned}
\left(\Delta_\omega \delta^\varsigma x\right)(k) &= \left\lceil \frac{x(k)+\varsigma}{\omega} \right\rceil \omega = \left\lceil \frac{x(k)}{\omega} + \frac{\varsigma}{\omega} + \left\lceil \frac{\varsigma}{\omega} \right\rceil - \left\lceil \frac{\varsigma}{\omega} \right\rceil \right\rceil \omega \\
&= \left\lceil \frac{\varsigma}{\omega} \right\rceil \omega + \left\lceil \frac{x(k)+\varsigma-\omega\lceil(\varsigma/\omega)\rceil}{\omega} \right\rceil \omega \\
&= \left(\delta^{\lceil \frac{\varsigma}{\omega} \rceil \omega} \Delta_\omega \delta^{\varsigma - \lceil \frac{\varsigma}{\omega} \rceil \omega} x\right)(k).
\end{aligned}$$

Second,

$$\begin{aligned}
\left(\delta^\varsigma \Delta_\omega x\right)(k) &= \varsigma + \left\lceil \frac{x(k)}{\omega} \right\rceil \omega \\
&= \varsigma - \left\lceil \frac{\varsigma}{\omega} \right\rceil \omega + \left\lceil \frac{\varsigma}{\omega} \right\rceil \omega + \left\lceil \frac{x(k)}{\omega} \right\rceil \omega \\
&= \varsigma - \left\lceil \frac{\varsigma}{\omega} \right\rceil \omega + \left\lceil \frac{x(k)+\lceil \varsigma/\omega \rceil \omega}{\omega} \right\rceil \omega \\
&= \left(\delta^{\varsigma - \lceil \frac{\varsigma}{\omega} \rceil \omega} \Delta_\omega \delta^{\lceil \frac{\varsigma}{\omega} \rceil \omega} x\right)(k).
\end{aligned}$$

To prove (17), note that $\lceil (a+\varsigma)/\omega \rceil \omega = \lceil \varsigma/\omega \rceil \omega + \lceil (a+\varsigma-\omega\lceil \varsigma/\omega \rceil)/\omega \rceil \omega$, and therefore

$$\begin{aligned}
\left(\Delta_\omega \delta^\varsigma \Delta_\omega x\right)(k) &= \left\lceil \frac{\lceil x(k)/\omega \rceil \omega + \varsigma}{\omega} \right\rceil \omega \\
&= \left\lceil \frac{\varsigma}{\omega} \right\rceil \omega + \left\lceil \left\lceil \frac{x(k)}{\omega} \right\rceil + \frac{\varsigma - \omega\lceil \varsigma/\omega \rceil}{\omega} \right\rceil \omega
\end{aligned}$$

since: $\lceil x(k)/\omega \rceil \in \mathbb{Z}$ and $-1 < (\varsigma - \omega\lceil \varsigma/\omega \rceil)/\omega \le 0$, finally,

$$\left(\Delta_\omega \delta^\varsigma \Delta_\omega x\right)(k) = \left\lceil \frac{\varsigma}{\omega} \right\rceil \omega + \left\lceil \frac{x(k)}{\omega} \right\rceil \omega = \left(\delta^{\lceil \frac{\varsigma}{\omega} \rceil \omega} \Delta_\omega x\right)(k).$$

C.2 Proof of Proposition 2 (Operator representation of a release-time-function):

First recall that release-time-functions are nondecreasing. Hence, in (9), $n_{\omega-1} - \omega \le n_0 \le n_1 \le \cdots \le n_{\omega-1} \le n_0 + \omega$. Moreover, recall that the release-time-function $\mathcal{R}_{\delta^\varsigma \Delta_\omega \delta^{\varsigma'}}(\xi)$ of an operator $\delta^\varsigma \Delta_\omega \delta^{\varsigma'}$ is defined by

$$\mathcal{R}_{\delta^\varsigma \Delta_\omega \delta^{\varsigma'}}(\xi) = \varsigma + \lceil (\xi + \varsigma')/\omega \rceil \omega,$$

where $\xi = x(k)$ is a date. Thus, $\mathcal{R}_p$ associated with (21) is

$$\begin{aligned}
\mathcal{R}_p(\xi) = \max(&n_0 + \lceil (\xi - (\omega-1))/\omega \rceil \omega, n_1 - \omega + \lceil \xi/\omega \rceil \omega, \\
&\cdots, n_{\omega\text{-}1} - \omega + \lceil (\xi - (\omega-2))/\omega \rceil \omega).
\end{aligned} \tag{35}$$

We can evaluate the expression (35) for all dates ξ. If we choose $\xi = j\omega, \quad \forall j \in \mathbb{Z}_{max}$, we have

$$\begin{aligned}\mathcal{R}_p(j\omega) &= \max(n_0 + \lceil (j\omega - (\omega-1))/\omega \rceil \omega, n_1 - \omega + \lceil j\omega/\omega \rceil \omega,\\ &\quad \cdots, n_{\omega\text{-}1} - \omega + \lceil (j\omega - (\omega-2))/\omega \rceil \omega)\\ &= \max(n_0 + j\omega, n_1 - \omega + j\omega, \cdots, n_{\omega\text{-}1} - \omega + j\omega)\\ &= n_0 + j\omega.\end{aligned}$$

Similarly $\forall i = \{1, \cdots, (\omega-1)\}$,

$$\begin{aligned}\mathcal{R}_p(i + j\omega) = \max(&n_0 + \lceil (i + j\omega - (\omega-1))/\omega \rceil \omega,\\ &n_1 - \omega + \lceil (i + j\omega)/\omega \rceil \omega,\\ &\cdots, n_{\omega\text{-}1} - \omega + \lceil (i + j\omega - (\omega-2))/\omega \rceil \omega)\\ = &\; n_i + \lceil (i + j\omega - (\omega-1))/\omega \rceil\, \omega = n_i + j\omega.\end{aligned}$$

Hence we have shown that,

$$\mathcal{R}_p(\xi) = \begin{cases} n_0 + \omega j & \text{if } \xi = 0 + \omega j,\\ n_1 + \omega j & \text{if } \xi = 1 + \omega j,\\ \quad\vdots & \\ n_{\omega-1} + \omega j & \text{if } \xi = (\omega-1) + \omega j.\end{cases}$$

C.3 Proof of Proposition 5 (Product of polynomials):

Due to (23) $p_1 = \bigoplus_{i=1}^{I} v_i \gamma^{n_i}$ and $p_2 = \bigoplus_{l=1}^{L} \bar{v}_l \gamma^{\nu_l}$ can be expressed with a common period $\omega = lcm(\omega_1, \omega_2)$:

$$p_1 = \bigoplus_{i=1}^{I} \Big(\bigoplus_{j=1}^{J_i} \delta^{\varsigma_{i_j}} \Delta_\omega \delta^{\varsigma'_{i_j}} \Big) \gamma^{n_i}, \quad p_2 = \bigoplus_{l=1}^{L} \Big(\bigoplus_{k=1}^{K_l} \delta^{\tau_{l_k}} \Delta_\omega \delta^{\tau'_{l_k}} \Big) \gamma^{\nu_l}.$$

Then the product is obtained by

$$\begin{aligned}p_1 \otimes p_2 &= \left(\bigoplus_{i=1}^{I} \Big(\bigoplus_{j=1}^{J_i} \delta^{\varsigma_{i_j}} \Delta_\omega \delta^{\varsigma'_{i_j}} \Big) \gamma^{n_i} \right) \left(\bigoplus_{l=1}^{L} \Big(\bigoplus_{k=1}^{K_l} \delta^{\tau_{l_k}} \Delta_\omega \delta^{\tau'_{l_k}} \Big) \gamma^{\nu_l} \right)\\ &= \bigoplus_{i=1}^{I} \bigoplus_{l=1}^{L} \left(\Big(\bigoplus_{j=1}^{J_i} \delta^{\varsigma_{i_j}} \Delta_\omega \delta^{\varsigma'_{i_j}} \Big) \Big(\bigoplus_{k=1}^{K_l} \delta^{\tau_{l_k}} \Delta_\omega \delta^{\tau'_{l_k}} \Big) \right) \gamma^{n_i + \nu_l}\\ &= \bigoplus_{i=1}^{I} \bigoplus_{l=1}^{L} \left(\bigoplus_{j=1}^{J_i} \bigoplus_{k=1}^{K_l} \delta^{\varsigma_{i_j}} \Delta_\omega \delta^{\varsigma'_{i_j}} \delta^{\tau_{l_k}} \Delta_\omega \delta^{\tau'_{l_k}} \right) \gamma^{n_i + \nu_l}\\ &= \bigoplus_{i=1}^{I} \bigoplus_{l=1}^{L} \left(\bigoplus_{j=1}^{J_i} \bigoplus_{k=1}^{K_l} \delta^{\varsigma_{i_j} + \lceil (\varsigma'_{i_j} + \tau_{l_k})/\omega \rceil \omega} \Delta_\omega \delta^{\tau'_{l_k}} \right) \gamma^{n_i + \nu_l},\end{aligned}$$

with $J_i \leq \omega$, $K_l \leq \omega$ and complexity $\mathcal{O}(2\omega IL)$.

C.4 Proof of Lemma 1 (Ultimate domination):

Recall that $(\gamma^{\nu}\delta^{\tau})^{*}\delta^{\varsigma}\Delta_{\omega}\delta^{\varsigma'} = \delta^{\varsigma}\Delta_{\omega}\delta^{\varsigma'}(\gamma^{\nu}\delta^{\tau})^{*}$ (Proposition 4, therefore $\tau_1 = k_1\omega,\ k_1 \in \mathbb{N}$ (resp. $\tau_2 = k_2\omega, k_2 \in \mathbb{N}$) and inequality (26) can be expressed by

$$\bigoplus_{j\geq K} \delta^{\varsigma_2+j\tau_2}\Delta_{\omega}\delta^{\varsigma_2'}\gamma^{n_2+j\nu_2} \preceq \bigoplus_{i\geq 0}\delta^{\varsigma_1+i\tau_1}\Delta_{\omega}\delta^{\varsigma_1'}\gamma^{n_1+i\nu_1}.$$

It exists a positive integer K such that inequality (26) holds, if and only if $x \in \mathbb{N}, \forall x \geq K,\ \exists y \in \mathbb{N}$ such that

$$\delta^{x\tau_2}\delta^{\varsigma_2}\Delta_{\omega}\delta^{\varsigma_2'} \preceq \delta^{y\tau_1}\delta^{\varsigma_1}\Delta_{\omega}\delta^{\varsigma_1'};\quad n_2 + x\nu_2 \geq n_1 + y\nu_1. \tag{36}$$

Since $\delta^{\varsigma_1}\Delta_{\omega}\delta^{\varsigma_1'}$ and $\delta^{\varsigma_2}\Delta_{\omega}\delta^{\varsigma_2'}$ are assumed to be canonical monomials then $\varsigma_1' < \omega$ and $\varsigma_2' < \omega$. Furthermore, since s_1 is in the commute form τ_1 is a multiple of ω and therefore $\tau_1 + \varsigma_1' > \varsigma_2'$. We can now rewrite (36),

$$\begin{aligned} &\delta^{x\tau_2}\delta^{\varsigma_2}\Delta_{\omega}\delta^{\varsigma_2'} \preceq \delta^{(y-1)\tau_1}\delta^{\varsigma_1}\Delta_{\omega}\delta^{\varsigma_1'+\tau_1};\ n_2 + x\nu_2 \geq n_1 + y\nu_1 \\ \Leftrightarrow\ &\varsigma_2 + x\tau_2 \leq \varsigma_1 + (y-1)\tau_1;\quad n_2 + x\nu_2 \geq n_1 + y\nu_1 \\ \Leftrightarrow\ &\frac{\varsigma_2 + x\tau_2 - \varsigma_1 + \tau_1}{\tau_1} \leq y \leq \frac{n_2 + x\nu_2 - n_1}{\nu_1}. \end{aligned}$$

Such an integer $y \in \mathbb{Z}$ exists, if

$$1 \leq \frac{n_2 + x\nu_2 - n_1}{\nu_1} - \frac{\varsigma_2 + x\tau_2 - \varsigma_1 + \tau_1}{\tau_1}.$$

This holds for a sufficiently large x, given by

$$x \geq K_1 = \left\lceil \frac{2\nu_1\tau_1 + \nu_1(\varsigma_2 - \varsigma_1) + \tau_1(n_1 - n_2)}{\tau_1\nu_2 - \tau_2\nu_1} \right\rceil.$$

In addition y has to be positive, which is guaranteed, if $x \geq K_2 = \lceil (n_1 - n_2)/v_2 \rceil$. Hence, we can give an upper bound for K in (26), i.e., $K = \max(0, K_1, K_2)$.

C.5 Proof of Proposition 6 (Sum of ultimately periodic series):

We distinguish two cases first: $\sigma(s_1) = \sigma(s_2)$. By defining $N = lcm(\nu_1, \nu_2) = k_1\nu_1 = k_2\nu_2$ and $T = k_1\tau_1 = k_2\tau_2$, then $(\gamma^{\nu_1}\delta^{\tau_1})^{*}$ and $(\gamma^{\nu_2}\delta^{\tau_2})^{*}$ can be written as

$$\begin{aligned} q_1'(\gamma^{N}\delta^{T})^{*} &= (e \oplus \gamma^{\nu_1}\delta^{\tau_1} \oplus \cdots \oplus \gamma^{(k_1-1)\nu_1}\delta^{(k_1-1)\tau_1})(\gamma^{k_1\nu_1}\delta^{k_1\tau_1})^{*}, \\ q_2'(\gamma^{N}\delta^{T})^{*} &= (e \oplus \gamma^{\nu_2}\delta^{\tau_2} \oplus \cdots \oplus \gamma^{(k_2-1)\nu_2}\delta^{(k_2-1)\tau_2})(\gamma^{k_2\nu_2}\delta^{k_2\tau_2})^{*}. \end{aligned}$$

Thus the sum can be written as: $s_1 \oplus s_2 = p_1 \oplus p_2 \oplus (q_1q_1' \oplus q_2q_2')(\gamma^{N}\delta^{T})^{*}$.
Second, $\sigma(s_1) > \sigma(s_2)$. Note that series s_1, s_2 can be expressed with a common period thus one can write,

$$\begin{aligned} s_1 \oplus s_2 =& \tilde{p}_1 \oplus \tilde{p}_2 \oplus \bigoplus_{i=1}^{I} \delta^{\varsigma_{1i}}\Delta_{\omega}\delta^{\varsigma_{1i}'}\gamma^{n_{1i}}(\gamma^{k_1\nu_1}\delta^{\bar{\tau}_1})^{*} \oplus \\ &\bigoplus_{j=1}^{J} \delta^{\varsigma_{2j}}\Delta_{\omega}\delta^{\varsigma_{2j}'}\gamma^{n_{2j}}(\gamma^{k_2\nu_2}\delta^{\bar{\tau}_2})^{*}. \end{aligned}$$

Due to Lemma 1, we can show that $s_1 \oplus s_2$ is ultimately dominated by s_1.

C.6 Proof of Proposition 7 (Product of ultimately periodic series):

Recall that s_1 and s_2 can be expressed in the commute form, Proposition 4. Then product of two series $s_1 = p_1 \oplus q_1(\gamma^{\nu_1}\delta^{\tau_1})^*$ and $s_2 = p_2 \oplus (\gamma^{\nu_2}\delta^{\tau_2})^* q_2$ can be written as

$$s_1 \otimes s_2 = p_1 p_2 \oplus p_1 q_2 (\gamma^{\nu_2}\delta^{\tau_2})^* \oplus p_2 q_1 (\gamma^{\nu_1}\delta^{\tau_1})^* \oplus q_1 (\gamma^{\nu_1}\delta^{\tau_1})^* (\gamma^{\nu_2}\delta^{\tau_2})^* q_2.$$

Clearly, p_1p_2 is a polynomial (Proposition 5). $(\gamma^{\nu_1}\delta^{\tau_1})^*(\gamma^{\nu_2}\delta^{\tau_2})^* = (\gamma^{\nu_1}\delta^{\tau_1} \oplus \gamma^{\nu_2}\delta^{\tau_2})^* = s_3$ is an ultimately periodic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, therefore it is also a series in $\mathcal{T}^*[\![\gamma]\!]$ and $q_1 s_3 q_2 = \tilde{s}_3$ as well. $p_1 q_2(\gamma^{\nu_2}\delta^{\tau_2})^* = \tilde{s}_2$ (resp. $p_2 q_1 (\gamma^{\nu_1}\delta^{\tau_1})^* = \tilde{s}_1$) are two series in $\mathcal{T}^*[\![\gamma]\!]$. Finally we have a sum $p_1p_2 \oplus \tilde{s}_1 \oplus \tilde{s}_2 \oplus \tilde{s}_3$ of ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$, Proposition 6 Appendix C.5.

C.7 Proof of Proposition 8 (Kleene star of a polynomial):

We first investigate a particular case, in which the star of a series in $\mathcal{T}^*[\![\gamma]\!]$ can be calculated similarly to the star of a simple monomial in $\mathcal{T}^*[\![\gamma]\!]$, see (24). Consider the following series $s \in \mathcal{T}^*[\![\gamma]\!]$ where w.l.o.g. τ is a multiple of ω, see Proposition 4 commute form,

$$s = \tilde{S}\Delta_\omega \delta^{\varsigma'} = \Big(\bigoplus_{i=1}^{I} \gamma^{n_{1i}}\delta^{\varsigma_{1i}} \oplus \bigoplus_{j=1}^{J} \gamma^{n_{2j}}\delta^{\varsigma_{2j}} (\gamma^{\nu}\delta^{\tau})^* \Big) \Delta_\omega \delta^{\varsigma'},$$

where $\tilde{S} = P \oplus Q(\gamma^\nu\delta^\tau)^* \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The product ss can be written as

$$\begin{aligned}
ss &= (P \oplus Q(\gamma^\nu\delta^\tau)^*)\Delta_\omega\delta^{\varsigma'}(P \oplus Q(\gamma^\nu\delta^\tau)^*)\Delta_\omega\delta^{\varsigma'} \\
&= \tilde{S}\Delta_\omega\delta^{\varsigma'} P \Delta_\omega \delta^{\varsigma'} \oplus \tilde{S}\Delta_\omega\delta^{\varsigma'} Q(\gamma^\nu\delta^\tau)^*)\Delta_\omega\delta^{\varsigma'} \\
&\quad \text{since, } \Delta_\omega(\gamma^\nu\delta^\tau)^* = (\gamma^\nu\delta^\tau)^*\Delta_\omega \\
&= \tilde{S}\Delta_\omega\delta^{\varsigma'} P\Delta_\omega\delta^{\varsigma'} \oplus \tilde{S}(\gamma^\nu\delta^\tau)^*\Delta_\omega\delta^{\varsigma'} Q\Delta_\omega\delta^{\varsigma'} \\
&\quad \text{due to (17), } \Delta_\omega\delta^{\varsigma'} P\Delta_\omega = P'\Delta_\omega, \quad \Delta_\omega\delta^{\varsigma'} Q\Delta_\omega = Q'\Delta_\omega \\
&= \tilde{S}(P' \oplus Q'(\gamma^\nu\delta^\tau)^*)\Delta_\omega\delta^{\varsigma'} = \tilde{S}\hat{S}\Delta_\omega\delta^{\varsigma'}
\end{aligned}$$

where $\hat{S} = P' \oplus Q'(\gamma^\nu\delta^\tau)^* \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is a series given by

$$\hat{S} = \bigoplus_{i=1}^{I} \gamma^{n_{1i}}\delta^{\lceil(\varsigma_{1i}+\varsigma')/\omega\rceil\omega} \oplus \bigoplus_{j=1}^{J} \gamma^{n_{2j}}\delta^{\lceil(\varsigma_{2j}+\varsigma')/\omega\rceil\omega}(\gamma^\nu\delta^\tau)^*.$$

The star s^* is an ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$, which can be obtained by

$$\begin{aligned}
s^* &= e \oplus \tilde{S}\Delta_\omega\delta^{\varsigma'} \oplus \underbrace{\tilde{S}\Delta_\omega\delta^{\varsigma'}\tilde{S}\Delta_\omega\delta^{\varsigma'}}_{\tilde{S}\hat{S}\Delta_\omega\delta^{\varsigma'}} \oplus \underbrace{\tilde{S}\Delta_\omega\delta^{\varsigma'}\tilde{S}\Delta_\omega\delta^{\varsigma'}\tilde{S}\Delta_\omega\delta^{\varsigma'}}_{\tilde{S}\hat{S}^2\Delta_\omega\delta^{\varsigma'}} \oplus \cdots \\
&= e \oplus \hat{S}^*\tilde{S}\Delta_\omega\delta^{\varsigma'} = e \oplus \hat{S}^* s. \qquad (37)
\end{aligned}$$

Second, a polynomial in $\mathcal{T}^*[\![\gamma]\!]$ can be partitioned into a sum of sub-polynomials in the following form

$$\begin{aligned}
p &= \left(\bigoplus\nolimits_{i=1}^{I} \gamma^{\nu_{1i}}\delta^{\varsigma_{1i}} \right) \Delta_\omega \oplus \left(\bigoplus\nolimits_{j=1}^{J} \gamma^{\nu_{2j}}\delta^{\varsigma_{2j}} \right) \Delta_\omega\delta^{-1} \cdots \\
&\oplus \left(\bigoplus\nolimits_{k=1}^{K} \gamma^{\nu_{\omega k}}\delta^{\varsigma_{\omega k}} \right) \Delta_\omega\delta^{1-\omega}, \\
&= \bigoplus\nolimits_{l=0}^{\omega-1} p_l = p_0 \oplus p_1 \oplus \cdots \oplus p_{\omega-1}.
\end{aligned}$$

where, $p_l = \bigoplus_i \gamma^{\nu_i}\delta^{\varsigma_i}\Delta_\omega\delta^{-l}$. Since $(a \oplus b)^* = (a^*b)^*a^*$,

$$p^* = \Big(\underbrace{(p_0 \oplus \cdots \oplus p_{\omega-2}}_{\bar{p}_{\omega-2}})^* p_{\omega-1}\Big)^* \underbrace{(p_0 \oplus \cdots \oplus p_{\omega-2}}_{\bar{p}_{\omega-2}})^*.$$

Let us define by $\bar{p}_l := p_0 \oplus \cdots \oplus p_l$, thus we can write the star $\bar{p}_l^*$ in a recursive form

$$\bar{p}_l^* = \left(\bar{p}_{l-1}^* p_l\right)^* \bar{p}_{l-1}^*. \tag{38}$$

When we choose $l = 1$ we obtain $\bar{p}_1^* = \left(p_0^* p_1\right)^* p_0^*$, since $\bar{p}_0 = p_0 = \bigoplus_{i=1}^I \gamma^{\nu_{1i}}\delta^{\varsigma_{1i}}\Delta_\omega$. Due to (37), p_0^* is given by

$$p_0^* = \mathrm{e} \oplus \underbrace{\Big(\bigoplus_{i=1}^I \gamma^{\nu_i}\delta^{\lceil \varsigma_i/\omega \rceil \omega}\Big)^* \bigoplus_{i=1}^I \gamma^{\nu_{1i}}\delta^{\varsigma_{1i}}}_{\tilde{S}_0}\Delta_\omega.$$

This star can be rewritten as $p_0^* = \mathrm{e} \oplus (\tilde{S}_0)\Delta_\omega$ where $\tilde{S}_0$ is a series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The product $p_0^* p_1$ is ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$, since

$$\begin{aligned}
p_0^* p_1 &= (\mathrm{e} \oplus (\tilde{S}_0)\Delta_\omega)\left(\bigoplus_{j=1}^J \gamma^{\nu_{2j}}\delta^{\varsigma_{2j}}\Delta_\omega\delta^{-1}\right),\\
&= \bigoplus_{j=1}^J \gamma^{\nu_{2j}}\delta^{\varsigma_{2j}}\Delta_\omega\delta^{-1} \oplus \tilde{S}_0\Delta_\omega\left(\bigoplus_{j=1}^J \gamma^{\nu_{2j}}\delta^{\varsigma_{2j}}\Delta_\omega\delta^{-1}\right),\\
&= \left(\bigoplus_{j=1}^J \gamma^{\nu_{2j}}\delta^{\varsigma_{2j}} \oplus \tilde{S}_0 \bigoplus_{j=1}^J \gamma^{\nu_{2j}}\delta^{\lceil \varsigma_{2j}/\omega \rceil \omega}\right)\Delta_\omega\delta^{-1},\\
&= \tilde{S}_{01}\Delta_\omega\delta^{-1},
\end{aligned}$$

where $\tilde{S}_{01}$ is again a series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Therefore, the star $(p_0^* p_1)^*$ can be calculated by using (37). It is an ultimately periodic series $\mathcal{T}^*[\![\gamma]\!]$. Then $\bar{p}_1^* = (p_0^* p_1)^* p_0^*$ is the product of two ultimately periodic series in $\mathcal{T}^*[\![\gamma]\!]$, see Proposition 7 Appendix C.6. In a similar way with $\bar{p}_1^*$ we can solve successively the recursive equation (38) $\forall i \in \{1, \cdots, \omega - 1\}$.

C.8 Proof of Proposition 9 (Kleene star of an ultimately periodic series):

Recall that for $r = (\gamma^\nu\delta^\tau)$, $qr^* = r^*q$, Proposition 4. The star of ultimately periodic series can be rewritten as a star of polynomials $s^* = (p \oplus qr^*)^* = p^*(qr^*p^*)^* = p^*(q(r \oplus p)^*)^* = p^*(\mathrm{e} \oplus q(q \oplus r \oplus p)^*)$, (Baccelli et al (1992)).

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