Control Synthesis for P-Temporal Event Graphs

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***Abstract*— P-temporal Petri Nets are convenient tools to model manufacturing systems whose activities times are included between a minimum and a maximum value. The typical feature of these nets is that a control of the firing dates of the transitions is required for the controlled system to behave as close as possible to the specified outputs and to ensure the liveness of tokens, when possible. This paper aims at designing a control law which is obtained by using residuated and dual residuated mappings. An example is given to illustrate the proposed approach.**

I. INTRODUCTION

This work aims at computing a control law for some systems characterized by synchronization and delay phenomena. A broad variety of applications are concerned. Among them, one can cite:
- Transportation systems [1],
- Production systems [2],
- Interface checking of hardware modules ([3],[4]),
- Software checking [5].

In order to achieve this objective we consider Timed-Event Graphs (TEG) which are Timed Petri Nets in which each place has a single upstream and a single downstream transition. TEGs appropriately model DES characterized by delay and synchronization phenomena. The events considered are the transitions firing. Therefore we will handle sequences (set of firing dates), called dater in the sequel. By considering these trajectories for each transition, the behavior of a TEG can be described by linear equations in some idempotent semiring such as (max,+) algebra. These linear models have allowed many achievements on the performance evaluation [6] and on the control of DES modeled by TEG. Control of TEG consists in the control of the dates of tokens input in the graph. Classically the control synthesis is done in order to optimize the just-in-time criterion. The objective is then to compute the latest dates of tokens input while the tokens outputs occur before the desired output dates. Many strategies exist in literature (see [7] for a survey). The historical problem of control was the optimal open loop control (see [8],[9] and [10]). In this paper, we give an extension of TEG models which is called P-temporal event graphs (p-TEG) in order to model the sojourn time of tokens in the places. The model is linear in the algebraic structure considered and must satisfy constraints which are non linear in this same algebraic setting. The control problem of these model is considered in order to optimize the just-in-time criterion. The design goal is to achieve some performance while minimizing internal stocks and ensuring the liveness of tokens. Optimal open-loop control and closed-loop control are considered in this paper.

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II. ALGEBRAIC PRELIMINARIES

Definition 2.1: A dioid $\mathscr{D}$ is a set endowed with two internal operations denoted by $\oplus$ (addition) and $\otimes$ (multiplication), both associative and both having neutral elements denoted by ε and e respectively, such that $\oplus$ is also commutative and idempotent (i.e. $a\oplus a = a$). The $\otimes$ operation is distributive with respect to $\oplus$, and ε is absorbing for the product (i.e. $\varepsilon\otimes a = a\otimes\varepsilon = \varepsilon$, $\forall a$). Dioids can be endowed with a natural order : $a \succeq b$ iff $a = a\oplus b$. A dioid is complete if every subset $\mathscr{A}\subseteq\mathscr{D}$ admits a least upper bound equal to $\bigoplus_{x\in\mathscr{A}} x$ and if $\otimes$ distributes over finite and infinite sums. The greatest element of a Dioid is denoted $\top = \bigoplus_{x\in\mathscr{D}} x$. A complete dioid have a complete lattice structure, and then $a\succeq b \Leftrightarrow b = a\wedge b$.

Theorem 2.1: ([8],[9]) Over a complete dioid $\mathscr{D}$, the implicit equation $x = ax\oplus b$ admits $x = a^*b$ as least solution, where $a^* = \oplus_{i\in\mathbb{N}} a^i$ with $a^0 = e$. Furthermore, if $x,\ y\in\mathscr{D}$, we have

$$x(yx)^* = (xy)^*x, \tag{1}$$

$$x^*\otimes x^* = x^* \text{ and} \tag{2}$$

$$(x\oplus y)^* = (x^*y)^*x^* = x^*(yx^*)^*. \tag{3}$$

Remark 2.1: (Matrix dioid) We can extend the notion of scalar dioid to matrix dioid by considering the following two internal operations;
Let $A,B\in\mathscr{D}^{n\times p}$ and $C\in\mathscr{D}^{p\times q}$

$$(A\oplus B)_{ij} = A_{ij}\oplus B_{ij}$$
and
$$(A\otimes C)_{ij} = \bigoplus_{k=1}^{k=p} A_{ik}\otimes C_{kj}$$

Definition 2.2: (Isotone mapping) A mapping Π from an ordered set $\mathscr{D}$ into an ordered set $\mathscr{C}$ such that:
$$\forall\ a,\ b\in\mathscr{D},\ a\preceq b \Rightarrow \Pi(a)\preceq\Pi(b).$$

Lemma 2.1: ([8]) Let Π be a mapping from a dioid $\mathscr{D}$ into another dioid $\mathscr{C}$. The following statements are equivalent:

1. the mapping Π is isotone;
2. the mapping Π is a $\oplus$-supermorphism, that is,
$$\forall\ a,\ b\in\mathscr{D},\ \Pi(a\oplus b)\succeq\Pi(a)\oplus\Pi(b).$$

3. if lower bounds exist in $\mathscr{D}$ and $\mathscr{C}$, Π is a $\wedge$-submorphism, that is,

$$\forall\ a,\ b \in \mathscr{D},\ \Pi(a\wedge b) \preceq \Pi(a)\wedge\Pi(b).$$

Lemma 2.2: ([8]) If a admits a left inverse b and a right inverse c, then

- $b = c$ and this unique inverse is denoted a^{-1};
- moreover, $\forall x,y,\ a(x\wedge y) = ax\wedge ay$.

The same holds true for right multiplication by a, and also for right and left multiplication by a^{-1}.

Definition 2.3: ([11],[8]) A multiplicative lattice-ordered group $\mathscr{G}$, means that, in addition to being a group and a lattice, the multiplication is isotone, and

- the multiplication is necessarily distributive with respect to both the upper and the lower bounds ($\mathscr{G}$ is called a reticulated group),
- moreover, the lattice is distributive (that is, upper and lower bounds are distributive with respect to one another).

Theorem 2.2: ([8][§4.3.5]) Let $\mathscr{G}$ be a multiplicative lattice-ordered group, and $a,b \in \mathscr{G}$. Since each element of $\mathscr{G}$ admits an inverse, one has the remarkable formulae:

$$(a\wedge b)^{-1} = a^{-1}\oplus b^{-1},\ (a\oplus b)^{-1} = a^{-1}\wedge b^{-1},$$
$$a\wedge b = a(a\oplus b)^{-1}b,$$

Proposition 2.1: Let $\mathscr{G}$ be a multiplicative lattice-ordered group, a and b in $\mathscr{G}$ and $x\in\mathscr{D}$ with $\mathscr{D}$ a dioid and $\mathscr{G}\subseteq\mathscr{D}$ then

$$ax\wedge bx = (a\wedge b)x.$$

Proof: First, using lemma (2.1) we have

$$(a\wedge b)x \preceq ax\wedge bx.$$

Second, since a and b are in $\mathscr{G}$ from theorem (2.2), we have

$$(a\wedge b)^{-1} = a^{-1}\oplus b^{-1},\ \text{therefore,}$$

$$\begin{aligned} ax\wedge bx &= (a\wedge b)(a^{-1}\oplus b^{-1})(ax\wedge bx)\\ &= (a\wedge b)[(x\wedge a^{-1}bx)\oplus(b^{-1}ax\wedge x)]\\ &\preceq (a\wedge b)[x\oplus x] = (a\wedge b)x \end{aligned}$$

this leads to

$$ax\wedge bx = (a\wedge b)x.$$

■

Definition 2.4: In the sequel, we will endow the dioid $\mathscr{D}$ with the product $a\odot b = a\otimes b$ with the following convention $\varepsilon\odot\top = \top$ (we recall that $\varepsilon\otimes\top = \varepsilon$). And over matrix dioid, $(A\odot B)_{ij} = \bigwedge_{k=1}^{n} A_{ik}\odot B_{kj}$ with $A\in\mathscr{D}^{p\times n}$ and $B\in\mathscr{D}^{n\times q}$.

In the sequel, $e^{\odot}\in\mathscr{D}^{n\times n}$ is the identity matrix of the law $\odot$, i.e, $e^{\odot}_{ii} = e$, and $e^{\odot}_{ij} = \top$ if $i\neq j$.

Definition 2.5: In the next, we will consider the dual star operator which is given by:

$g_* = \bigwedge_{i\in N} g^{\odot i}$ with $g^{\odot i} = \underbrace{g\odot\cdots\odot g}_{\text{i times}}$ and $g^{\odot 0} = e^{\odot}$.

Proposition 2.2: Let $\mathscr{G}$ be a reticulated group and $A,B\in\mathscr{D}^{p\times n}$ be two matrices with each entry in $\mathscr{G}$ and $x\in\mathscr{D}^{n\times q}$ then we have

$$(A\wedge B)\odot x = A\odot x\wedge B\odot x$$

Proof: Let $A,B\in\mathscr{D}^{p\times n}$ be two matrices with each entry in $\mathscr{G}$ then

$$\begin{aligned} \big((A\wedge B)\odot x\big)_{ij} &= \bigwedge_{k=1}^{k=n}(A\wedge B)_{ik}\odot x_{kj}\\ &= \bigwedge_{k=1}^{k=n}(A_{ik}\wedge B_{ik})\odot x_{kj}\\ &= \bigwedge_{k=1}^{k=n}(A_{ik}\odot x_{kj})\wedge(B_{ik}\odot x_{kj})\\ &\quad \text{thanks to proposition (2.1)}\\ &= \Big(\bigwedge_{k=1}^{k=n}(A_{ik}\odot x_{kj})\Big)\wedge\Big(\bigwedge_{k=1}^{k=n}(B_{ik}\odot x_{kj})\Big)\\ &= (A\odot x\wedge B\odot x)_{ij}. \end{aligned}$$

■

Residuation theory allows a kind of pseudo-inversion of mapping defined over lattices, it plays a central role in the control of systems. For (max,+) linear systems. We refer the reader to [9] and [7] for an introduction.

Definition 2.6: Let f be a mapping from a complete dioid $\mathscr{D}$ to a complete dioid $\mathscr{C}$, f is lower semi-continuous (l.s.c), respectively, upper semi-continuous (u.s.c), if for all subsets (finite or infinite) X of $\mathscr{D}$

$$f\Big(\bigoplus_{x\in X} x\Big) = \bigoplus_{x\in X} f(x),$$

respectively

$$f\Big(\bigwedge_{x\in X} x\Big) = \bigwedge_{x\in X} f(x).$$

The function f is continuous if it is both l.s.c and u.s.c.

Definition 2.7: An isotone mapping $f:\mathscr{D}\to\mathscr{C}$, where $\mathscr{D}$ and $\mathscr{C}$ are ordered sets, is a residuated mapping if for all $y\in\mathscr{C}$, the least upper bound of the subset $\{x|f(x)\preceq y\}$ exists and belongs to it. It is denoted by $f^{\#}(y)$, and $f^{\#}$ is called the residual of f.

An isotone mapping $g:\mathscr{D}\to\mathscr{C}$ is a dual residuated mapping if for all $y\in\mathscr{C}$, the greatest lower bound of the subset $\{x|g(x)\succeq y\}$ exists and belongs to it. It is then denoted by $g^{\flat}(y)$, and $g^{\flat}$ is called the dual residual of g.

Theorem 2.3: ([8]) Let f,g be isotone mappings from : $\mathscr{D}$ to $\mathscr{C}$, where $\mathscr{D}$ and $\mathscr{C}$ are ordered sets, the following equivalences holds true:

f is a residuated $\Leftrightarrow$ $f\circ f^{\#}\preceq Id_{\mathscr{C}}$ and $f^{\#}\circ f\succeq Id_{\mathscr{D}}$ $\Leftrightarrow$ f is l.s.c and $f(\varepsilon) = \varepsilon$.

g is dual residuated $\Leftrightarrow$ $g\circ g^{\flat}\succeq Id_{\mathscr{C}}$ and $g^{\flat}\circ g\preceq Id_{\mathscr{D}}$, $\Leftrightarrow$ g is u.s.c and $g(\top) = \top$.

Example 2.1: ([8]) The mapping $L_a:\mathscr{D}\to\mathscr{D},\ x\mapsto a\otimes x$ is isotone and l.s.c (*i.e.*, $L_a(\bigoplus_{x\in X} x) = \bigoplus_{x\in X} L_a(x)$), then it is residuated. The residual is denoted $L_a^{\#}(x) = a\backslash x$ in (max, +) literature. We recall that $\varepsilon\backslash x = \top$, $\top\backslash x = \varepsilon$ and $\top\backslash\top = \top$.

Many properties are proposed in ([8] Table 4.1) and ([7] Annex A). In particular:

$$(a\backslash b)/c = a\backslash(b/c) = a\backslash b/c \tag{4}$$

$$b/b = (b/b)^*. \tag{5}$$

Proposition 2.3: ([8, §4.4.2]) If $\Pi:\mathscr{D}\to\mathscr{C}$ and $\Phi:\mathscr{C}\to\mathscr{B}$ are dually residuated mappings, then $\Phi\circ\Pi$ is also dually residuated and

$$(\Phi\circ\Pi)^{\flat} = \Pi^{\flat}\circ\Phi^{\flat}.$$

$\Pi\wedge\Phi$ is also dually residuated and

$$(\Pi\wedge\Phi)^{\flat} = \Pi^{\flat}\oplus\Phi^{\flat}.$$

Proposition 2.4: If each entry of A admits an inverse then the mapping $\Gamma_A : x \mapsto A \odot x$ is u.s.c, with x an element of $\mathscr{D}^{n\times q}$, that is:

$$\Gamma_A(\bigwedge_{x\in X} x) = \bigwedge_{x\in X} \Gamma_A(x)$$

Proof: $\Gamma_A(\bigwedge_{x\in X} x) = A \odot (\bigwedge_{x\in X} x)$

$\Rightarrow (\Gamma_A(\bigwedge_{x\in X} x))_{ij} = \bigwedge_{k=1}^{n} A_{ik} \odot (\bigwedge_{x\in X} x_{kj})$ see proposition (2.1),

$$\begin{aligned}(\Gamma_A(\bigwedge_{x\in X} x))_{ij} &= \bigwedge_{k=1}^{n} \bigwedge_{x\in X} (A_{ik} \odot x_{kj})\\ &= \bigwedge_{x\in X} \bigwedge_{k=1}^{n} (A_{ik} \odot x_{kj})\\ &= \bigwedge_{x\in X} (\Gamma_A(x))_{ij}\end{aligned}$$

then

$$\Gamma_A(\bigwedge_{x\in X} x) = \bigwedge_{x\in X} \Gamma_A(x)$$

■

Corollary 2.1: Let $A \in \mathscr{D}^{n\times n}$, $X \in \mathscr{D}^{n\times q}$ be two matrices, if each entry of A admits an inverse then the mapping $\Gamma_A : x \mapsto A \odot x$ is dually residuated and its dual residual is given by $\Gamma_A^{\flat} : x \mapsto A \mathbin{\bullet\!\!\!\backslash} x$ with:

$$(A \mathbin{\bullet\!\!\!\backslash} x)_{ij} = \bigoplus_{l=1}^{l=n} A_{li} \mathbin{\bullet\!\!\!\backslash} x_{lj} = \bigoplus_{l=1}^{l=n} A_{li}^{-1} \odot x_{lj}$$

and by respecting the following rules:

$\top \mathbin{\bullet\!\!\!\backslash} x = \varepsilon$, $\varepsilon \mathbin{\bullet\!\!\!\backslash} x = \top$ and $\varepsilon \mathbin{\bullet\!\!\!\backslash} \varepsilon = \varepsilon$ (*i.e.,* $\varepsilon^{-1} \odot \varepsilon = \varepsilon$).

Proof: The result is direct application of theorem (2.3). ■

It is important to note that:

$$a \succeq b \Rightarrow a \mathbin{\bullet\!\!\!\backslash} x \preceq b \mathbin{\bullet\!\!\!\backslash} x.$$

Furthermore, if b admits an inverse we have:

$$b \mathbin{\bullet\!\!\!\backslash} (a \otimes c) = (b \mathbin{\bullet\!\!\!\backslash} a) \otimes c \text{ (i.e., } b^{-1} \odot (a \otimes c) = (b^{-1} \odot a) \otimes c).$$

Proposition 2.5: Let $\mathscr{G}$ be a reticulated group, $A, B \in \mathscr{D}^{p\times n}$ two matrices with each entry in $\mathscr{G}$ and $x \in \mathscr{D}^{n\times q}$ then we have

$$(A \wedge B) \mathbin{\bullet\!\!\!\backslash} x = A \mathbin{\bullet\!\!\!\backslash} x \oplus B \mathbin{\bullet\!\!\!\backslash} x.$$

Proof: The result is a direct application of proposition (2.3). ■

Theorem 2.4: ([8, §4.5]) Let $A \in \mathscr{D}^{n\times n}$, the following equivalences holds true:

$$x = A^* \otimes x \Leftrightarrow x \succeq A \otimes x \Leftrightarrow A \mathbin{\circ\!\!\!\backslash} x \succeq x \Leftrightarrow A^* \mathbin{\circ\!\!\!\backslash} x = x.$$

Corollary 2.2: The greatest solution of $Ax \preceq x$ and $x \preceq B$ is $A^* \mathbin{\circ\!\!\!\backslash} B$.

Proposition 2.6: Let $G \in \mathscr{D}^{n\times n}$ with each entry in a reticulated group then the following equivalences holds true:

$$x \preceq G \odot x \Leftrightarrow G \mathbin{\bullet\!\!\!\backslash} x \preceq x \Leftrightarrow G_* \mathbin{\bullet\!\!\!\backslash} x = x \Leftrightarrow G_* \odot x = x.$$

Proof: First, we prove that:

$$x \preceq G \odot x \Rightarrow G \mathbin{\bullet\!\!\!\backslash} x \preceq x.$$

If $x \preceq G \odot x$ then $G \mathbin{\bullet\!\!\!\backslash} x \preceq G \mathbin{\bullet\!\!\!\backslash} (G \odot x)$ since $(G \mathbin{\bullet\!\!\!\backslash} .)$ is isotone, furthermore theorem (2.3) yields $G \mathbin{\bullet\!\!\!\backslash} (G \odot x) \preceq x$, then

$$G \mathbin{\bullet\!\!\!\backslash} x \preceq x.$$

Second, we prove that:

$$G \mathbin{\bullet\!\!\!\backslash} x \preceq x \Rightarrow G_* \mathbin{\bullet\!\!\!\backslash} x = x.$$

If $x \succeq G \mathbin{\bullet\!\!\!\backslash} x \Rightarrow x \succeq (e^{\odot} \mathbin{\bullet\!\!\!\backslash} x) \oplus (G \mathbin{\bullet\!\!\!\backslash} x) \oplus (G^{\odot 2} \mathbin{\bullet\!\!\!\backslash} x) \oplus \cdots$
and thanks to proposition (2.5)
$x \succeq G \mathbin{\bullet\!\!\!\backslash} x \Rightarrow x \succeq (e^{\odot} \wedge G \wedge G^{\odot 2} \wedge \cdots) \mathbin{\bullet\!\!\!\backslash} x = G_* \mathbin{\bullet\!\!\!\backslash} x \succeq e^{\odot} \mathbin{\bullet\!\!\!\backslash} x = x$
then

$$x = G_* \mathbin{\bullet\!\!\!\backslash} x.$$

Third, we prove that

$$x = G_* \mathbin{\bullet\!\!\!\backslash} x \Rightarrow x = G_* \odot x$$

$x = G_* \mathbin{\bullet\!\!\!\backslash} x \Rightarrow G_* \odot x = G_* \odot (G_* \mathbin{\bullet\!\!\!\backslash} x) \succeq x$ (see theorem (2.3)), but $G_* \odot x \preceq e^{\odot} \odot x = x$, then $G_* \odot x = x$.
Fourth, we prove that $G_* \odot x = x \Rightarrow x \preceq G \odot x$.
Thanks to proposition (2.4),
$G_* \odot x = (x \wedge G \odot x \wedge G^{\odot 2} \odot x \wedge \cdots) \preceq G \odot x$. ■

Proposition 2.7: Let $A \in \mathscr{D}^{n\times p}$, $X \in \mathscr{D}^{p\times q}$ and $B \in \mathscr{D}^{n\times n}$ be three matrices. If each element B_{ij} admits an inverse then we have:

$$B \mathbin{\bullet\!\!\!\backslash} (A \otimes X) = (B \mathbin{\bullet\!\!\!\backslash} A) \otimes X.$$

Proof: $(B \mathbin{\bullet\!\!\!\backslash} (A \otimes X))_{ij} = \bigoplus_{l=1}^{l=n} B_{li} \mathbin{\bullet\!\!\!\backslash} (A \otimes X)_{lj}$

$$\begin{aligned}&= \bigoplus_{l=1}^{l=n} B_{li} \mathbin{\bullet\!\!\!\backslash} (\bigoplus_{k=1}^{k=p} A_{lk} \otimes X_{kj})\\ &= \bigoplus_{l=1}^{l=n} \bigoplus_{k=1}^{k=p} B_{li} \mathbin{\bullet\!\!\!\backslash} (A_{lk} \otimes X_{kj}) \text{ since } \Gamma_B^{\flat} \text{ is l.s.c}\\ &= \bigoplus_{k=1}^{k=p} \bigoplus_{l=1}^{l=n} (B_{li} \mathbin{\bullet\!\!\!\backslash} A_{lk}) \otimes X_{kj} \text{ since } B_{li} \text{ admits an inverse}\\ &= \bigoplus_{k=1}^{k=p} (B \mathbin{\bullet\!\!\!\backslash} A)_{ik} \otimes X_{kj} = ((B \mathbin{\bullet\!\!\!\backslash} A) \otimes X)_{ij}.\end{aligned}$$

■

Proposition 2.8: Let us consider a dioid $\mathscr{D}$, a reticulated group $\mathscr{G} \subseteq \mathscr{D}$ and two matrices $A, G \in \mathscr{D}^{n\times n}$ and each entry of G in $\mathscr{G}$. The greatest x such that:

$$A \otimes x \preceq x \preceq G \odot x \text{ and } x \preceq B$$

is given by

$$\hat{x} = ((G_* \mathbin{\bullet\!\!\!\backslash} A^*)^*) \mathbin{\circ\!\!\!\backslash} B$$

Proof: First, we prove that: $A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B \Rightarrow x \preceq \hat{x}$.
Second we prove that $\hat{x}$ satisfy the following properties

(i)	$\hat{x}$	$\preceq$	B
(ii)	$\hat{x}$	$=$	$A^* \otimes \hat{x}$
(iii)	$\hat{x}$	$=$	$G_* \odot \hat{x}$

By considering propositions (2.6) and theorem (2.4), $A \otimes x \preceq x \preceq G \odot x$ implies that
$x = G_* \odot x = G_* \mathbin{\bullet\!\!\!\backslash} x = A^* \otimes x$, which means that:

$$x \in ImG_* \cap ImA^*.$$

Then, x must be such that $x = G_* \mathbin{\bullet\!\!\!\backslash} (A^* x)$. The assumption about entries of G and proposition (2.7) leads to
$x = (G_* \mathbin{\bullet\!\!\!\backslash} A^*) \otimes x \Rightarrow x \preceq (G_* \mathbin{\bullet\!\!\!\backslash} A^*) \mathbin{\circ\!\!\!\backslash} x$, which is equivalent to $x = ((G_* \mathbin{\bullet\!\!\!\backslash} A^*)^*) \mathbin{\circ\!\!\!\backslash} x$ (see theorem (2.4)).
Then $A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B \Leftrightarrow x = ((G_* \mathbin{\bullet\!\!\!\backslash} A^*)^*) \mathbin{\circ\!\!\!\backslash} x$ and $x \preceq B \Leftrightarrow x \preceq \hat{x} = ((G_* \mathbin{\bullet\!\!\!\backslash} A^*)^*) \mathbin{\circ\!\!\!\backslash} B$. According to theorem (2.4) $\hat{x}$ is such that

$$(G_* \mathbin{\bullet\!\!\!\backslash} A^*) \otimes \hat{x} \preceq \hat{x} \preceq (G_* \mathbin{\bullet\!\!\!\backslash} A^*) \mathbin{\circ\!\!\!\backslash} \hat{x} \qquad (6)$$

Now it suffices to prove that (i), (ii) and (iii) are respected
First we prove that $\hat{x} \in ImA^*$, according to theorem (2.4),

this is equivalent to $\hat{x} = A^* \otimes \hat{x} = A^* \circ\!\!\!\backslash \hat{x}$
$\hat{x} \preceq (G_* \bullet\!\!\!\backslash A^*) \circ\!\!\!\backslash \hat{x}$
$\Rightarrow A^* \circ\!\!\!\backslash \hat{x} \preceq A^* \circ\!\!\!\backslash ((G_* \bullet\!\!\!\backslash A^*) \circ\!\!\!\backslash \hat{x}) = ((G_* \bullet\!\!\!\backslash A^*) \otimes A^*) \circ\!\!\!\backslash \hat{x}$
$= (G_* \bullet\!\!\!\backslash (A^* \otimes A^*)) \circ\!\!\!\backslash \hat{x}$ (see proposition (2.7))
$= (G_* \bullet\!\!\!\backslash A^*) \circ\!\!\!\backslash \hat{x} \succeq \hat{x}$ since $(G_* \bullet\!\!\!\backslash A^*) \succeq e$ (see theorem (2.1)).
Furthermore, $\hat{x} \succeq A^* \circ\!\!\!\backslash \hat{x}$ (since $A^* \succeq e$), then $\hat{x} = A^* \circ\!\!\!\backslash \hat{x}$, *i.e.*, $\hat{x} \in ImA^*$.
Second, we prove that $\hat{x} \in ImG_*$, *i.e.*, $\hat{x} = G_* \odot \hat{x} = G_* \bullet\!\!\!\backslash \hat{x}$ (see proposition (2.6)), from equation (6),
$\hat{x} \succeq (G_* \bullet\!\!\!\backslash A^*) \otimes \hat{x} = G_* \bullet\!\!\!\backslash (A^* \otimes \hat{x}) = G_* \bullet\!\!\!\backslash \hat{x}$ since $\hat{x} = A^* \otimes \hat{x}$.
On the other side $G_* \preceq e^{\odot}$ then $\hat{x} \preceq G_* \bullet\!\!\!\backslash \hat{x}$ so
$\hat{x} = G_* \bullet\!\!\!\backslash \hat{x} = G_* \odot \hat{x}$.
Third, since $(G_* \bullet\!\!\!\backslash A^*)^* \succeq e$, $\hat{x} \preceq B$.
To summarize, $\hat{x}$ is the greatest solution of
$A \otimes x \preceq x \preceq G \odot x$ and $x \preceq B$. ■

Corollary 2.3: The previous result means that a state trajectory x respects the constraints of the system
$\Leftrightarrow x \in ImA^* \cap ImG_*$
$\Leftrightarrow x = G_* \odot x$ and $x = A^* x \Leftrightarrow x = (G_* \bullet\!\!\!\backslash A^*)^* x$.

III. CONTROL FOR P-TEMPORAL EVENT GRAPHS

A. P-temporal event graphs

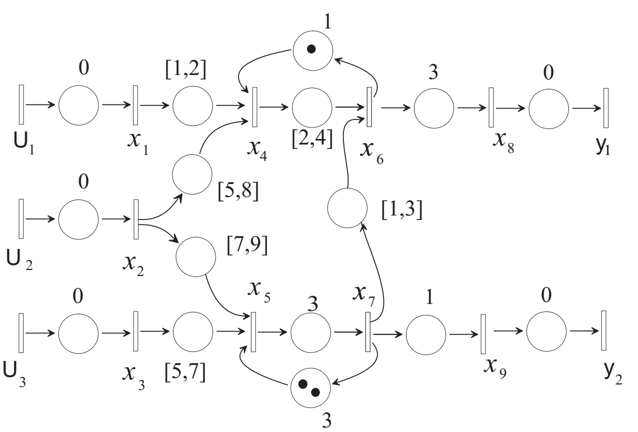

Figure 1: A p-temporal event graph

The P-temporal Petri net model defined in [13], enables to model manufacturing systems whose activities times are included between a minimum and a maximum value, that is the sojourn time associated to each place is included in an interval $[s_{\min_i}, s_{\max_i}]$ with $0 \leq s_{\min_i} \leq s_{\max_i}$. Before the duration $s_{\min_i}$, the token in p_i is in the non-available state. After $s_{\min_i}$ and before $s_{\max_i}$, the token in p_i is in the available state for the firing of a transition. After $s_{\max_i}$, the token in p_i is dead. The "death" of a token has to be seen as a constraint violation.

A transition is fired as soon as there is an available token in each upstream place. The behavior of a transition may be described as a sequence of firing dates. The variable $x(k)$ is a "dater" and it represents the k^{th} firing date of the transition labeled x. For each increasing sequence $x(k)$, it is possible to define the transformation $x(\gamma) = \bigoplus_{k \in \mathbb{Z}} x(k) \otimes \gamma^k$, where γ is a backward shift operator in event domain (that is $y(\gamma) = \gamma x(\gamma) \Leftrightarrow y(\gamma) = x(k-1)$, see [8], p. 228). This transformation is analogous to the Z-transform used in discrete-time classical control theory and the formal series $x(\gamma)$ is a synthetic representation of the trajectory $\{x(k)\}_{k \in \mathbb{Z}}$. The set of formal series in γ is denoted $\mathbb{Z}_{\max}[\![\gamma]\!]$ and constitutes a dioid. In general, the behavior of a P-TEG (*i.e.*, the firing sequence of each transition) can be represented by linear relations over this dioid:

$$\begin{cases} x(\gamma) &= \underline{A} x(\gamma) \oplus \underline{B} u(\gamma) = \underline{A}^* \underline{B} u(\gamma), \\ y(\gamma) &= \underline{C} x(\gamma) = \underline{C}\underline{A}^* \underline{B} u(\gamma), \end{cases} \tag{7}$$

but trajectories must respect non-linear constraints which can be given by:

$$\begin{cases} x(\gamma) &\preceq \overline{A} \odot x(\gamma) \wedge \overline{B} \odot u(\gamma), \\ y(\gamma) &\preceq \overline{C} \odot x(\gamma). \end{cases} \tag{8}$$

in which $\odot$ is given in definition (2.4), and $\overline{A}$ represents the constraints relations between internal transitions, $\overline{B}$ represents the constraints relations between internal transitions and input transitions and $\overline{C}$ represents the constraints relations between output transitions and internal transitions. Entries of matrices $\overline{A}$, $\overline{B}$ and $\overline{C}$ are assumed to be in the reticulated group $\overline{\mathbb{Z}}_{\max} \subset \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

Example 3.1: Figure 1 gives an example of P-temporal timed event graph. We suppose that there is no constraint on the input and output transition (this condition does not affect the generality of results).
The behavior of the P-temporal event graph can be represented by

$$\underline{A} \otimes X(\gamma) \preceq X(\gamma) \preceq \overline{A} \odot X(\gamma).$$

$$\underline{A} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & 5 & \varepsilon & \varepsilon & \varepsilon & 1\gamma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 7 & 5 & \varepsilon & \varepsilon & \varepsilon & 3\gamma^2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \end{pmatrix},$$

$$\overline{A} = \begin{pmatrix} \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ 2 & 8 & \top & \top & \top & \top & \top & \top & \top \\ \top & 9 & 7 & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & 4 & \top & \top & 3 & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \\ \top & \top & \top & \top & \top & \top & \top & \top & \top \end{pmatrix},$$

$$B = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, C = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}.$$

Entry $\underline{A}(5,7) = 3\gamma^2$, corresponds to the place between transition x_7 and x_5, and means that there are two tokens in the place and that the minimum sojourn time is 3 time units. $\overline{A}(6,4) = 4$ means that the tokens in the place between transitions x_4 and x_6 must not stay more than 4 time units. The entries equal to $\top = +\infty$, mean that there is no constraint on the sojourn time. It must be noted that each entry admits an inverse (with the convention $\top^{-1} = \varepsilon$). This assumption is essential to solve the control problem.

B. Optimal control of p-temporal event graphs

The linear model in dioid has allowed many developments on their control [7]. TEG control problems are usually stated in a just-in-time context, that is we seek the greatest firing dates of the input transitions in order to obtain output transition firing dates lower than the given desired outputs. The inputs achieving this objective are computed thanks to the desired output; roughly speaking it corresponds to invert the system.
From a mathematical point of view, as long as the transfer function (matrix, in the MIMO case) has to be inverted, it is no surprise that residuation plays an essential role.

Similarly, let us suppose a sequence of dates for which one would like to see events occur at the latest, and we are asked to provide the latest input dates that would meet this objective under the constraints of the p-temporal event graph.

This means that we seek the greatest control u which satisfies the constraints on the output:

$$y = C\underline{A}^*\underline{B}u \preceq z, \tag{9}$$

and which respects the constraints imposed by the model, that is:

$$\begin{cases} x(\gamma) & \preceq & \overline{A} \odot x(\gamma) \wedge \overline{B} \odot u(\gamma), \\ y(\gamma) & \preceq & \overline{C} \odot x(\gamma). \end{cases} \tag{10}$$

Example 3.2: Let us consider the following desired outputs

k	0	1	2	3
$z_1(k)$	17	21	22	$+\infty$
$z_2(k)$	17	22	25	$+\infty$

z can be expressed as a formal power series in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ as the following

$$z = \begin{pmatrix} 17\gamma^0 \oplus 21\gamma^1 \oplus 22\gamma^2 \oplus +\infty\gamma^3 \\ 17\gamma^0 \oplus 22\gamma^1 \oplus 25\gamma^2 \oplus +\infty\gamma^3 \end{pmatrix}.$$

Thanks to proposition (2.8), the greatest X such that:

$$y = C\underline{A}^*x \preceq z,$$

and which respects the constraints imposed by the model,
$x_{opt} \preceq x_0 = \underline{A}^*B(C\underline{A}^*B)\circ\backslash z$
$\underline{A} \otimes x_{opt}(\gamma) \preceq x_{opt}(\gamma) \preceq \overline{A} \odot x_{opt}(\gamma)$.
is given by:
$x_{opt} = ((\overline{A}_* \bullet\backslash \underline{A}^*)^*)\circ\backslash x_0 = \overline{A}^*\circ\backslash x_0 = (\overline{A}^*)\circ\backslash(\underline{A}^*B(C\underline{A}^*B)\circ\backslash z)$
so $u_{opt} \preceq (\underline{A}^*B)\circ\backslash X_{opt}$.

A sufficient condition witch ensures that u_{opt} is the optimal control is:

$$\underline{A}^*Bu_{opt} = \overline{A}^*\underline{A}^*Bu_{opt}$$

For the p-temporal event graph given in figure (1), the control

$$u = (\underline{A}^*B)\circ\backslash X_{opt} = \begin{pmatrix} 7\gamma^0 \oplus 11\gamma^1 \oplus 15\gamma^2 \oplus +\infty\gamma^3 \\ 0\gamma^0 \oplus 4\gamma^1 \oplus 8\gamma^2 \oplus +\infty\gamma^3 \\ 2\gamma^0 \oplus 6\gamma^1 \oplus 10\gamma^2 \oplus +\infty\gamma^3 \end{pmatrix},$$

respects the condition given below, so it is the optimal control.

C. State feedback control

In this section, we propose a closed-loop control strategy for P-temporal event graphs. We assume that all internal transitions are both controllable and observable i.e, the control law is given by $x = Fx$ (see [16]).
Under our assumptions the state-space representation is then given by:

$$x = (\underline{A} \oplus F)x \oplus Bv$$
$$y = Cx.$$

We seek a controller F which achieves the model matching problem ([9], [16]) and which respects the constraints. The model matching consists in controlling the system such that the behavior of controlled system G_c be as close as possible to a desired behavior described by a reference model denoted G_{ref}. The very nature of (max, +) linear systems (synchronization and delay) leads to a control which can only delays the inputs, therefore the controlled system G_c is necessarily greater or equal than the nominal system. For P-temporal event graphs the dynamic of the system wich ensures liveness is given by matrix $\overline{\underline{A}}^*$ (see corollary 2.3). Consequently, specification G_{ref} is assumed greater than $C\overline{\underline{A}}^*B$.
Formally we seek a controller F such that:
$x = (\underline{A} \oplus F)x \oplus Bv$
$y = Cx \preceq G_{ref}v \qquad \forall\ v$ with $C\overline{\underline{A}}^*B \preceq G_{ref}$ and
$x \preceq \overline{A} \odot x$
First,
$y = Cx = C(\underline{A} \oplus F)^*Bv \preceq G_{ref}v\ \forall v$
$\Leftrightarrow C(\underline{A} \oplus F)^*B \preceq G_{ref}$
$\Leftrightarrow X = (\underline{A} \oplus F)^* \preceq C\circ\backslash G_{ref} \circ/ B = X_0$.
Furthermore,
$X = (\underline{A} \oplus F)^* \Rightarrow X = (\underline{A} \oplus F)^*X$
$\Rightarrow X \succeq (\underline{A} \oplus F)X$
$\Rightarrow X \succeq \underline{A}X$ and $X \succeq FX$.
According to proposition (2.8), the greatest X such that
$\underline{A}X \preceq X \preceq \overline{A} \odot X$ and $X \preceq X_0$ is given by $X_{opt} = \overline{\underline{A}}^*\circ\backslash X_0$.

Proposition 3.1: The greatest F such that $X = X_{opt}$ and $X \succeq (A \oplus F)X$ is given by

$$F_{max} = X_{opt} \circ/ X_{opt} \succeq \underline{A}.$$

Proof: First, $X \succeq (\underline{A} \oplus F)X \Leftrightarrow X \succeq \underline{A}X$ and $X \succeq FX$.
According to proposition (2.8), $X_{opt} \succeq \underline{A}X_{opt}$.
Furthermore, $X \succeq FX \Leftrightarrow F \preceq X \circ/ X$, then
$F_{max} = X_{opt} \circ/ X_{opt}$ and $F_{max}X_{opt} = X_{opt}$ (since $(x \circ/ x)x = x$).

To conclude $X_{opt} \succeq \underline{A}X_{opt} \Leftrightarrow F_{max} = X_{opt} \circ\!\!\!/ X_{opt} \succeq \underline{A}$, *i.e.*, $(\underline{A} \oplus F_{max})^* = F_{max}^* = F_{max}$ (see equation (5)). ■

Proposition 3.2: Controller F_{max} ensures that

$$C(\underline{A} \oplus F_{max})^* B \preceq G_{ref}.$$

Proof: First, $C(\underline{A} \oplus F_{max})^* B \preceq G_{ref}$

$$\Leftrightarrow (\underline{A} \oplus F_{max})^* \preceq C \circ\!\!\!\backslash G_{ref} \circ\!\!\!/ B = X_0.$$

Otherwise, F_{max} is such that
$(\underline{A} \oplus F_{max})^* X_{opt} = X_{opt} = \overline{\underline{A}}^* \circ\!\!\!\backslash X_0 \preceq X_0$ (since $\overline{\underline{A}}^* \succeq e$).
By assumption, $C\overline{\underline{A}}^* B \preceq G_{ref}$ then

$$\begin{aligned} X_{opt} = (C\overline{\underline{A}}^*) \circ\!\!\!\backslash G_{ref} \circ\!\!\!/ B &\succeq (C\overline{\underline{A}}^*) \circ\!\!\!\backslash (C\overline{\underline{A}}^* B) \circ\!\!\!/ B \\ &\succeq B \circ\!\!\!/ B \\ &\succeq e \end{aligned}$$

therefore $X_0 \succeq X_{opt} = (\underline{A} \oplus F_{max})^* X_{opt} \succeq (\underline{A} \oplus F_{max})^* e$ then,

$$G_{ref} \succeq CX_0 B \succeq CX_{opt} B \succeq C(\underline{A} \oplus F_{max})^* B.$$

■

Proposition 3.3: Controller F_{max} ensures that the constraints are respected. Formally,

$$x = (\underline{A} \oplus F_{max})^* Bv \preceq \overline{A} \odot x \ \forall v. \quad (11)$$

Proof: Thanks to proposition (2.8), inequality (11) is equivalent to

$$x = \overline{A}_* \odot x = \underline{A}^* x = \overline{\underline{A}}^* x$$

then the constraints will be respected if and only if

$$x = (\underline{A} \oplus F)^* Bv \ \in Im\overline{\underline{A}}^* \ \forall v.$$

According to proposition (3.1),

$$\begin{aligned} F_{max} &= X_{opt} \circ\!\!\!/ X_{opt} \\ &= (\overline{\underline{A}}^* \circ\!\!\!\backslash X_0) \circ\!\!\!/ (\overline{\underline{A}}^* \circ\!\!\!\backslash X_0) \\ &= \overline{\underline{A}}^* \circ\!\!\!\backslash X_0 \circ\!\!\!/ (\overline{\underline{A}}^* \circ\!\!\!\backslash X_0) \text{ (see equation 4)} \\ &= \overline{\underline{A}}^* (\overline{\underline{A}}^* \circ\!\!\!\backslash X_0 \circ\!\!\!/ (\overline{\underline{A}}^* \circ\!\!\!\backslash X_0)) \text{ (see theorem (2.4))} \end{aligned}$$

then

$$F_{max} = \overline{\underline{A}}^* F_{max}.$$

Furthermore, $(\underline{A} \oplus F_{max})^* = F_{max}$, then

$$x = (\underline{A} \oplus F_{max})^* Bv = F_{max} Bv = \overline{\underline{A}}^* F_{max} Bv \in Im\overline{\underline{A}}^*.$$

■

Example 3.3: Let us consider the P-TEG given in figure 1 and a transfer function

$G_{ref} = C\overline{\underline{A}}^* B = \begin{pmatrix} 7(4\gamma)^* & 14(4\gamma)^* & 12(4\gamma)^* \\ 4(4\gamma)^* & 11(4\gamma)^* & 9(4\gamma)^* \end{pmatrix}$. The maximum controller F_{max} such that $C(\underline{A} \oplus F)^* B \preceq G_{ref}$ is given by:

$$F_{max} = X_{opt} \circ\!\!\!/ X_{opt}$$

and we have:

$$C(\underline{A} \oplus F_{max})^* B = \begin{pmatrix} 7(4\gamma)^* & 14(4\gamma)^* & 12(4\gamma)^* \\ 4(4\gamma)^* & 11(4\gamma)^* & 9(4\gamma)^* \end{pmatrix}$$

which is equal to G_{ref} so, the controller F_{max} preserves the performances of the system.

Remark 3.1: Computation of these example may be obtained by considering Scilab toolboxes (see [18]).

IV. CONCLUSION

In this paper we assumed that the TEG includes some sojourn time constraints. These graphs are called P-temporal event graphs (P-TEG). We have given an algebraic model of P-TEG and an optimal control law synthesis in a just in time context. This one ensures that the controlled system behaves as close as possible to the specified outputs and ensures the liveness of tokens, when possible. Next, we have given an optimal state feedback control which ensures that the firing of transition respects the constraints and the closed-loop transfer function is less than the transfer of a model reference. The next step is to extend this work to other control structures such as the ones given in [14]. The traditional interval theory is very effective for parameter estimation, it would be interesting to apply the results of this paper to P-TEG which includes some parametric uncertainties as the one studied in [15].

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