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EVALUATION OF A NONLINEAR BISTABLE FILTER FOR BINARY SIGNAL DETECTION

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We consider the nonlinear bistable dynamic system that is the archetypal system giving way to the phenomenon of stochastic resonance for noise-improved signal processing. Independently of a strict stochastic resonance effect, we use this bistable system as a nonlinear filter for a detection task on a binary signal. We expose a methodology to tune the nonlinear filter at its best performance that minimizes its probability of detection error. The optimally tuned nonlinear filter is then compared to the ideal matched filter, which is the optimal filter for the detection with Gaussian noise. We show that the performance of the nonlinear filter, although (expectedly) not as good, comes close to that of the ideal matched filter operating in its strict nominal conditions. We next examine several possible departures, quite plausible in practical operation, from the nominal conditions of the ideal matched filter. We demonstrate that in such degraded conditions, the nonlinear filter can catch up and surpass the performance of the matched filter. This reveals a robustness superiority of the nonlinear filter, compared to the matched filter operating outside its strict nominal conditions.

Keywords: Detection; bistable dynamic system; stochastic resonance; matched filter; desynchronization.

1. Introduction

Nonlinear systems present rich potentialities for signal and information processing. Among their interesting properties, it has recently been shown that nonlinear systems can give way to the phenomenon of stochastic resonance (SR). SR is a nonlinear effect which describes the possibility to enhance the processing of a signal thanks to the action of the noise (see [Moss *et al.*, 1994; Gammaitoni *et al.*, 1998] for overviews). Since its introduction in climate dynamics [Benzi *et al.*, 1981], SR has been reported both experimentally and theoretically in many different processes [McNamara *et al.*, 1988; Anishchenko *et al.*, 1992; Pantazelou *et al.*, 1995; Chapeau-Blondeau & Godivier, 1996, 1997; Chapeau-Blondeau & Rojas-Varela, 2000; Chapeau-Blondeau, 2003]. One of the earliest and most studied systems for SR, is the nonlinear bistable dynamic system governed by a double-well potential (see [Moss *et al.*, 1994; Gammaitoni *et al.*, 1998]). When this system is driven by a small sinusoidal input added to noise, the detectability at the output of the periodic signal (usually measured by a signal-to-noise ratio in the frequency domain) is maximized by an optimal nonzero level of the input noise. This was the original form of SR reported in this sytem. Its early observation triggered many other studies to further analyze the potentialities of such nonlinear systems for signal processing [Jung & Hänggi, 1991; Anishchenko *et al.*, 1994; Dykman & McClintock, 1998; Godivier & Chapeau-Blondeau, 1998;

Gingl *et al.*, 2000; Xu *et al.*, 2002; Morfu *et al.*, 2003], and especially for detection problems [Inchiosa & Bulsara, 1996; Galdi *et al.*, 1998; Bulsara *et al.*, 2002; Gammaitoni & Bulsara, 2002; Zozor & Amblard, 2002]. Some of these attempts are promising. For example, [Zozor & Amblard, 2002] proposes to use, in a discrete-time representation, with non-Gaussian noise, the bistable dynamic system as a preprocessor amplifying a sine to be detected. This could be relevant for the design of locally optimum detectors in underwater-acoustics situations where sinewaves are corrupted by non-Gaussian noises (as proposed in [Saha & Anand, 2003], with another kind of stochastic resonator). Still, many issues remain open for investigation about the bistable dynamic system applied for detection.

In this report, we shall complement the analysis of the bistable system for detection, through a direct comparison to the ultimate optimal detector provided by the matched filter. Such a comparison, which has never been explicitly undertaken, is important as a reference for a better appreciation of the potentialities of the bistable system. We consider the scheme used in [Godivier & Chapeau-Blondeau, 1998], where the bistable dynamic system transmits a broadband binary signal. The study of [Godivier & Chapeau-Blondeau, 1998] shows the possibility of improving the transmission of a small signal via addition of noise, as measured by an input–ouput mutual information. Here, in the present study, the bistable system is used for a detection task, in the sense of classical detection theory [Kay, 1998], and the assessment is made through the standard probability of detection error.

The optimal detector (matched filter) is taken as a reference for comparison, in its strict nominal conditions, and in degraded conditions resulting from limitations arising in its practical implementation. Another incompletely clarified point about bistable dynamic systems used in detection is to determine if the regime where noise helps signal transmission (SR regime) is the best regime. In most SR studies, the parameters of the bistable dynamic system are fixed; the signal is subthreshold, i.e. too small to elicit by itself a strong response from the system. Addition of noise then brings assistance to the signal in eliciting a more efficient response from the fixed bistable dynamic system. Here, for the bistable system used as a nonlinear detector, instead of tuning the level of the input noise with a fixed system, we tune the system parameters in order to optimize the detection at a fixed given input noise level. This is somehow the classical way of optimizing a tunable processing device. In the process, it will be interesting to examine if SR appears naturally when the system is tuned (i.e. if the system naturally tries to operate in the SR region where the input signal is subthreshold), or on the contrary if the SR region turns out to be a suboptimal regime for a tunable bistable dynamic system.

2. A Bistable Dynamic System used as a Nonlinear Filter

We consider the earliest system to have revealed SR, a nonlinear bistable dynamic system governed by the quartic potential. An input signal $u(t)$ is applied to the dynamic system whose internal state $x(t)$ evolves according to

$$\tau_a \frac{dx(t)}{dt} = x(t) - \frac{x^3(t)}{X_b^2} + u(t), \tag{1}$$

with the parameters $\tau_a > 0$ and $X_b > 0$. The free relaxation of the system $\tau_a \dot{x} = -dU/dx$ is governed by the potential $U(x) = -x^2/2 + x^4/(4X_b^2)$. Such a system has two stationary stable states $x = \pm X_b$ corresponding to the two minima of the potential $U(x = \pm X_b) = -X_b^2/4$ separated by a potential barrier with height $U_0 = X_b^2/4$. Seeing things in a mechanical way, Eq. (1) describes the overdamped motion of a particule in the potential $U(x)$ when forced by $u(t)$. The internal states $x(t)$ determine the ouput $y(t)$ of the system, through a single-bit quantization expressed by

$$y(t) = \mathrm{sign}[x(t)]. \tag{2}$$

In most SR studies, $u(t)$ is an additive signal-noise mixture, with $u(t) = s(t) + \eta(t)$; $s(t)$ is an information-carrying signal and $\eta(t)$ a stationary random noise. Then, always in the SR scheme, the parameters (τ_a, X_b) of the bistable dynamic system are fixed; the information-carrying signal $s(t)$ is too weak to transmit any variations to the output $y(t)$. In such conditions, addition of noise can improve the transmission of $s(t)$ from the fixed bistable dynamic system. A measure of performance is chosen to quantify the improvement. The occurrence of a maximum in the measure of performance at a nonzero noise level is the manifestation of

SR. In the present paper, we analyze the system described by Eqs. (1) and (2) differently. Unlike the SR case, where the system parameters (τ_a, X_b) are imposed and only the noise $\eta(t)$ is tunable, we consider a fixed given input noise level; the system parameters are no longer fixed and can be tuned to optimize a measure of performance. This starting proposal is, in fact, the usual optimization method of any tunable system in signal processing. Yet, this approach is here unusual, and even new, for the bistable dynamic system mostly considered in the SR perspective.

3. The Nonlinear Filter in a Detection Process

Within this framework, we choose to use the nonlinear filter of the previous section in a detection task. The information-carrying signal $s(t)$ is a random binary signal made of rectangular pulses of duration T_p and amplitude $\pm A$ (see line A in Fig. 1). This signal $s(t)$ is corrupted by an additive zero-mean Gaussian white noise $\eta(t)$, with autocorrelation function $\langle \eta(t)\eta(0)\rangle = 2D\delta(t)$ (where D denotes noise power spectral density). At a given time t, the detection problem on the signal-noise mixture $u(t)$ is

$$H_0 : u(t) = -A + \eta(t) \text{ with prior probability } \mathrm{P}_0$$
$$H_1 : u(t) = +A + \eta(t) \text{ with prior probability } \mathrm{P}_1$$

The signal-noise mixture $u(t)$ is applied at the input of the bistable dynamic system described by Eqs. (1) and (2); at every time multiple of the pulse duration T_p, the output of the system $y(t)$ is read in order to make a decision

$$D_0 : u(t) = -A + \eta(t) \quad \text{if } y(t) < 0$$
$$D_1 : u(t) = +A + \eta(t) \quad \text{if } y(t) > 0.$$

The time of decision is assumed perfectly synchronized with the end of a binary pulse on the information-carrying signal $s(t)$. In the above detection process, the bistable dynamic system of Eqs. (1) and (2) is considered as a specific nonlinear filter. We will devise a methodology to tune the filter parameters (τ_a, X_b) to make the best detection, and will then study the performance of this nonlinear filter. The performance will be assessed by the probability of detection error P_{er},

$$P_{\text{er}} = P_1 \times \Pr\{\mathrm{D}_0\,|\,\mathrm{H}_1\} + \mathrm{P}_0 \times \Pr\{\mathrm{D}_1\,|\,\mathrm{H}_0\}. \qquad (3)$$

For the sequel, we assume $P_0 = P_1 = 1/2$.

4. Optimal Tuning of the Nonlinear Filter

In this section, we derive an optimal tuning methodology of the nonlinear filter parameters (τ_a, X_b) minimizing the error probability P_{er} of Eq. (3). Our investigation is based on a numerical simulation of the continuous process of Eq. (1) by means of a Euler–Maruyama discretization [Gardiner, 1985] at a small time step Δt much smaller than the characteristic times τ_a and T_p.

4.1. *Qualitative observations*

In this study, the input signal-noise mixture characteristics D, T_p and A are assumed fixed, as imposed by external conditions of operation. Then, the influence of the internal parameters (τ_a, X_b) of the nonlinear filter is illustrated qualitatively in Fig. 1. In all reports, units for τ_a and X_b are naturally taken as T_p and A respectively, and D is measured in units T_pA^2. If the system characteristic time τ_a is small, the system reaches the stationary state quickly, but fluctuations around this stationary state are large (see line B1 in Fig. 1). The detection performance in this configuration will be poor. On the contrary, if τ_a is large, the system does not have the time to reach the stationary state during T_p (see line C1 in Fig. 1). In this case, even though the system fluctuations are small, detection will not be efficient. This entails that there must exist a intermediate τ_a minimizing the probability of detection error, large enough to smooth the fluctuations and small enough to let the system switch fast enough from one stable state to the other. X_b rules the height of the potential barrier U_0 to be overcome. This parameter is related to the filter capability to switch between the potential wells, measured by the Kramer rate [Moss, 1994],

$$R_k = \frac{\sqrt{|U''(0)|U''(X_b)}}{2\pi\tau_a} \times \exp\left(\frac{-U_0}{D\tau_a}\right)$$
$$= \frac{1}{\pi\tau_a\sqrt{2}} \times \exp\left(\frac{-X_b^2}{4D\tau_a}\right). \qquad (4)$$

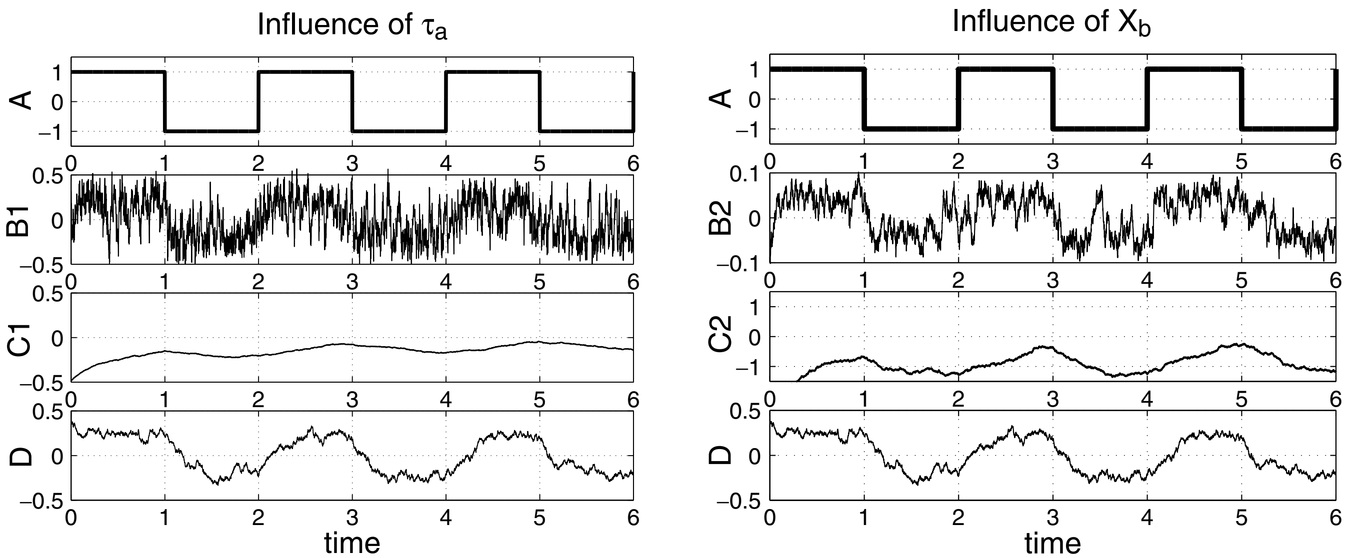


Fig. 1. Qualitative influence of the filter parameters τ_a and X_b. $D = 0.0275, T_p = 1, A = 1$ are fixed in both panels of this figure. Influence of τ_a for a given $X_b = 0.1$; line A is the input signal $s(t)$, line B1 nonlinear filter internal state $x(t)$ for $\tau_a = 0.1$, line C1 for $\tau_a = 10$, line D for $\tau_a = 1$. Influence of X_b for a given $\tau_a = 1$; line A is the input signal, line B2 the nonlinear filter internal state $x(t)$ for $X_b = 0.01$, line C2 for $X_b = 1$, line D for $X_b = 0.1$.

As suggested by Eq. (4), if X_b is too small for a given τ_a, the system presents many inter-well transitions but most of them are not due to the information-carrying signal $s(t)$ (see line B2 in Fig. 1). This situation induces a large error probability P_{er}. At the extreme opposite, if X_b is too large, the input signal-noise mixture $u(t)$ might not be sufficient to jump over the potential barrier (see line C2 in Fig. 1). In such a case, the detection is not efficient. One can notice that it is this regime where stochastic resonance takes place; if the bistable dynamic system parameters are fixed with a large X_b, a judicious amount of added noise can enhance the detection performance. This is of no interest in the present report, because we let ourselves free to adjust optimally the system parameters.

4.2. *Quantitative observations*

In Fig. 2, we present the quantitative influence of the nonlinear filter parameters (τ_a, X_b) on the probability of error P_{er} of Eq. (3), numerically evaluated. Three different noise power densities are tested ($D = 0.0275, D = 0.0524$ and $D = 0.304$). A justification for the choice of these specific values will be given in Sec. 5. In the three presented cases, the probability of error P_{er} is plotted as a function of τ_a and X_b; the surfaces present a valley shape. This confirms the observations of Fig. 1 that for a given X_b there exists a unique optimal τ_a in minimizing P_{er}.

Figures 3(a) and 3(b) display the minimal probability of error of Fig. 2 in plane (τ_a, X_b) and respectively in plane (P_{er}, X_b). As visible in Fig. 3(b), the minimal error probability is decreasing as X_b is decreasing. The optimal tuning of the nonlinear filter implies to consider X_b as small as possible. Figure 3(a) exhibits, as a function of X_b, the corresponding τ_a minimizing the error probability P_{er}. This is the tuning curve of the nonlinear filter; once one parameter is fixed, the other one has to be deduced from this curve. A qualitative justification of this tuning method can be given: X_b/A has to be small to enable the internal state signal $x(t)$ to switch around the detection threshold easily. At the same time, τ_a/T_p has to be large to diminish the impact of the fluctuations on the detector performance.

At this step of the report, we can consider the question raised in the introduction, to determine if SR appears as a natural favorable regime for detection with a bistable dynamic system. SR takes place when the amplitude A of the information-carrying signal $s(t)$ is too small compared to the height of the

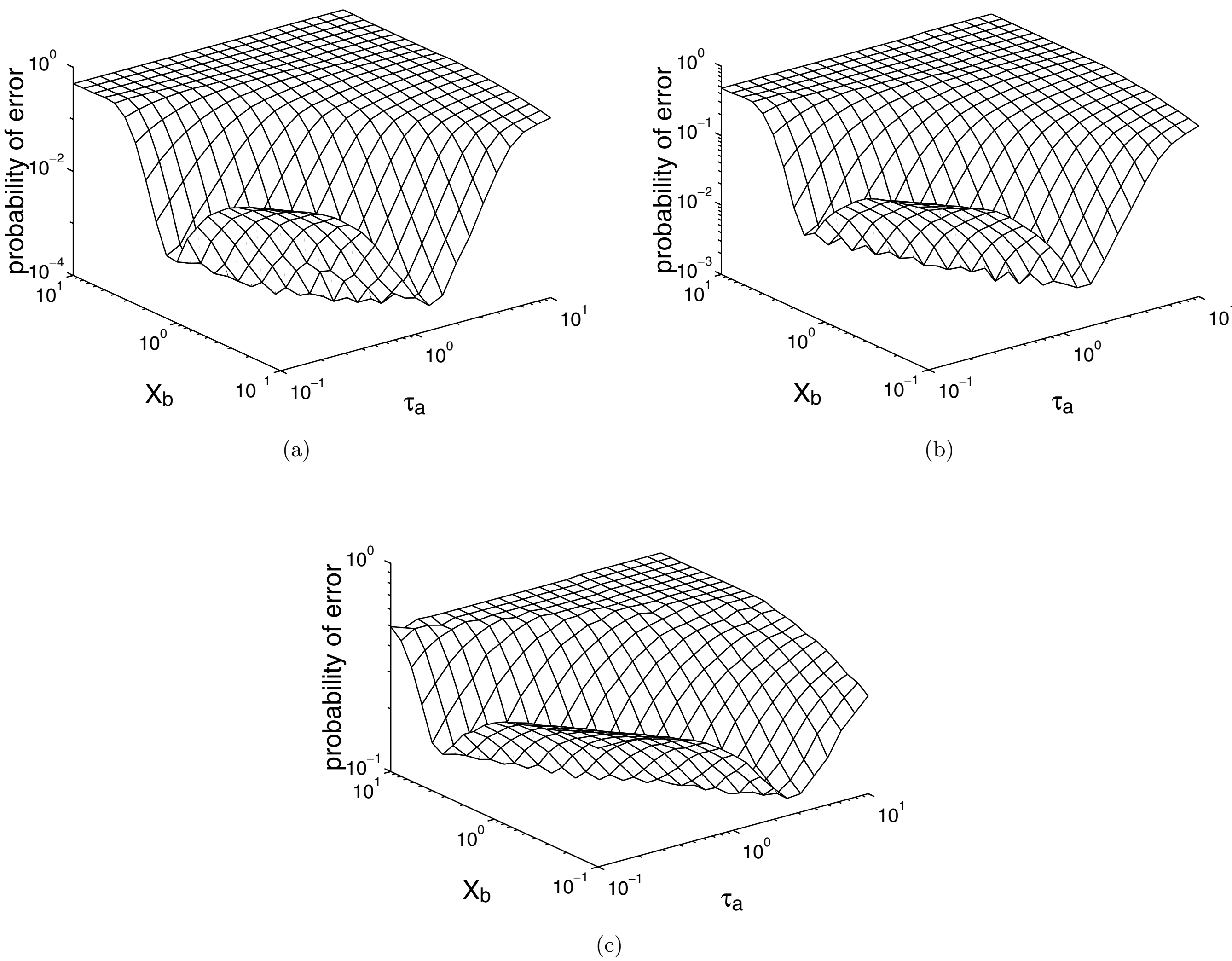


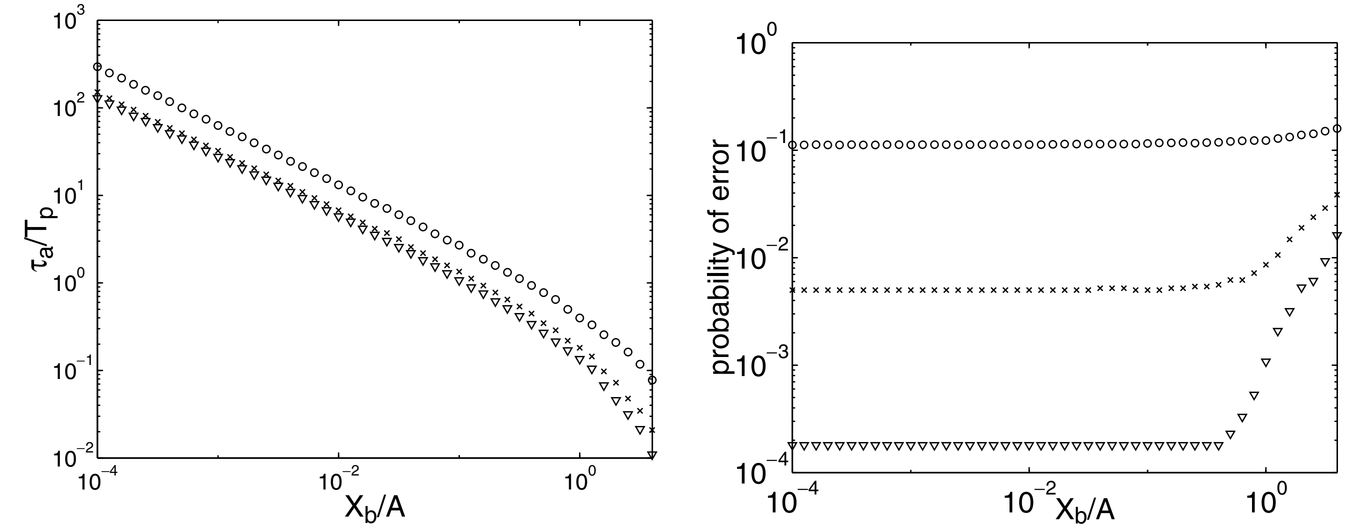
Fig. 2. Quantitative influence of the nonlinear filter parameters τ_a and X_b on the probability of detection error $P_{\rm er}$ of Eq. (3), for different noise power densities D. The results are obtained by Monte Carlo simulations. (a) $D = 0.0275$. (b) $D = 0.0524$. (c) $D = 0.304$.

potential barrier. This happens as soon as $X_b/A > \sqrt{27/4}$ (see e.g. [Godivier & Chapeau-Blondeau, 1998] for a justification of this specific value). On the contrary, Fig. 3 illustrates that the most efficient regime for detection (i.e. the conditions that minimize the detection error probability $P_{\rm er}$ for a given noise density D) would rather be for $X_b/A \ll 1$. Therefore, SR lies outside the interesting domain to tune an adjustable bistable nonlinearity for detection. A confirmation is given in Fig. (3a), where the optimal tuning curve of the bistable dynamic system is almost not sensitive to the noise density at small values of D (the tuning curve for $D = 0.0275$ and $D = 0.0524$ are very close). SR may be of interest only with a fixed nonadjustable nonlinearity, if the signal amplitude is too small, addition of noise via SR can help the detection.

4.3. *Tuning methodology of the nonlinear filter*

The results of Figs. 2 and 3 provide an optimal tuning methodology to configure the nonlinear filter to work at its best (minimizing the error probability $P_{\rm er}$):

Tuning methodology of the nonlinear filter
Take X_b the smallest as possible
Deduce the optimal corresponding τ_a from the tuning curve Fig. 3(a).



(a) (b)

Fig. 3. Empirical results for the optimal tuning of the nonlinear filter minimizing the error probability P_{er}. $\circ$ stands for $D = 0.304$, $\times$ for $D = 0.0524$ and ∇ for $D = 0.0275$. (a) The tuning curve; the optimal τ_a for a given X_b. (b) The corresponding minimal probability of error for each X_b.

In Fig. 3(b), the minimal probability of error decreases very slowly but monotonically as X_b decreases. One could ask for a limit of the minimal error probability achievable by the nonlinear filter. The dynamics based on Eq. (1) have been the subject of numerous studies. Yet, because of the nonlinearity and the statistically nonstationary input of the nonlinear filter, a full theoretical description has never been given at the moment. Thus, there exists no known theoretical limit for the minimal error probability achievable by the nonlinear filter. It would be interesting to search for this bound, but our purpose here is not to get into a complete theoretical modeling of the nonlinear filter. We want to keep the reader focused on the detection task. However, in practice, taking X_b as close as possible to zero is not without problems. When X_b tends to zero, the output of the nonlinear filter also tends to zero. This is a manageable situation in numerical simulations, but an output signal getting arbitrarily small would be problematic in an analog physical implementation. Moreover, ensuring that X_b/A tends to zero means that we can control more and more accurately the level of X_b in comparison with the level of the information-carrying signal $s(t)$. These practical considerations have necessarily to be taken into account in a physical implementation of the process, and they will entail for X_b a minimum floor value.

5. Performance of Nonlinear Filter versus Matched Filter

5.1. *Comparison with the ideal matched filter*

From Sec. 4, we now know how to tune the nonlinear filter to its best performance for detection. We next wish to compare the nonlinear filter to the matched filter. For the detection problem defined in Sec. 3, the matched filter is the ultimate optimal detector achieving the overall minimal probability of detection error [Kay, 1998]. The matched filter is a replica-correlar: it correlates the received signal (the signal-noise mixture) $u(t)$ with a replica of a binary pulse of the information-carrying signal $s(t)$. In our case, the impulse response of the matched filter is $h(t) = A$ if $t \in [0, T_p]$ and $h(t) = 0$ elsewhere (see solid line in Fig. 5). The signal at the ouput of the matched filter $y'(t)$ is

$$y'(t) = \int_{-\infty}^{t} h(t - t')\, u(t')\, dt'. \tag{5}$$

At every time multiple of input pulse duration T_p, the output of the matched filter $y'(t)$ is read in order to make a decision

$$D_0 : u(t) = -A + \eta(t) \text{ if } y'(t) < 2D \times \log(P_0/P_1)$$
$$D_1 : u(t) = +A + \eta(t) \text{ if } y'(t) > 2D \times \log(P_0/P_1),$$

where we recall that we have assumed $P_0 = P_1$. The instant of decision is at the end of each input binary pulse. The instant of decision is assumed perfectly synchronized with the end of a binary pulse of the information-carrying signal $s(t)$. Under these conditions, the performance of the matched filter is given by

$$P_{\text{er}} = \frac{1}{2}\left[1 - \text{erf}\left(\frac{AT_p}{\sqrt{2}\sqrt{2DT_p}}\right)\right]. \tag{6}$$

Results given in Sec. 4.3 for three noise densities ($D = 0.0275$, 0.0524, 0.304) correspond to error probabilities of the matched filter of, respectively, $10^{-5}, 10^{-3}$ and 10^{-1} (for $T_p = 1$ and $A = 1$ expressed in arbitrary units). These orders of magnitude are the typical ones found in practical digital communications dealing with speech (10^{-3} per bit) or image (10^{-5} per bit). In Table 1, we compare the performance of the nonlinear filter to that of the matched filter, with these relevant conditions of detection. This comparison with the optimal detector constitutes in itself a new result. This contributes to complement the assessment of the bistable dynamic system described by Eqs. (1) and (2).

It can be seen from Table 1 that the matched filter is, expectedly, always better than the nonlinear filter. One can note that the relative difference in error probability (relative to the ideal matched filter) increases as the noise density D decreases. Although not as good, the performance of the optimally-tuned nonlinear filter comes close to that of the matched filter, and as much as the noise level D increases. Furthermore, the matched filter somehow shows an idealistic character. To achieve the overall best performance described by Eq. (6), the matched filter must be perfectly synchronized, i.e. the readings of its output $y'(t)$ must be performed exactly at the end of each rectangular pulse on $s(t)$. In addition, the matched filter because of its impulse response being a strict rectangular pulse, turns out to be a linear filter of infinite order. These theoretical specifications of the matched filter will usually not be exactly reachable in practice, and a practical implementation will have to cope with some plausible departures from the ideal specifications. We will now study the impact of some plausible practical departures.

Table 1. Error probability P_{er} of the matched filter and the optimally tuned nonlinear filter. $X_b = 10^{-3}$ and τ_a is adjusted according to the optimal tuning methodology described in Sec. 4.3.

Noise density	0.0275	0.0524	0.304
Ideal matched filter	1.0×10^{-5}	1.0×10^{-3}	1.00×10^{-1}
Nonlinear filter	18×10^{-5}	5.0×10^{-3}	1.13×10^{-1}
Relative difference	17%	4%	0.13%

5.2. *Comparison in presence of desynchronization*

The matched filter previously described assumes perfect synchronization of the time of decision with the end of each binary pulse on $s(t)$. In practice, this condition can never be perfectly satisfied. Practically, in digital communications, synchronization is achieved by an electronic device (typically a phase locked loop) which like any electronic device admits its own limitations. We propose to take into account this practical difficulty of implementation. We compare the nonlinear filter and the matched filter in the presence of desynchronization between the time of decision and the end of a binary pulse on $s(t)$. Let ΔT be the temporal absolute value of the difference between the end of a pulse on $s(t)$ and the time of decision (this describes a decision which can either be late or in advance with the end of a pulse on $s(t)$). For this desynchronized matched filter, it is possible to obtain the exact analytical expression of the probability of error as

$$P_{\text{er}} = P_1 \times \Pr\{D_0|H_1, \triangle T\} + P_0 \times \Pr\{D_1|H_0, \triangle T\}, \tag{7}$$

$$\begin{aligned}\Pr\{D_0|H_1\} &= P_0 \times \frac{1}{2}\left[1 + \text{erf}\left(\frac{-A(T_p - 2\Delta T)}{\sqrt{2}\sqrt{2DT_p}}\right)\right] \\ &\quad + P_1 \times \frac{1}{2}\left[1 + \text{erf}\left(\frac{-AT_p}{\sqrt{2}\sqrt{2DT_p}}\right)\right],\end{aligned} \tag{8}$$

$$\begin{aligned}\Pr\{D_1|H_0\} &= P_1 \times \frac{1}{2}\left[1 - \text{erf}\left(\frac{A(T_p - 2\Delta T)}{\sqrt{2}\sqrt{2DT_p}}\right)\right] \\ &\quad + P_0 \times \frac{1}{2}\left[1 - \text{erf}\left(\frac{AT_p}{\sqrt{2}\sqrt{2DT_p}}\right)\right].\end{aligned} \tag{9}$$

Recalling $P_0 = P_1 = 1/2$ in our case, Eq. (7) can be simplified as

$$P_{\text{er}} = \frac{1}{2} - \frac{1}{4}\text{erf}\left(\frac{AT_p(1-2\Delta T/T_p)}{\sqrt{2}\sqrt{2DT_p}}\right) - \frac{1}{4}\text{erf}\left(\frac{AT_p}{\sqrt{2}\sqrt{2DT_p}}\right). \tag{10}$$

Figure 4 shows, in the presence of a desynchronization ΔT, the evolution of the performances of the desynchronized optimally tuned nonlinear filter and of the desynchronized matched filter. The probability of error is, naturally, in both cases an increasing function of the desynchronization ΔT. However, the nonlinear filter turns out to be less sensitive to desynchronization than the matched filter. Therefore, even if (as seen in Table 1) the matched filter does better than the nonlinear filter in perfect condition of synchronization, there exists a desynchronization beyond which the nonlinear filter catches up and even outperforms the matched filter. As visible in Fig. 4, for any of the three tested

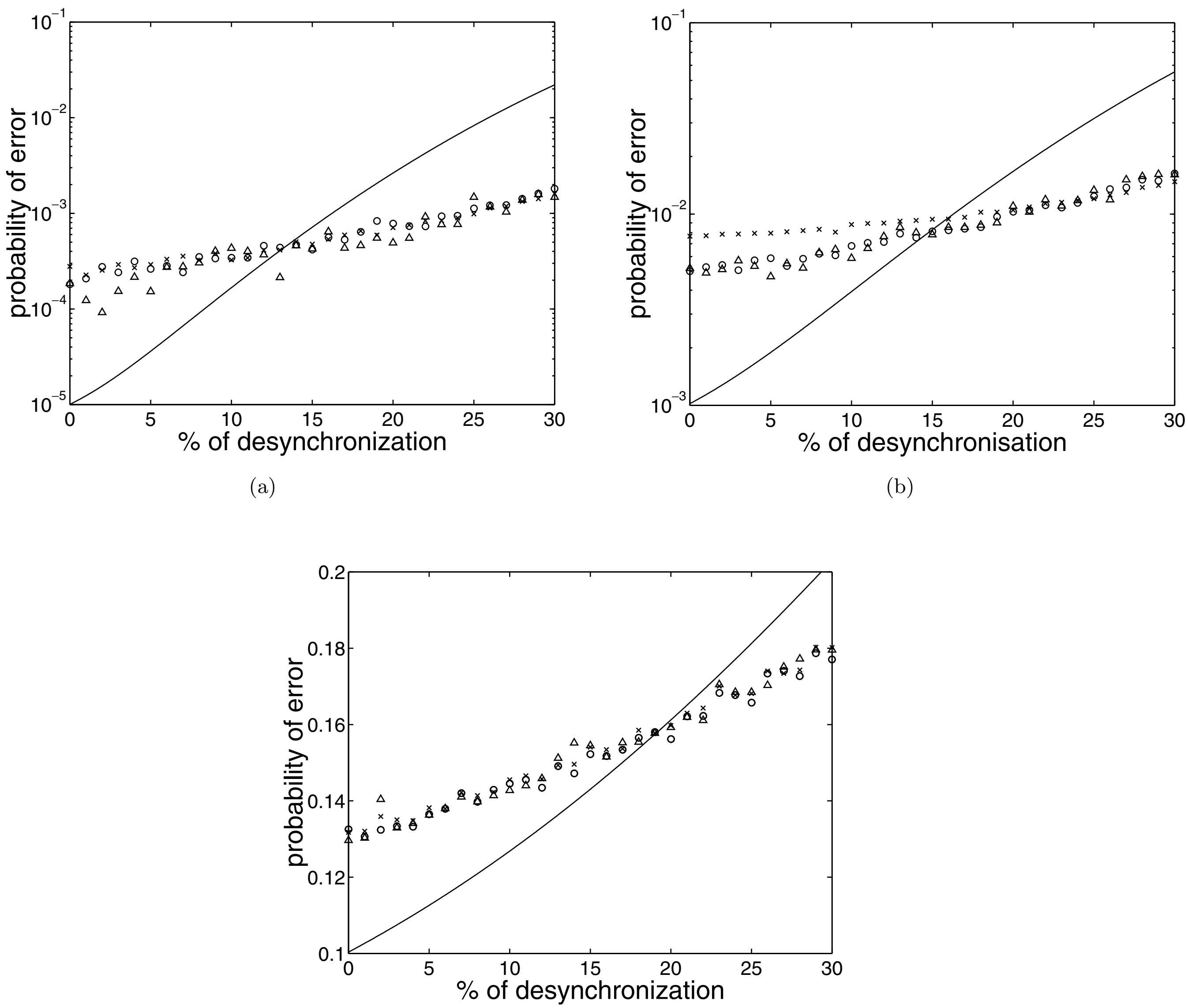


Fig. 4. Probability of error P_{er} as a function of desynchronization measured by $\Delta T/T_p$. (a) $D = 0.0275$. (b) $D = 0.0524$. (c) $D = 0.304$. $A = 1$ and $T_p = 1$ in the three cases. Solid line is the theoretical result for the matched filter given by Eq. (10). $\circ$ stand for the numerical result of the optimally tuned nonlinear filter with $X_b = 10^{-3}$, $\times$ for $X_b = 10^{-2}$, $\triangle$ for $X_b = 10^{-1}$.

noise densities D, the nonlinear filter performance surpasses that of the matched filter for a desynchronization $\Delta T/T_p$ of about 15% to 20%. Figure 4 therefore demonstrates that the nonlinear filter is more robust than the matched filter toward desynchronization. It is to be noticed that this result does not depend on the value chosen for X_b as long as $X_b/A \ll 1$ (as shown in Fig. 4).

Besides, in Fig. 4, the nonlinear filter has been tuned optimally by using the tuning methodology presented in Sec. 4.3 for perfect synchronization. Hence, in Fig. 4, the choice made for X_b and τ_a has been fixed for each curve in order to minimize the error probability P_{er} at zero desynchronization; each parameter X_b and τ_a remained the same even when some desynchronization was introduced. Nevertheless, we have also tried to readjust optimally the nonlinear filter for each desynchronization with the three tested noise densities D (not presented here). Up to 30% of desynchronization, no significant modification of parameters X_b and τ_a was found necessary between tuning in synchronized or desynchronized conditions. This can be considered as another interesting result: the tuning methodology described in Sec. 4.3 is also robust to desynchronization.

5.3. *Comparison with practical implementation of the matched filter*

We come back to the ideal conditions where the matched filter is the optimal filter. The ideal matched filter is a linear filter having an impulse response $h(t)$ which is a rectangular pulse of duration T_p. In practice, such a rectangular response can never be perfectly realized with a physical analog filter operating in continuous time as does the nonlinear filter of Eq. (1). Practically, an analog implementation of the matched filter, for instance as an electronic circuit, will have to rely on a finite-order analog linear filter. It is only at the limit of an infinite order that the rectangular impulse response will be reached. But for physical implementation, the order has to remain finite, and even small, for practicality and simplicity of the associated electronics.

In generality, the input–output relation of an analog linear filter is of a form given by

$$a_n \frac{d^n y''(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y''(t)}{dt^{n-1}} + \cdots + a_0 y''(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_0 u(t), \tag{11}$$

where $y''(t)$ is the filter output and $u(t)$ the input; $(a_n, a_{n-1}, \ldots, a_0)$ and $(b_m, b_{m-1}, \ldots, b_0)$ are the analog filter parameters; n sets the order of the filter, m is the number of zeros present in the transfer function of the filter. The analog filter parameters $(a_n, a_{n-1}, \ldots, a_0)$ and $(b_m, b_{m-1}, \ldots, b_0)$ have to be adjusted to ensure that the impulse response of the analog filter $h''(t)$ will fit the impulse response of the matched filter $h(t)$. Therefore, the analog filter parameters $(a_n, a_{n-1}, \ldots, a_0)$ and $(b_m, b_{m-1}, \ldots, b_0)$ are found by minimizing the following integral,

$$[(a_n, a_{n-1}, \ldots, a_0), (b_m, b_{m-1}, \ldots, b_0)] = \arg_{(a_i, b_j)} \min \left[\int_0^{+\infty} (h(t) - h''_{(a_i,b_j)}(t))^2\, dt \right]. \tag{12}$$

In order to evaluate the nonlinear filter in a definite context we choose to approximate the matched filter with the simplest analog filter; we consider filters of order $n = 1$ and $n = 2$ with no zeroes in the transfer function ($m = 0$). In addition to being the simplest designable analog filters, their impulse responses present the advantage of being totally and exactly expressible (see e.g. [Levine, 1996]). The results of minimization given in Eq. (12) are presented in Table 2 and the corresponding impulse responses are plotted in Fig. 5. We observe, in Fig. 5, how the impulse responses

Table 2. Approximation of the matched filter by first- and second-order linear filters with no zeroes. Results are obtained by achieving the minimization of Eq. (12).

Analog Filter Parameters	First-Order Filter	Second-Order Filter
a_i coefficients	$(a_1 = 0.5207; a_0 = 1)$	$(a_2 = 0.0977; a_1 = 0.4858; a_0 = 1)$
b_j coefficient	$b_0 = 0.7945$	$b_0 = 0.8253$

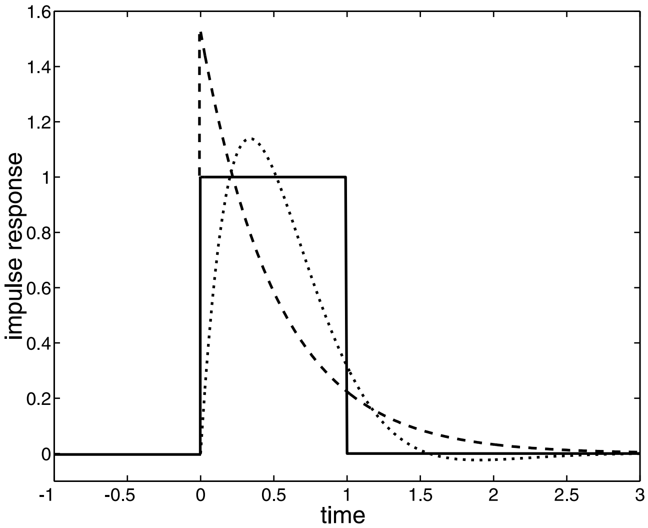


Fig. 5. Impulse response $h''(t)$ of the closest approximation of the matched filter by a first- and second-order linear filter with no zeroes. Solid line represents the impulse response of the ideal matched filter $h(t)$, dotted line stands for the first-order filter and dashed line for the second-order filter. T_p and A are taken equal to unity.

of the first- and second-order filters approximate the ideal impulse response of the matched filter.

In Table 3, we compare the performance of the optimally tuned nonlinear filter (already mentioned in Table 1) with that of the approximate version of the matched filter described by Eqs. (11) and (12). It appears that the performances of the first- and second-order filter implementation of the matched filter are degraded with regards to the ideal version of the matched filter. The first-order filter presents higher probability of error than the second-order filter. It appears also that the nonlinear filter is less efficient than the second-order filter but, nevertheless, outperforms the first-order filter. This is another new outcome of this report. A bistable dynamic system used as nonlinear filter in a detection scheme can do better than a first-order analog implementation of the optimal filter.

Table 3. Comparison of error probability $P_{\rm er}$ of the optimally tuned nonlinear filter with approximate versions of the matched filter.

Noise density	0.0275	0.0524	0.304
Ideal matched filter	1.0×10^{-5}	1.0×10^{-3}	1.00×10^{-1}
First order filter	45×10^{-5}	5.9×10^{-3}	1.13×10^{-1}
Second order filter	6×10^{-5}	2.6×10^{-3}	1.11×10^{-1}
Nonlinear filter	20×10^{-5}	5.0×10^{-3}	1.13×10^{-1}

Finally, Fig. 6 shows the evolution of the performance in the presence of desynchronization, for the optimally tuned nonlinear filter and for the different tested versions of the matched filter. Two important properties are visible in Fig. 6. First, the ideal matched filter, although the most efficient in its strict nominal conditions, is not strongly robust against desynchronization. The second-order linear filter, and then the nonlinear filter, progressively catch up and surpass the performance of the ideal matched filter as desynchronization increases.

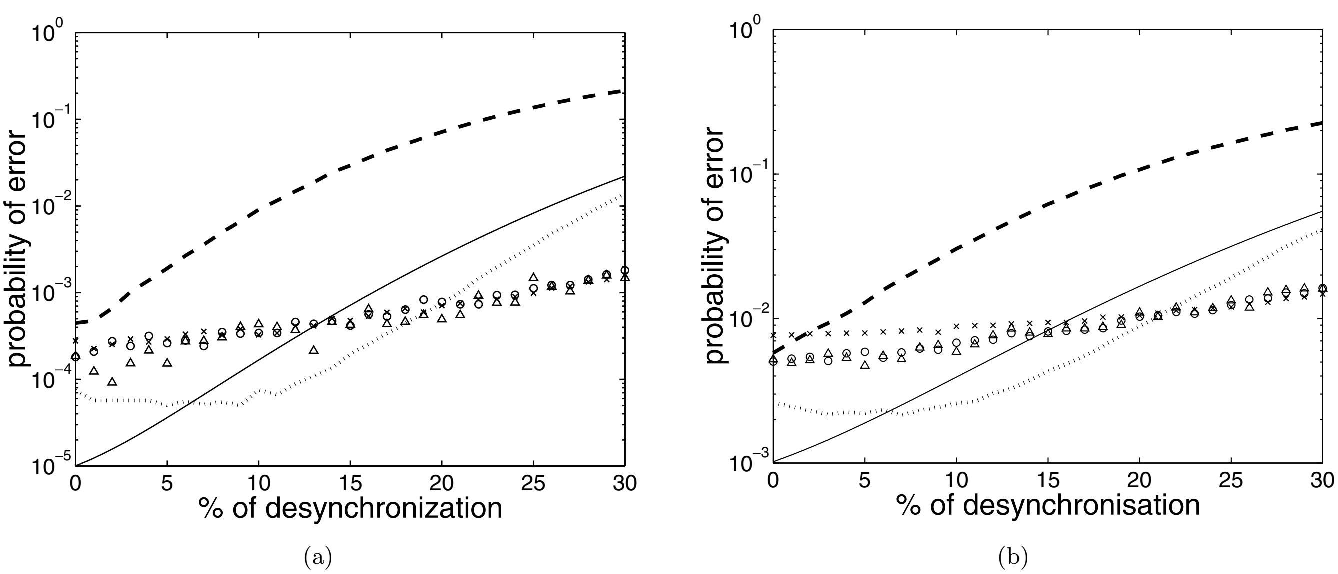
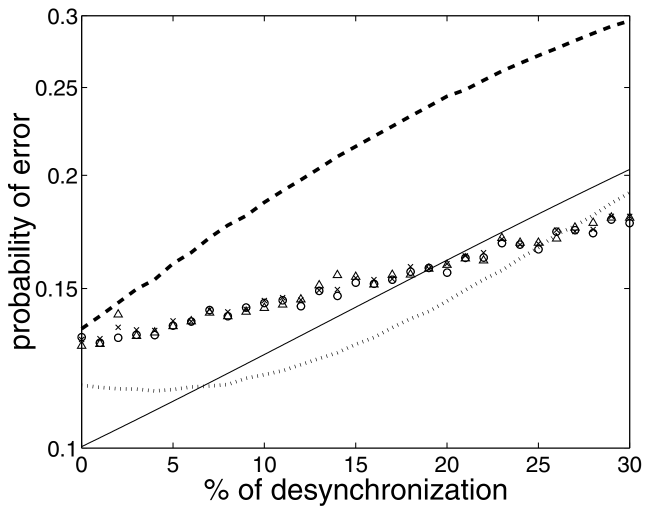


Fig. 6. Same as Fig. 4 with the first (dashed line) and second (dotted line) order analog implementations of the matched filter.



(c)

Fig. 6. (*Continued*)

Second, at large levels of desynchronization, around 20% or above, the optimally tuned nonlinear filter achieves the best performance. Figure 6 illustrates the main message of this study: ideal optimal filters are useful in their strict nominal conditions, but they may not be maximally robust against departures from their nominal conditions; other filters, like the bistable nonlinear filter, although suboptimal, may be more robust and maintain a better performance in varying conditions.

6. Discussion

We have considered the nonlinear bistable dynamic system that is the archetypal system giving way to the phenomenon of stochastic resonance. We have used this nonlinear system outside the scope of strict stochastic resonance. We did not operate with a fixed nonlinear system excited by a small (subthreshold) input signal to observe how addition of noise can improve the performance (i.e. stochastic resonance). Instead, we operated with fixed signal and fixed noise and we optimized the parameterization of the nonlinear system for the best efficiency. Our test problem was a detection task on a binary signal corrupted by additive white Gaussian noise. We exposed a methodology to tune the parameters of the bistable nonlinear filter at its best performance that minimizes its probability of detection error. We observed that the optimally tuned nonlinear filter is found to operate outside the domain where stochastic resonance takes place. At the optimal tuning of the nonlinear filter, the input signal is not subthreshold and noise addition cannot bring further improvement beyond the optimal tuning of the filter, but only degradation. Stochastic resonance is a useful nonlinear property for small signals having to cope with nonadjustable systems. Next, we compared the performance of the optimally tuned nonlinear filter to the performance of the matched filter, which is the ultimate optimal system for our detection problem. We observed that, although (expectedly) not as good, the performance of the optimally tuned nonlinear filter comes relatively close to that of the matched filter, and as much as the noise level increases. Next we examined several possible departures, quite plausible in practical operation, from the nominal conditions of the ideal matched filter, and concerning the synchronization and the finite-order implementation of the matched filter. In such degraded conditions we demonstrated that the matched filter can be caught up and outperformed by other suboptimal filters like the bistable nonlinear filter. This tells us that ideal optimal filters are useful in their nominal conditions, but they may not be maximally robust against departures from strict nominal conditions; other filters, although suboptimal, may be more robust and maintain a better performance in varying conditions.

The bistable nonlinear system may belong to this class of robust nonlinear filters. The nonlinear filter of Eq. (1), with time constant τ_a, has a smoothing ability, capable of reducing the noise to bring out the signal. This is essentially also how the linear matched filter operates. In addition, the nonlinear filter of Eq. (1) has a bistable character, which may be useful to restore a binary signal, here, or to resist to some loss of synchronization. Bistability is an additional property, which is not present in the linear matched filter. This somehow expresses richer or more versatile dynamics in nonlinear systems, compared to linear systems. We recall that in other domains of operation, nonlinear systems (in contrast to linear systems) can give way to stochastic resonance for noise-enhanced signal processing. This richness in dynamic behaviors and properties may be the source of the ability of nonlinear systems to maintain good performance in broad conditions.

Beyond the present detection of a binary signal, many other situations could be investigated to complement the analysis of the potentialities of the bistable nonlinear filter in relation to optimal or other systems.

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