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Stochastic resonance and the benefit of noise in nonlinear systems

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Abstract. Stochastic resonance is a nonlinear effect wherein the noise turns out to be beneficial to the transmission or detection of an information-carrying signal. This paradoxical effect has now been reported in a large variety of nonlinear systems, including electronic circuits, optical devices, neuronal systems, material-physics phenomena, chemical reactions. Stochastic resonance can take place under various forms, according to the types considered for the noise, for the information-carrying signal, for the nonlinear system realizing the transmission or detection, and for the quantitative measure of performance receiving improvement from the noise. These elements will be discussed here so as to provide a general overview of the effect. Various examples will be treated that illustrate typical types of signals and nonlinear systems that can give rise to stochastic resonance. Various measures to quantify stochastic resonance will also be presented, together with analytical approaches for the theoretical prediction of the effect. For instance, we shall describe systems where the output signal-to-noise ratio or the input–output information capacity increase when the noise level is raised. Also temporal signals as well as images will be considered. Perspectives on current developments on stochastic resonance will be evoked.

1 Introduction

When linear coupling takes place between signal and noise, usually the noise acts as a nuisance degrading the signal. In contrast, when certain types of nonlinear interaction take place between signal and noise, there may exist a possibility of cooperation between the signal and the noise. The presence of the noise then becomes beneficial to the signal, up to a point where an increase of the noise may improve the performance for transmitting or detecting the signal. Stochastic resonance (SR) designates this type of nonlinear effect whereby the noise can benefit to the signal [53, 70].

This paradoxical effect was first introduced some twenty years ago in the domain of climate dynamics, as an explanation for the regular recurrences of ice ages [2]. Since this origin, SR has been largely developed and extended to a broad variety of domains [52, 25, 51]. Today, it is possible to synthesize the various forms observed for SR by means of the scheme of Fig. 1.

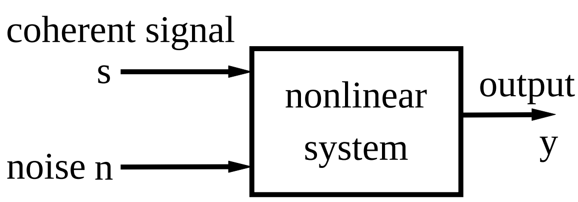

Figure 1: A general scheme for stochastic resonance, which consists in the possibility of increasing the "similarity" between the information-carrying or coherent input signal s and the output signal y by means of an increase of the level of the noise η.

Stochastic resonance, as illustrated by Fig. 1, involves four essential elements:
(i) an information-carrying or coherent signal s: it can be deterministic, periodic or non, or random;
(ii) a noise η, whose statistical properties can be of various kinds (white or colored, Gaussian or non, ...);
(iii) a transmission system, which generally is nonlinear, receiving s and η as inputs under the influence of which it produces the ouput signal y;
(iv) a performance or efficacy measure, which quantifies some "similarity" between the output y and the coherent input s (it may be a signal-to-noise ratio, a correlation coefficient, a Shannon mutual information, ...).
SR takes place each time it is possible to increase the performance measure by means of an increase in the level of the noise η. Historically, the developments of SR have proceeded through variations and extensions over these four basic elements.

From the origin, and for a relatively long period of time, SR studies have concentrated on a *periodic* coherent signal $s(t)$, transmitted by nonlinear systems of a *dynamic* and *bistable* type [26, 50]. This form of SR now appears simply as a special form of SR. Nevertheless, it bears important historical and conceptual significance, and we shall present in the next Section this form of (periodic) SR as our first detailed example of an SR phenomenon.

2 Periodic SR in bistable dynamic systems

This form of SR is based on the evolution equation

$$\tau_a \dot{x}(t) = -\frac{dU(x)}{dx} + s(t) + \eta(t) \ . \tag{1}$$

Such an equation represents a dynamic system whose state $x(t)$ is forced by the input $s(t)+\eta(t)$, and whose free relaxation $\tau_a \dot{x} = -dU/dx$ is governed by a potential $U(x)$ which generally is a double-well potential. A form frequently chosen is the "quartic" potential

$$U(x) = -\frac{x^2}{2} + \frac{x^4}{4X_b^2} \tag{2}$$

with parameter $X_b > 0$, whose shape is depicted in Fig. 2.

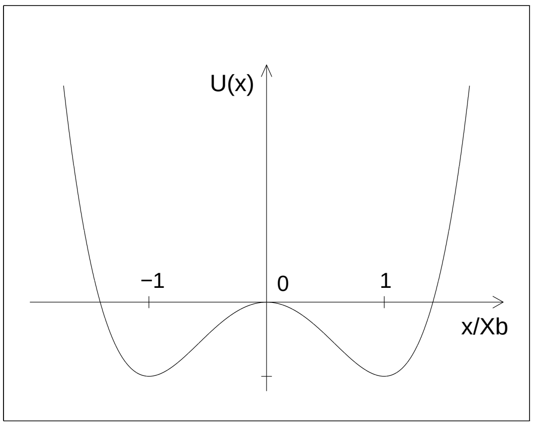

Figure 2: Double-well bistable potential $U(x)$ of Eq. (2).

Because of its double-well potential $U(x)$, the dynamic system of Eq. (1) has two stable stationary states ($x = \pm X_b$ in the case of Eq. (2)) corresponding to the two minima of the potential ($U(x = \pm X_b) = -X_b^2/4$ in the case of Eq. (2)), and separated by a potential barrier (with height $X_b^2/4$ in the case of Eq. (2)).

A mechanical interpretation of this system allows a concrete description of the occurence of the SR phenomenon. In such an interpretation, Eq. (1) describes the motion in an overdamped regime (inertia $\ddot{x}$ is neglected relative to viscous friction $\dot{x}$), of a particle in a potential $U(x)$ subjected to the external force $s(t) + \eta(t)$. If a periodic input $s(t) = A\cos(2\pi t/T_s)$ is applied alone and with a too weak amplitude A, then the particle cannot jump over the potential barrier between the two wells; it remains confined in one of the wells around a potential minimum, with no transitions between wells. One can introduce here a binary output signal $y(t)$, with two states say $y(t) = \pm 1$, indicating which of the two wells the particle is in at time t, for instance

$$y(t) = \mathrm{sign}[x(t)] \tag{3}$$

with the potential of Eq. (2). As no transitions take place between wells, $y(t)$ remains stuck in one of its two states. Then, if a small noise $\eta(t)$ is added, a cooperative effect between the signal $s(t)$ and the noise becomes possible, enabling occasionally the particule to jump over the potential barrier. This translates into transitions between wells which are correlated with the periodic input $s(t)$ as it plays a part in their production (in conjunction with the noise). When the noise level is raised, the probability of occurence of such coherent transitions first increases, thus reinforcing the correlation of the output $y(t)$ with the periodic input $s(t)$. For stronger noise levels, incoherent transitions induced by the noise alone will become more and more frequent, and will gradually destroy the correlation of the output with the periodic input. The noise thus has a nonmonotonic influence, first enhancing the correlation of the output with the periodic input, up to an optimum, after which the correlation is gradually destroyed.

The output $y(t)$ is a random signal, because of the influence of the noise input $\eta(t)$, yet it bears correlation with the periodic input $s(t)$. To quantify the correlation of $y(t)$ with $s(t)$, the standard method starts with the calculation of the autocorrelation function of $y(t)$, and then through Fourier transform, to its power spectral density [53, 70, 25]. In the power spectral density of $y(t)$, the influence of the periodic input $s(t)$ shows up as spectral lines at integer multiples of the coherent frequency $1/T_s$. These lines emerge out of a broadband continuous noise background stemming from incoherent transitions due to $\eta(t)$. This typical constitution of the output power spectral density in periodic SR is depicted in Fig. 3A. The SR effect is identified by coherent spectral lines whose emergence out of the noise background gets more pronounced when the level of the input noise is raised. This is quantified [53, 70, 25] by a signal-to-noise ratio (SNR) at the output, defined as the power contained in the spectral line at $1/T_s$ divided by the power contained in the noise background in the region of $1/T_s$. When the level of the input noise is raised, the ouput SNR experiences a nonmonotonic evolution culminating at a maximum for an optimal nonzero noise level, whence the term resonance.

For the case of the quartic potential of Eq. (2) with a Gaussian white noise $\eta(t)$ of autocorrelation function $\mathrm{E}[\eta(t)\eta(t+\tau)] = 2D\delta(\tau)$, in the regime where the coherent signal is both small and slow, the theory of [50] (based on a master equation for the output state probabilities governed by transition rates of a Kramers type) derives an explicit expression for the SNR under the form:

$$\mathcal{R}_{\mathrm{out}} = \sqrt{2}\frac{A^2 X_b^2/4}{(D/\tau_a)^2}\exp\left(-\frac{X_b^2/4}{D/\tau_a}\right) . \tag{4}$$

A typical evolution of the output SNR of Eq. (4) is depicted in Fig. 3B. The nonmonotonic evolution of the SNR in Fig. 3B with the noise power density D is the signature of SR, which

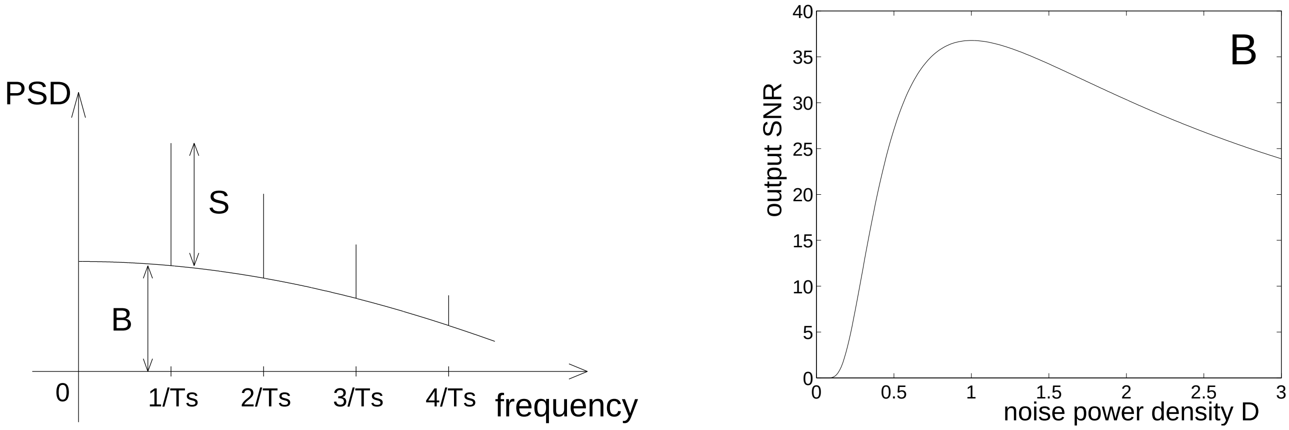

Figure 3: (A) Power spectral density at the output serving for the definition of the signal-to-noise ratio $\mathcal{R}_{\rm out} = S/B$ offering a quantitative measure to periodic SR. (B) Output signal-to-noise ratio $\mathcal{R}_{\rm out}$ from Eq. (4) as a function of the input noise power density D, demonstrating SR in the bistable dynamic system of Eqs. (1)–(3).

is also verified by simulation of the system of Eqs. (1)–(3) [50]. This theory of [50] is applicable for any double-well bistable potential $U(x)$, but with a sinusoidal modulation $s(t)$ both slow and small and with a white Gaussian input noise. This treatment was for example used with a double-well potential differing from the quartic potential, and which describes certain regimes of operation of neurons in networks [4].

It is under this form, associated to a bistable dynamic system of the type of Eq. (1) excited by a noisy sinusoid, that the phenomenon of SR was originally introduced [2]. Under this form also, that SR has received the most sustained attention for a long period of time [25]. This form of SR has been gradually observed experimentally in various processes obeying the bistable dynamics of Eq. (1), and including electronic circuits [23], lasers [69], electron paramagnetic resonance [27], neurons [4, 17], a magnetoelastic pendulum [63], chemical reactions [43, 20], superconducting devices [36]. In these observations, the characterization scheme exposed above for SR, which proceeds through the computation of the output autocorrelation function then to the power spectral density to access the SNR in the frequency domain, can be numerically realized on measured signals. In addition, the theory of [50] can be used to derive an approximate expression of the SNR for a theoretical prediction of SR.

Other theories have been proposed to predict the behavior of the SNR in SR based on Eq. (1). Some approaches consider the Fokker-Planck equation [28] associated to the stochastic differential equation (1) [25, 42, 41, 38]. In this context which is both nonlinear and nonstationary, these approaches lead to approximate expressions for the SNR.

Another method proposed for the theoretical analysis of SR is linear-response theory [21, 19, 22, 55]. This theory is a perturbative method based on the linearization of the response of a nonlinear system for a small coherent input added to the noise. In principle, it can be applied to any nonlinear system, not necessarily bistable dynamic; an application is performed in [65] exhibiting periodic SR in a monostable nonlinear dynamic system. But linear-response theory is restricted to perturbative conditions defined by a small coherent input. However, SR can well take place outside these perturbative conditions [10], especially if one wants to obtain the important property of an output SNR larger than the input SNR in SR [7].

3 Periodic SR in static nonlinear systems

SR in nonlinear dynamic systems as those based on Eq. (1) is usually difficult to theoretically analyze, and deriving an explicit expression for the SNR in given conditions usually requires to ressort to approximations (like those of a small or a slow coherent signal $s(t)$). There exists another class of nonlinear dynamic systems where the theoretical analysis of SR can be performed exactly in many conditions [6, 10]. This class consists of the nonlinear dynamic systems for which the nonlinear and the dynamic characters are separated. These systems are formed by the association of a static or memoryless nonlinearity followed by an arbitrary linear dynamic system, according to Fig. 4, and they are amenable to a general theory [10, 7].

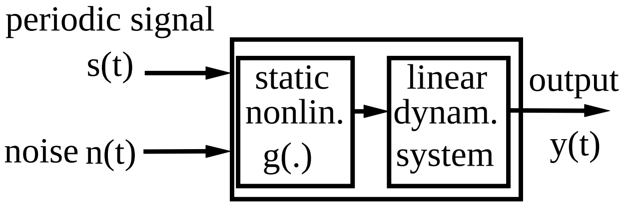

Figure 4: A general class of nonlinear dynamic systems exhibiting SR, and describable by the theory of [10, 7].

We shall now expose an example of periodic SR in a simple instance of the class of Fig. 4, with a complete theoretical description based on the approach of [10]. We shall deal with a nonlinear electronic circuit involving one of the most common nonlinearities found in electronic devices, i.e. the threshold nonlinearity of a PN junction diode [32].

We consider the circuit of Fig. 5, where the input voltage consists of the sum $s(t) + \eta(t)$, with $s(t)$ a T_s-periodic signal and $\eta(t)$ a stationary white noise with the probability density function $f_\eta(u)$. The output voltage $y(t)$ results as a nonstationary, yet cyclostationary [58], random signal bearing correlation with the periodic input $s(t)$.

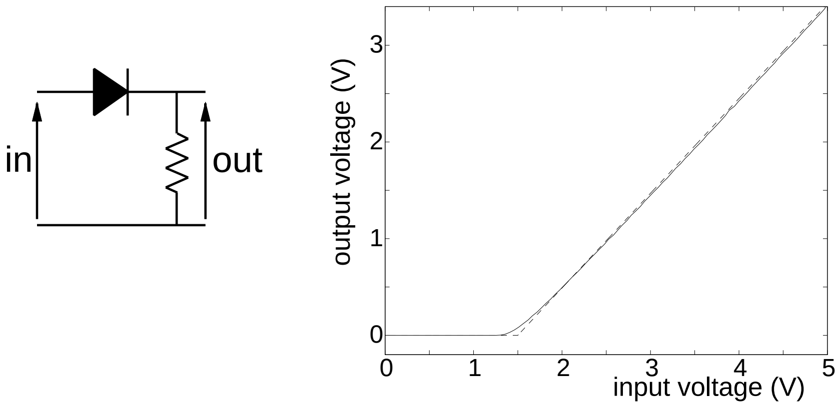

Figure 5: Diode circuit and its input–output static characteristics experimentally recorded (solid line) and its simple piecewise-linear theoretical model (dashed line).

This correlation is quantified here by means of the output autocorrelation function [10]

$$R_{yy}(k\Delta t) = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{E}[y(j\Delta t)y(j\Delta t + k\Delta t)] \ , \tag{5}$$

where, for comparing theory and experiment, we use a discrete-time formulation in which the signals are sampled at a step $\Delta t \ll T_s = N\Delta t$.

The output power spectral density follows, from a discrete Fourier transform of R_{yy} over an integer number $2M$ of period T_s, as

$$P_{yy}(\ell\Delta\nu) = \sum_{k=-MN}^{MN-1} R_{yy}(k\Delta t)\exp\left(-i2\pi\frac{k\ell}{2MN}\right) , \tag{6}$$

with the frequency resolution $\Delta\nu = 1/(2MN\Delta t)$.

The theory of [10] leads in these conditions to an explicit expression for the output power spectral density as

$$P_{yy}(\ell\Delta\nu) = \overline{\mathrm{var}(y)} + \sum_{n=-\infty}^{+\infty} |\overline{Y}_n|^2\, \widehat{\delta}(\ell - 2Mn) , \tag{7}$$

with $\widehat{\delta}(0) = 2MN$ and $\widehat{\delta}(j) = 0$ for an integer $j \neq 0$. In Eq. (7),

$$\overline{Y}_n = \frac{1}{N}\sum_{j=0}^{N-1} \mathrm{E}[y(j\Delta t)]\exp\left(-i2\pi\frac{jn}{N}\right) \tag{8}$$

is the order n Fourier coefficient of the T_s-periodic output expectation $\mathrm{E}[y(j\Delta t)]$ computable as

$$\mathrm{E}[y(t)] = \int_{-\infty}^{+\infty} g(u) f_\eta[u - s(t)]du . \tag{9}$$

Also in Eqs. (7),

$$\overline{\mathrm{var}(y)} = \frac{1}{N}\sum_{j=0}^{N-1} \mathrm{var}[y(j\Delta t)] , \tag{10}$$

with the T_s-periodic output variance $\mathrm{var}[y(t)] = \mathrm{E}[y^2(t)] - \mathrm{E}^2[y(t)]$ computable with

$$\mathrm{E}[y^2(t)] = \int_{-\infty}^{+\infty} g^2(u) f_\eta[u - s(t)]du . \tag{11}$$

In Eqs. (9) and (11), the function $g(u)$ represents the input–output static characteristics of the nonlinear circuit realizing $y = g(s + \eta)$. For the circuit of Fig. 5, the experimental characteristics has been measured and is represented in Fig. 5. We chose the diode (a red LED TLHR 5400 from TEMIC) so as to have an almost linear characteristics above the conduction threshold, well approximated by the simple model $g(u) = u - V_{\mathrm{th}}$ for $u > V_{\mathrm{th}}$, and $g(u) = 0$ otherwise, with $V_{\mathrm{th}} = 1.5\,\mathrm{V}$.

The output power spectral density P_{yy} of Eq. (7) has a shape which conforms to the general model of Fig. 3A. It is formed by spectral lines with amplitude $|\overline{Y}_n|^2$ at integer multiples of the coherent frequency n/T_s, emerging out of a continuous noise background whose magnitude is measured by $\overline{\mathrm{var}(y)}$ constant in the present case of a white noise. The standard ouput SNR given by the power in the line at frequency n/T_s divided by the power in the noise background in a small frequency band ΔB around n/T_s, results as

$$\mathcal{R}_{\mathrm{out}}\left(\frac{n}{T_s}\right) = \frac{|\overline{Y}_n|^2}{\overline{\mathrm{var}(y)}\Delta t\Delta B} . \tag{12}$$

In the case where $\eta(t)$ is a zero-mean Gaussian noise with rms amplitude σ_η, Eq. (9) leads to

$$\mathrm{E}[y(t)] = \frac{\sigma_\eta}{\sqrt{2}}\left\{\frac{1}{\sqrt{\pi}}\exp[-z^2(t)] - z(t)\,\mathrm{erfc}[z(t)]\right\} , \tag{13}$$

and Eq. (11) to

$$\mathrm{E}[y^2(t)] = \frac{\sigma_\eta^2}{2} \left\{ [1 + 2z^2(t)] \,\mathrm{erfc}[z(t)] - \frac{2}{\sqrt{\pi}} z(t) \exp[-z^2(t)] \right\} , \tag{14}$$

with $z(t) = [V_{\mathrm{th}} - s(t)]/(\sigma_\eta \sqrt{2})$.

From Eqs. (13) and (14), the output SNR of Eq. (12) has been computed. Also, this quantity has been experimentally evaluated on the circuit of Fig. 5 in the interesting regime of a subthreshold coherent input $s(t)$. Both results are shown in Fig. 6 (with $\Delta B = 1/T_s$) revealing very good agreement between theory and experiment, taking into account the simple piecewise-linear model adopted for the static characteristics. The nonmonotonic evolution of the output SNR with the input noise level, which culminates at a maximum value for an optimal nonzero noise level, demonstrates the SR effect, by which here a subthreshold coherent input $s(t)$ is aided by noise to overcome the threshold.

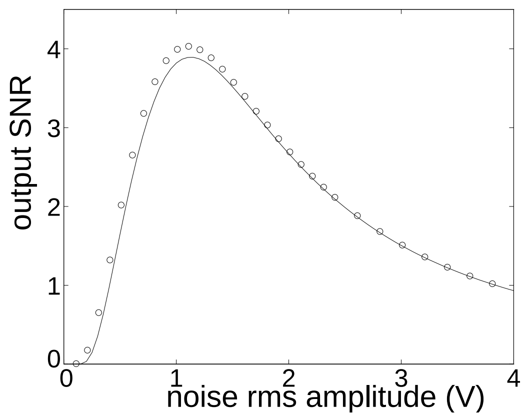

Figure 6: Experimental (circles) and theoretical from Eq. (12) (solid line) ouput SNR $\mathcal{R}_{\mathrm{out}}(1/T_s)$ at the fundamental $1/T_s$, as a function of the rms amplitude σ_η of a zero-mean Gaussian input noise $\eta(t)$, for the diode circuit of Fig. 5 with a sinusoidal input signal $s(t) = A\cos(2\pi t/T_s)$ with $A = 1\,\mathrm{V}$.

The present electronic circuit stands for one of the simplest conceivable stochastic resonators. In particular, it comes here with a complete theoretical analysis, which is not always feasible for more complex stochastic resonators. Other electronic circuits also shown to exhibit SR, like a Schmitt trigger [23] or a chaotic Chua circuit [1], have more complicated nonlinearities that hinder an exact theoretical treatment of the effect, which was essentially exhibited through experiments or numerical simulations.

SR in our diode circuit relies here on the presence of a threshold and a subliminal signal whose transmission is only possible in the presence of the noise, with maximum efficacy for an optimum noise level explicitly predictable with the theory. Other situations exist where a subliminal signal or a threshold are not necessary for SR [10, 3]. Also for SR, the periodic signal $s(t)$ can be replaced either by a high-frequency carrier modulated by a low-frequency message, or by a broadband aperiodic signal [1, 13, 8].

4 Aperiodic SR in a nonlinear information channel

We shall now expose an example of aperiodic SR, i.e. SR with an aperiodic coherent signal $s(t)$ in the scheme of Fig. 1. At the same time, we shall introduce another measure of the performance receiving enhancement from the noise, under the form of an information-theoretic quantity. This will result in an information channel whose capacity is improvable via noise addition.

We consider the transmission of information over a memoryless channel. The input to the channel is a discrete random variable, that we write s to conform to the general scheme of Fig. 1, and which assumes values 1 or -1 respectively with probabilities p_1 and $p_{-1} = 1 - p_1$. Transmission over the channel involves two effects. First a noise η is added to the input s to produce $s + \eta$. Next, $s + \eta$ is compared to a fixed double threshold $\theta > 0$ to determine the discrete output y of the channel according to

$$\begin{aligned} s+\eta < -\theta &\Rightarrow y = -1, \\ -\theta \le s+\eta \le \theta &\Rightarrow y = 0, \\ s+\eta > \theta &\Rightarrow y = 1. \end{aligned} \tag{15}$$

The noise η is a random variable, continuous or discrete, with the cumulative distribution function $F_\eta(u) = \Pr\{\eta \le u\}$. The successive realizations of the noise η are independent and identically distributed, and the same for the random input s. The noise η and the input s are statistically independent.

The present channel can be considered as a binary channel with erasure [15] where the input binary information $s = \pm 1$ can be received as $y = \pm 1$ possibly with an error, or erased when $y = 0$.

For the input–output transfer probabilities of this channel we have, for example, the probability $p_{11} = \Pr\{y = 1 \,|\, s = 1\}$ which is also $\Pr\{s + \eta > \theta \,|\, s = 1\}$ amounting to $\Pr\{\eta > \theta - 1\} = 1 - F_\eta(\theta - 1)$. With similar rules we arrive at

$$p_{11} = \Pr\{y = 1 \,|\, s = 1\} = 1 - F_\eta(\theta - 1) \,, \tag{16}$$

$$p_{-1,1} = \Pr\{y = -1 \,|\, s = 1\} = F_\eta(-\theta - 1) \,, \tag{17}$$

$$p_{1,-1} = \Pr\{y = 1 \,|\, s = -1\} = 1 - F_\eta(\theta + 1) \,, \tag{18}$$

$$p_{-1,-1} = \Pr\{y = -1 \,|\, s = -1\} = F_\eta(-\theta + 1) \,, \tag{19}$$

$$p_{01} = \Pr\{y = 0 \,|\, s = 1\} = F_\eta(\theta - 1) - F_\eta(-\theta - 1) \,, \tag{20}$$

$$p_{0,-1} = \Pr\{y = 0 \,|\, s = -1\} = F_\eta(\theta + 1) - F_\eta(-\theta + 1) \,. \tag{21}$$

Once the transfer probabilities are known, the input–output mutual information $I(s; y)$ can be evaluated from the entropies [15] as

$$I(s; y) = H(y) - H(y \,|\, s) \,. \tag{22}$$

With $h(u) = -u \log_2(u)$, the output entropy $H(y) = \sum_y h(\Pr\{y\})$ is here

$$H(y) = h[p_1 p_{11} + (1 - p_1) p_{1,-1}] + h[p_1 p_{-1,1} + (1 - p_1) p_{-1,-1}] + h[p_1 p_{01} + (1 - p_1) p_{0,-1}] \,, \tag{23}$$

and the input–output conditional entropy $H(y|s) = p_1 \sum_y h(\Pr\{y|s{=}1\}) + (1-p_1) \sum_y h(\Pr\{y|s{=}-1\})$ is

$$H(y \,|\, s) = p_1 [h(p_{11}) + h(p_{01}) + h(p_{-1,1})] + (1 - p_1)[h(p_{1,-1}) + h(p_{0,-1}) + h(p_{-1,-1})] \,. \tag{24}$$

Equations (22), (23) and (24) provide an explicit expression for the mutual information $I(s; y)$ as a function of the transfer probabilities (16)–(21) and the input probability p_1. In the case where the input noise η is symmetric, i.e. with an even probability density, the transmission process is symmetric, and the mutual information is invariant in the exchange of the values of p_1 and $p_{-1} = 1 - p_1$. As the mutual information of a discrete memoryless channel is always a concave function of the input probability distribution [15], we deduce that for η symmetric, the maximum of $I(s; y)$ which defines the information capacity C of the channel, is reached for $p_1 = 0.5 = p_{-1}$. In case of a symmetric noise, Eqs. (22), (23) and (24) with $p_1 = 0.5$, thus allow an explicit evaluation of the capacity C of the channel. We can now examine, on the capacity C, the influence of the noise η as conveyed by the function $F_\eta(u)$.

Figure 7A shows the capacity C as a function of the rms amplitude σ_η of the noise η chosen zero-mean Gaussian. When the threshold $\theta < 1$, an input $s = 1$ (respectively $s = -1$) is alone sufficent to trigger an output $y = 1$ (resp. $y = -1$). In the absence of the noise η the capacity is thus $C = 1$ bit. Addition of noise is then only felt as a nuisance, entailing a decay of C as σ_η increases above zero. In contrast, when $\theta > 1$, and input $s = 1$ (resp. $s = -1$) is alone insufficent to trigger an output $y = 1$ (resp. $y = -1$). In the absence of the noise η the capacity is thus $C = 0$ and the channel is unable to transmit any information. It is the addition of noise which allows the transmission of information, through a cooperative effect where the noise and the input s collaborate to overcome the threshold. This translates into a nonzero capacity C of the channel in the presence of the noise, with a region where C increases as σ_η increases, up to an optimal noise level where C is maximized.

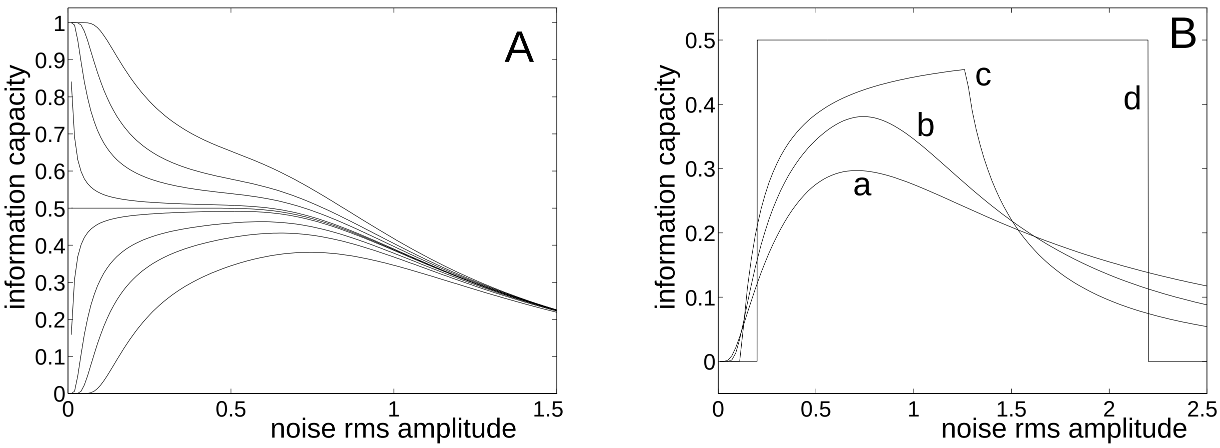

Figure 7: Information capacity C (in bits) of the channel of Eq. (15) as a function of the rms amplitude σ_η of the noise η. Panel A: η is zero-mean Gaussian; the 9 curves are obtained for 9 values of the threshold θ, with successively from the uppest curve to the lowest: $\theta = 0.8$, 0.9, 0.95, 0.99, 1, 1.01, 1.05, 1.1, 1.2. Panel B: $\theta = 1.2$ and the symmetric zero-mean noise η being (a) two-sided exponential, (b) Gaussian, (c) uniform, (d) two-level discrete.

Our model also predicts an influence of the noise distribution on the SR effect of noise-enhanced capacity. For illustration, we have tested four different symmetric noises sharing the same rms amplitude σ_η. Figure 7B shows that in each case the noise enhancement of the capacity is preserved, yet with an influence from the noise distribution.

The results of the present Section demonstrate that information-theoretic measures are capable of quantifying an effect of noise-enhanced transmission of an aperiodic random information-carrying signal. This is a form of aperiodic SR. Other examples of SR characterized by information-theoretic measures can be found in [44, 56, 5, 35, 8, 31, 16]. Various studies have used information-theoretic measures to establish that aperiodic SR can take place in information transmission by neurons [44, 5, 16].

5 Aperiodic SR in image transmission

Recently the phenomenon of SR has been applied to the noise-enhanced transmission of bidimensional spatial signals or images, instead of monodimensional temporal signals. Various studies have considered SR in image perception by the visual nervous system [62, 59, 71]. SR has also been reported in image transmission by nonlinear optical devices [68].

For illustration, we shall expose here a simple example of SR on images, which again has the advantage of allowing a complete theoretical analysis. Also, at this occasion we shall introduce a new class of quantitative measures for SR based on input–output cross-correlations [14, 13].

Leaning again on the general scheme of Fig. 1, we consider this time that the coherent information-carrying signal s is a bidimensional image where the pixels are indexed by integer coordinates (ℓ, m) and have intensity $s(\ell, m)$. For a simple illustration, we take here a binary image with $s(\ell, m) \in \{0, 1\}$. A noise $\eta(\ell, m)$, statistically independent of $s(\ell, m)$, linearly corrupts each pixel of image $s(\ell, m)$. The noise values are independent from pixel to pixel, and are identically distributed with the cumulative distribution function $F_\eta(u) = \Pr\{\eta(\ell, m) \leq u\}$. A nonlinear detector, that we take as a simple hard limiter with threshold θ, receives the sum $s(\ell, m) + \eta(\ell, m)$ and produces the output image $y(\ell, m)$ according to:

$$\begin{aligned} \text{If} \quad s(\ell, m) + \eta(\ell, m) > \theta \quad \text{then} \quad & y(\ell, m) = 1 \\ \text{else} \quad & y(\ell, m) = 0 \; . \end{aligned} \tag{25}$$

When the intensity of the input image $s(\ell, m)$ is low relative to the threshold θ of the detector, i.e. when $\theta > 1$, then $s(\ell, m)$ (in the absence of noise) remains undetected as the output image $y(\ell, m)$ remains a dark image. Addition of the noise $\eta(\ell, m)$ will then allow a cooperation between the intensities of images $s(\ell, m)$ and $\eta(\ell, m)$ to overcome the detection threshold. The result of this cooperative effect can be visually appreciated on Fig. 8, where an optimal nonzero noise level maximizes the visual perception.

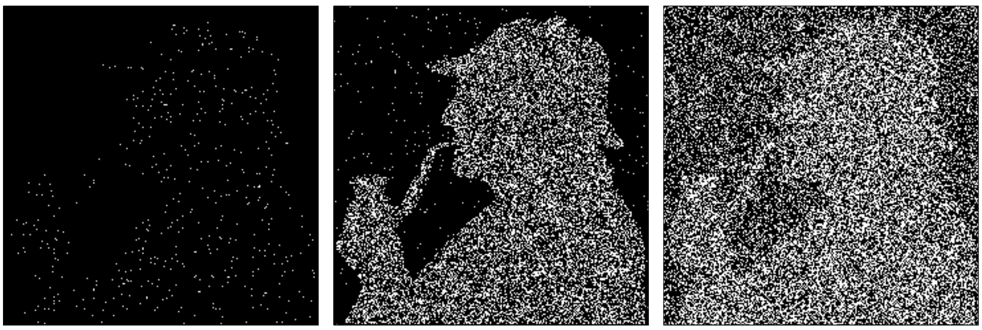

Figure 8: The 256×256 image $y(\ell, m)$ at the output of the detector of Eq. (25) with threshold $\theta = 1.2$, when $\eta(\ell, m)$ is a zero-mean Gaussian noise with rms amplitude 0.1 (left), 0.5 (center) and 2 (right).

It is possible to quantitatively characterize the effect visually perceived in Fig. 8. An appropriate quantitative measure of the similarity between input image $s(\ell, m)$ and output image $y(\ell, m)$, is provided by the normalized cross-covariance [68]

$$C_{sy} = \frac{\langle (s - \langle s \rangle)(y - \langle y \rangle) \rangle}{\sqrt{\langle (s - \langle s \rangle)^2 \rangle \, \langle (y - \langle y \rangle)^2 \rangle}} \; , \tag{26}$$

where $\langle . \rangle$ denotes an average over the images.

Another possibility is provided by the mutual information I_{sy} between the pixels of images $s(\ell, m)$ and $y(\ell, m)$,

$$I_{sy} = H(y) - H(y|s) , \quad (27)$$

with standard definitions [61] for the entropies $H(y)$ and $H(y|s)$.

Both measures C_{sy} and I_{sy} can be experimentally evaluated through pixels counting on images similar to those of Fig. 8. Also, for the simple transmission system of Eq. (25), both measures C_{sy} and I_{sy} can receive explicit theoretical expressions, as a function of $p_1 = \Pr\{s(\ell, m) = 1\}$ the probabilty of a pixel at 1 in the binary input image $s(\ell, m)$, and as a function of the properties of the noise conveyed by $F_\eta(u)$ [68].

For the scenario of Fig. 8, Fig. 9 represents these quantitative measures C_{sy} and I_{sy}, with both their experimental and theoretical evaluations which are in close agreement. Both measures identify a maximum efficacy in image transmission for an optimal nonzero noise level, another instance of SR.

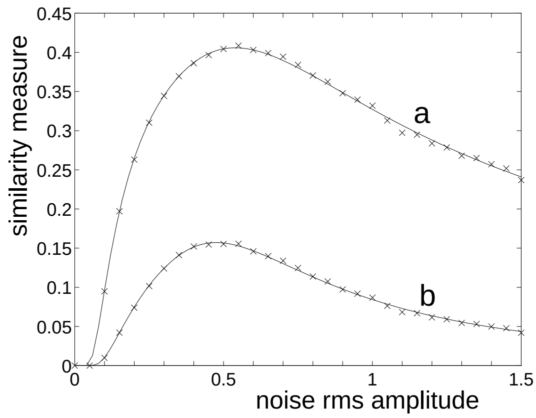

Figure 9: Input–output similarity measures between input image $s(\ell, m)$ and output image $y(\ell, m)$, as a function of the rms amplitude of the noise $\eta(\ell, m)$ chosen zero-mean Gaussian. (a) is the cross-covariance C_{sy} of Eq. (26), (b) is the mutual information I_{sy} of Eq. (27). The crosses are experimental evaluations through pixels counting on images, the solid lines are the theoretical predictions ($p_1 = 0.6$).

6 Perspectives

The above examples that we have treated are typical of the signals and nonlinear mechanisms by which SR can occur, and of how SR can be measured. Such processes have received numerous experimental embodiments materializing SR in many areas of physical sciences [52, 25]. Among the systems specially studied that exhibit SR, one finds electronic circuits [23, 1, 30], optical devices [69, 39, 18, 40, 68], neuronal systems [4, 17, 11, 9], phenomena from material physics [27, 54, 34], chemical reactions [43, 20, 37]. Most of the time SR is described at the macroscopic level by means of concepts from classical physics; but quantum forms of SR are also discussed [49, 25, 66].

SR, as presented here as an effect of noise-enhanced signal transmission, encompasses and generalizes other useful noise techniques that were previously known for specific purposes. This is the case with "dithering", which designates a technique of noise addition used in signal quantization [67, 33] or image coding [60]. An application of dithering aims at limiting the distortion to an analog signal in its analog-to-digital conversion. The technique specifies to add to the analog signal prior to its quantization a white noise uniform over the quantization step, this to allow exact recovery of the mean of the analog signal from its digital representation. Dithering can be seen as a special form of SR, on a special nonlinear system (i.e. analog-to-digital converter) with a special performance measure (i.e. minimum input–output distortion of the mean) [24, 10].

The "modern" approach to SR is thus, in the presence of given signals and nonlinear system, to select one specific performance measure endowed with a special significance in relation to the context or prospect involved, and then to examine whether conditions exist where the performance can be enhanced by addition of noise.

The examples of SR systems that we have treated here are nonlinear systems involving potential barriers or threshold nonlinearities. In such systems, SR takes place when the coherent signal alone is unsufficient to overcome the barrier or the threshold, and assistance to this aim is provided by the noise. Nevertheless, we emphasize that these nonlinear ingredients (barriers, thresholds) are not strictly necessary for the occurence of SR. The phenomenon of SR has been shown to operate in nonlinear systems without barriers [65] and without threshold [10, 3]: [65] in a single-well monostable dynamic system, [10] in smooth monotonic or nonmonotonic static nonlinearities, [3] in thermal kinetics driven by the Boltzmann exponential factor. At a general level, SR can be interpreted as a displacement, by means of noise addition, of the operating point of a nonlinear system into a region more favorable to the coherent signal. Standard techniques would use a constant bias for displacing the operating point, meanwhile SR achieves it by noise addition.

Current developments to SR are concerned with the extension of the "signal-enhanced-by-noise" effect to broader conditions and domains. Among others one can cite the domain of array-enhanced SR where SR systems are interconnected to interact in arrays [45, 47], the domain of noise-enhanced propagation of nonlinear waves [48, 72, 46], extensions of SR to multidimensional signals or images [68, 71]. SR studies are particularly active and relevant in the domain of neural information processing [29, 64, 12, 57, 16]. Neurons are examples of intrinsically nonlinear and noisy systems, in which SR operates, and that achieve very high performances for signal and information processing, through mechanisms that mainly remain to be understood.

In parallel to such analysis-oriented studies, another line of current development is the exploitation of SR for practical purposes and as a basis for competitive methods for signal and information processing. This direction of exploration is still in its infancy. It can have an impact if one is ready to depart, as neural systems suggest, from the realm of linear signal processing (as soon as the lower levels of processing) to come to the stage of nonlinear processing, more difficult to master but potentially rich of useful effects, as exemplified by SR.

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