EUSIPCO 2002, 11th European Signal Processing Conference, Toulouse, France, 3-6 Sept. 2002, Vol. 1, pp. 185–188

IDENTIFICATION AND MODELING OF FRACTAL SIGNALS WITH ITERATED FUNCTION SYSTEMS

Christophe PORTEFAIX, Christine CAVARO-MÉNARD, François CHAPEAU-BLONDEAU
Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers,
62 avenue Notre Dame du Lac, 49000 ANGERS, FRANCE.
christophe.portefaix@istia.univ-angers.fr, Christine.Menard@univ-angers.fr, chapeau@univ-angers.fr

ABSTRACT

We analyze a minimal model of iterated function systems (IFS) and show possibilities for controlling properties of the fractal signals they generate. Applications for signal modeling and spread-spectrum communications illustrate the potentialities of IFS for signal processing.

1 INTRODUCTION

Iterated function systems (IFS) have recently been introduced in the context of fractal geometry, and have found applications for image coding, signal interpolation or modeling [1, 2, 3]. Beyond, IFS present interesting potentialities for many areas of signal processing, still to be explored. We present and analyze here a simple model of IFS and show possibilities for controlling properties of the fractal signals they generate.

2 AN IFS MODEL

We consider one-dimensional signals $s(x) \in \mathbb{R}$ where the abscissa x is a time-like variable defined over the support $x \in [0, 1[$. Two transformations are introduced, both operating on the abscissa $x \in [0, 1[$ and on the corresponding signal amplitude $s(x)$:

$$T_1 \left| \begin{array}{ccc} [0,1[\times \mathbb{R} & \longrightarrow & [0,1/2[\times \mathbb{R} \\ (x,\, s(x)) & \longmapsto & (x/2,\ a_1 s(x) + b_1) \end{array} \right. \quad (1)$$

and

$$T_2 \left| \begin{array}{ccc} [0,1[\times \mathbb{R} & \longrightarrow & [1/2,1[\times \mathbb{R} \\ (x,\, s(x)) & \longmapsto & (1/2 + x/2,\ a_2 s(x) + b_2) \end{array} \right. \quad (2)$$

with real coefficients a_j and b_j verifying $0 < |a_j| < 1$, for $j = 1, 2$, so as to have contractive mappings.

The union $T_1 \cup T_2$ provides a transformation $s(x) \longmapsto T[s(x)]$ mapping a signal $s(x)$ with support $[0, 1[$ onto another signal $T[s(x)]$ with the same support $[0, 1[$. For T_1 and T_2 both parts of the transformation for abscissa x and for signal amplitude s are contractive affine transforms. Consequently, the mapping $s(x) \longmapsto T[s(x)]$ is also a contractive affine transform. It results that $s(x) \longmapsto T[s(x)]$ admits one single fixed point, i.e. a signal $\sigma(x)$ verifying $T[\sigma(x)] = \sigma(x)$ also called the attractor of transformation T [1]. Starting from any initial signal $s_0(x)$, iterative application of the transformation T defined by the scheme of Eqs. (1)–(2) realizes an IFS. The process converges to a unique attractor $\sigma(x)$ which is completely determined by the four parameters (a_1, b_1, a_2, b_2). An important property of this correspondence [1] is that small smooth changes in (a_1, b_1, a_2, b_2) are associated to small smooth changes in $\sigma(x)$.

An attractor $\sigma(x)$ verifies a self-transformability relation $T[\sigma(x)] = \sigma(x)$ through the affine transforms of Eqs. (1)–(2), which confers to it a property of self-affinity or a fractal character. This translates into very complicated shapes for $\sigma(x)$ with structures or details occuring at all scales. The set of attractors $\sigma(x)$ when (a_1, b_1, a_2, b_2) is varied forms a class of signals exhibiting properties not common to other models of signals. Based on their fractal character, such signals can offer models for complicated signals especially originating in complex natural processes [4, 5, 6]. IFS more elaborated than Eqs. (1)–(2), where the partioning of the signal support is modifiable, are at the root of fractal image coding [2]. Also, an application exists for fractal interpolation [1, 3], where the degrees of freedom offered by the parameters, analog to (a_1, b_1, a_2, b_2), of the IFS are used to force the attractor to pass through specific points $(x_j, s(x_j))$. This is one aspect of the important general issue concerning IFS, of controlling or imposing specific properties for the attractor through the choice of the transformation parameters. We address several other aspects of this issue in the following.

3 LINEAR COEFFICIENTS

For the control of the attractor $\sigma(x)$ via (a_1, b_1, a_2, b_2), it is in general very difficult to obtain an explicit expression for $\sigma(x)$ as a function of (a_1, b_1, a_2, b_2). Nevertheless, it is feasible to obtain expressions for various linear coefficients associated to $\sigma(x)$ as functions of (a_1, b_1, a_2, b_2). For any function $f(x)$, we define

$$\langle f(x) \rangle = \int_0^1 f(x)dx \ , \quad (3)$$

and consider the linear coefficient $\langle f(x)\sigma(x)\rangle$. Over $x\in[0,1/2[$, according to Eq. (1) the integrand $f(x)\sigma(x)$ reduces to $f(x)[a_1\sigma(2x)+b_1]$, for which the change of variable $u=2x$ is performed, and over $x\in[1/2,1[$, according to Eq. (2) the integrand $f(x)\sigma(x)$ reduces to $f(x)[a_2\sigma(2x-1)+b_2]$, for which the change of variable $v=2x-1$ is performed. This leads to

$$\begin{aligned}\langle f(x)\sigma(x)\rangle = \;& \frac{1}{2}\int_0^1 f\Big(\frac{u}{2}\Big)[a_1\sigma(u)+b_1]du + \\ & \frac{1}{2}\int_0^1 f\Big(\frac{1}{2}+\frac{v}{2}\Big)[a_2\sigma(v)+b_2]dv\ ,\end{aligned} \tag{4}$$

or equivalently

$$\begin{aligned}\langle f(x)\sigma(x)\rangle = \frac{1}{2}\Big[a_1\Big\langle f\Big(\frac{x}{2}\Big)\sigma(x)\Big\rangle + b_1\Big\langle f\Big(\frac{x}{2}\Big)\Big\rangle\Big] + \\ \frac{1}{2}\Big[a_2\Big\langle f\Big(\frac{1}{2}+\frac{x}{2}\Big)\sigma(x)\Big\rangle + b_2\Big\langle f\Big(\frac{1}{2}+\frac{x}{2}\Big)\Big\rangle\Big]\ .\end{aligned} \tag{5}$$

For appropriate choices of $f(x)$, Eq. (5) will provide useful linear equations relating various linear coefficients of $\sigma(x)$ to (a_1,b_1,a_2,b_2). For instance, when $f(x)=\exp(-in2\pi x)$, the quantities $S_n=\langle\exp(-in2\pi x)\sigma(x)\rangle$, for n integer are the Fourier coefficients of the attractor $\sigma(x)$ and are found to verify according to Eq. (5)

$$2S_n = \big[a_1+(-1)^n a_2\big]S_{n/2}+\big[b_1+(-1)^n b_2\big]\frac{(-1)^n-1}{n\pi}i\ . \tag{6}$$

Equation (6) can be exploited to impose, through (a_1,b_1,a_2,b_2), four coefficients among the harmonic Fourier coefficients S_n or the subharmonic coefficients $S_{n/2}$.

When $f(x)=x^n$, the moments $\mu_n=\langle x^n\sigma(x)\rangle$ (n integer) of the attractor $\sigma(x)$ are found to verify, according to Eq. (5),

$$\big(2^{n+1}-a_1-a_2\big)\mu_n = a_2\sum_{k=0}^{n-1}\binom{n}{k}\mu_k + \frac{b_1+(2^{n+1}-1)b_2}{n+1} \tag{7}$$

with the binomial coefficients $\binom{n}{k}$. Again, Eq. (7) can be exploited to impose, through (a_1,b_1,a_2,b_2), four moments among the μ_n. If the first four moments are chosen, one obtains

$$\big(2-a_1-a_2\big)\mu_0 = b_1+b_2 \tag{8}$$

$$\big(2^2-a_1-a_2\big)\mu_1 - a_2\mu_0 = \frac{b_1}{2}+3\frac{b_2}{2} \tag{9}$$

$$\big(2^3-a_1-a_2\big)\mu_2 - a_2(2\mu_1+\mu_0) = \frac{b_1}{3}+7\frac{b_2}{3} \tag{10}$$

$$\big(2^4-a_1-a_2\big)\mu_3 - a_2(3\mu_2+3\mu_1+\mu_0) = \frac{b_1}{4}+15\frac{b_2}{4} \tag{11}$$

Equations (8)–(11) are linear relations that force $(\mu_0,\mu_1,\mu_2,\mu_3)$ when (a_1,b_1,a_2,b_2) are chosen, or conversely. For instance, the average value $\langle\sigma(x)\rangle$ is $\mu_0=(b_1+b_2)/(2-a_1-a_2)$ according to Eq. (8).

4 IDENTIFICATION OF THE IFS

An application of the relations derived in Section 3 is the identification of the parameters (a_1,b_1,a_2,b_2) of the IFS from the observation of its attractor $\sigma(x)$. If it is known that a given signal $\sigma(x)$ is the attractor of a transformation as in Eqs. (1)–(2), then coefficients S_n or moments μ_n can be measured on $\sigma(x)$, and Eqs. (6) or (7) provide linear equations that can be solved to deduce the parameters (a_1,b_1,a_2,b_2). For illustration, we consider the transformation of Eqs. (1)–(2) with the choice $(a_1=-0.8,b_1=-1.1,a_2=-0.7,b_2=1.2)$, associated according to Eqs. (8)–(11), to the theoretical moments $(\mu_0=0.0286,\mu_1=0.2236,\mu_2=0.2211,\mu_3=0.1869)$. The resulting attractor $\sigma(x)$ of Fig. 1 has been produced through a numerical implementation of Eqs. (1)–(2) with 2000 equispaced values of x over $[0,1[$. The first four moments $(\mu_0,\mu_1,\mu_2,\mu_3)$ were then measured on $\sigma(x)$ and they matched their theoretical values with an accuracy of about 3×10^{-3}. These measured moments were then used in Eqs. (8)–(11) and allowed to identify the parameters (a_1,b_1,a_2,b_2) of the IFS with an accuracy of about 6×10^{-3} from its observed attractor $\sigma(x)$.

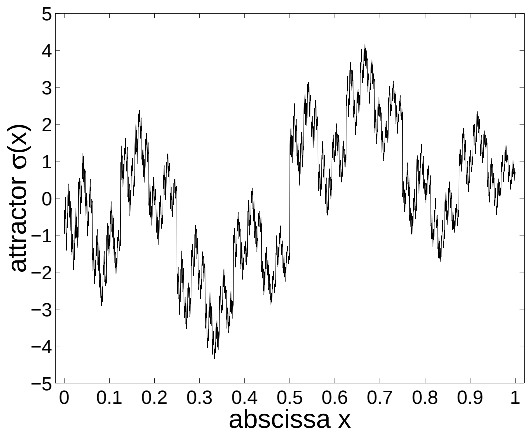

Figure 1: Attractor $\sigma(x)$ of Eqs. (1)–(2) with $(a_1=-0.8,b_1=-1.1,a_2=-0.7,b_2=1.2)$ used for the identification of the parameters of the IFS from the measurement of the first four moments $(\mu_0,\mu_1,\mu_2,\mu_3)$ of $\sigma(x)$.

The parameters (a_1,b_1,a_2,b_2) of the IFS could also be identified from measured Fourier coefficients S_n via Eq. (6). Furthermore, identification can be performed from a noisy version of the attractor $\sigma(x)$. Especially, good noise rejection capabilities can be expected if the four coefficients that are measured for the identification (among the μ_n or the S_n) are chosen such that the noise has statistically little impact on these coefficients. For instance, attractor $\sigma(x)$ of Fig. 1 has an rms value $\sigma_{\rm eff}=1.75$. On Fig. 2(a), it has been additively corrupted by a zero-mean Gaussian white noise with a standard deviation of 1.75. We thus have a signal-to-noise ratio of unity, but on the moments μ_n the statistical average of the noise vanishes. Measurement of the first four moments of the noisy attractor of Fig. 2(a) gave ($\mu_0 = -0.0147, \mu_1 = 0.2222, \mu_2 = 0.2282, \mu_3 = 0.1953$). Application of Eqs. (8)–(11) then yielded the identification of ($a_1 = -0.797, b_1 = -1.277, a_2 = -0.652, b_2 = 1.226$). Injection of these values in the IFS of Eqs. (1)–(2) finally led to the reconstructed attractor of Fig. 2b which realizes an rms error of 0.145 with the original attractor of Fig. 1. This result can be seen as a denoising of the noisy attractor of Fig. 2a, given that it is known that it belongs to the class of signals defined by Eqs. (1)–(2), where the signal-to-noise ratio has been improved by a factor of more than 10.

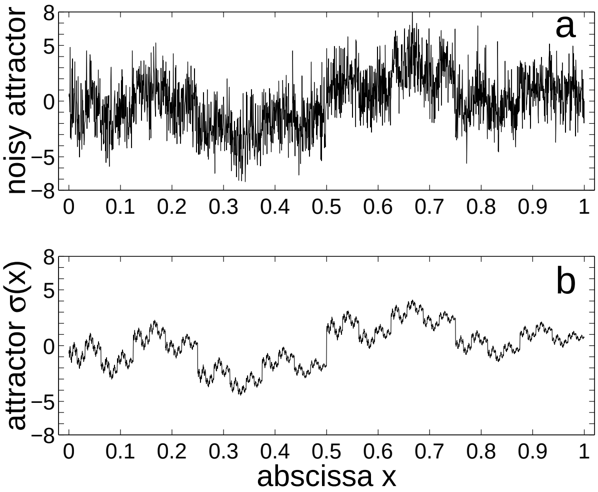

Figure 2: (a) Attractor $\sigma(x)$ of Fig. 1 additively corrupted by a zero-mean Gaussian white noise with a signal-to-noise ratio of unity. (b) Reconstructed attractor from the identification of the IFS parameters realizing an improvement by 10 of the signal-to-noise ratio.

5 SIGNAL MODELING

Another application of the relations of Section 3 is the modeling of signals as attractors of a given IFS as in Eqs. (1)–(2) specified by its parameters (a_1, b_1, a_2, b_2). Owing to the very specific self-transformability property verified by the attractor of Eqs. (1)–(2), not all signals can be expected to be modeled adequately in this way. Yet, signals from natural origin with fractal or self-similarity properties may constitute good candidates. It is the case for Brownian motions, a class of signals that can represent many natural processes. We observed that the attractors of the IFS defined by Eqs. (1)–(2) can provide very parsimonious models for realizations of Brownian motions.

In Fig. 3 are shown several realizations of a Brownian motion. For each realization, the four moments $(\mu_0, \mu_1, \mu_2, \mu_3)$ have been measured and Eqs. (8)–(11) have been used to determine the parameters (a_1, b_1, a_2, b_2) of an IFS whose attractor $\sigma(x)$ shares the same values for the moments $(\mu_0, \mu_1, \mu_2, \mu_3)$. The resulting attractor $\sigma(x)$ is also shown for each case in Fig. 3 and it serves as a model for the corresponding realization of the Brownian motion.

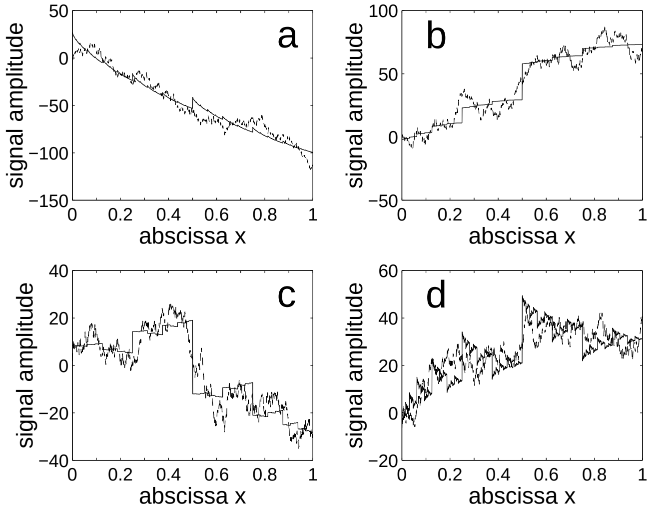

Figure 3: In each of the four panels: One realization of a Brownian motion (dotted line) whose moments $(\mu_0, \mu_1, \mu_2, \mu_3)$ are matched by the attractor (solid line) $\sigma(x)$ of an IFS as in Eqs. (1)–(2).

The ability of the attractors of the IFS Eqs. (1)–(2) to approximate realizations of a Brownian motion comes from their self-affinity property, which makes them suitable to capture the power-law structure $1/f^2$ of the power spectrum of the Brownian motion, as illustrated in Fig. 4.

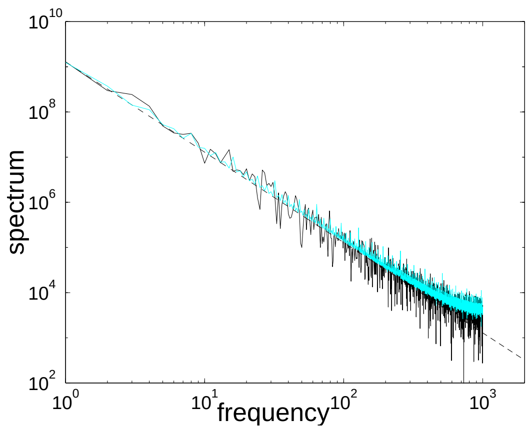

Figure 4: Estimation of the power spectrum (black) of the Brownian motion of Fig. 3a compared to the spectrum (gray) of the corresponding IFS attractor of Fig. 3a. The dotted line has slope -2.

Another possible application of the IFS of Eqs. (1)–(2) is to consider the attractor $\sigma(x)$ associated to the parameters (a_1, b_1, a_2, b_2) and an another attractor $\sigma'(x)$ associated to (a'_1, b'_1, a'_2, b'_2). If we operate in a similar way as in Section 3, it can be established that, between

$\sigma(x)$ and $\sigma'(x)$, the cross-correlation

$$C = \int_0^1 \sigma(x)\,\sigma'(x) dx \ , \qquad (12)$$

is expressable as

$$C = \frac{(a_1 b_1' + a_2 b_2')\mu_0 + (a_1' b_1 + a_2' b_2)\mu_0' + b_1 b_1' + b_2 b_2'}{2 - a_1 a_1' - a_2 a_2'} \qquad (13)$$

with the average values μ_0 and μ_0' given by Eq. (8).

Equation (13) can be used to select the parameters (a_1, b_1, a_2, b_2) and (a_1', b_1', a_2', b_2') so as to impose orthogonality, i.e. a zero cross-correlation C between $\sigma(x)$ and $\sigma'(x)$. An example of two orthogonal fractal waveforms generated in this way are shown in Fig. 5.

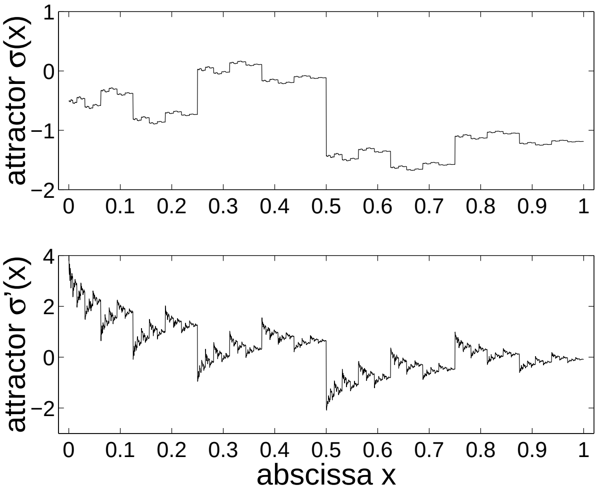

Figure 5: Two orthogonal fractal signals generated through Eq. (13) and the IFS of Eqs. (1)–(2).

Since four parameters are associated to one given attractor, it may be possible to impose that one attractor be orthogonal to four others, and construct a set of different attractors that will be mutually (pair-wise) orthogonal. At the same time, since these attractors are fractal compactly-supported signals with details at all scales, they have a broadband frequency spectrum and can form useful families of waveforms for spread-spectrum communications or CDMA techniques [7]. These broadband waveforms are defined by only four parameters (a_1, b_1, a_2, b_2) and their exchange over a communication channel in a spread-spectrum scheme with transmitted references can be made very cost-effective. Also, the number of sampling points to define the waveform through the IFS of Eqs. (1)–(2) is arbitrary; it can be made large to improve operation at the correlation receiver in a digital scheme, but at no extra cost other than the four parameters to describe the reference.

6 DISCUSSION

The IFS of Eqs. (1)–(2) with four parameters (a_1, b_1, a_2, b_2) is the simplest IFS that can be conceived to generate fractal signals. Understanding of these minimal IFS models is useful to shed light onto more general and composite conditions. We have seen here possibilities for controlling several properties of the fractal attractors through the choice of (a_1, b_1, a_2, b_2).

These simple IFS models can be extended in many different ways. For instance, the affine transformations of the signal amplitudes can be implemented in the complex plane, or they can be made dependent on x or nonlinear, provided they remain contractive. Also, the partioning of the support $[0, 1[$ can be devised with great flexibility, into more than two sub-intervals, and parameterized. It is this possibility that is exploited in fractal image coding [2], where the support (the image) has to be adequately partioned into "domain blocks" which are contractively transformed into smaller "range blocks". Once the "geometrical" mapping between one range block and one domain block has been settled, the signal amplitudes (the gray levels) are mapped through an affine transform of the type $a_1 s(x) + b_1$ where the parameters (a_1, b_1) are determined by minimizing the mean-squared difference between the gray-levels over the domain block and those over the associated range block being under coding. By contrast here, we address signal coding, not through minimization of a mean-squared difference, but through exact matching of coefficients (Fourier, moments), and also our partioning of the support $[0, 1[$ is fixed and not parameterized. We nevertheless obtain rich signal models under the form of fractal attractors, with properties we show controllable through (a_1, b_1, a_2, b_2).

In addition to image coding, IFS present interesting potentialities for many areas of signal processing, still to be explored, and based on their unique ability to model signals as attractors of an iterated convergent signal transformation, with control over the signals provided through the parameters of the transformation.

References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, New York, 1993.

[2] N. Lu, *Fractal Imaging*, Academic Press, New York, 1997.

[3] D. S. Mazel, M. H. Hayes, "Using iterated function systems to model discrete sequences", *IEEE Trans. on Signal Processing*, vol. 40, pp. 1724–1734, 1992.

[4] J. Lévy Véhel, K. Daoudi, E. Lutton, "Fractal modeling of speech signals", *Fractals*, vol. 2, pp. 379–382, 1994.

[5] F. Chapeau-Blondeau, "(max, +) dynamic systems for modeling traffic with long-range dependence", *Fractals*, vol. 6, pp. 305–311, 1998.

[6] F. Chapeau-Blondeau, A. Monir, "Generation of signals with long-range correlation", *Electronics Letters*, vol. 37, pp. 599–600, 2001.

[7] M. K. Simon, J. K. Omura, R. A. Scholtz, B. K. Levitt, *Spread Spectrum Communications Handbook*, McGraw-Hill, New York, 1994.