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# MODELLING OF FRACTAL IMAGES WITH ITERATED FUNCTION SYSTEMS: MOMENT MATCHING, CONTINUITY OF ATTRACTORS.

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## ABSTRACT

We describe and analyze a very parsimonious model of iterated function systems (IFS) and address the issue of controlling the properties of the fractal images they generate. We specially focus on image modelling through control of the geometrical moments of the fractal attractors and on conditions for imposing continuity to such fractal images.

## 1. INTRODUCTION

Iterated function systems (IFS) have recently been introduced in the context of fractal geometry. For image processing, IFS have found applications for image synthesis and image compression [1, 2, 3]. IFS as an emerging tool still contain rich potentialities to be explored for image processing. Here, we describe and analyze a very parsimonious model of IFS and address the issue of controlling the properties of the fractal images they generate. We specially focus on image modelling through control of the geometrical moments of the fractal attractors and on conditions for imposing continuity to such fractal images.

## 2. IFS MODEL

We consider the set $\mathcal{I}$ of two-dimensional signals or images $s(x, y) \in \mathbb{R}$ with spatial coordinates $(x, y)$ defined over the support $[0, 1[\times[0, 1[= \mathcal{S}$. A transformation $T$ is introduced which maps an initial image of $\mathcal{I}$ into another (final) image of $\mathcal{I}$. The final image is obtained as the union of 4 sub-images defined over the 4 quarters of support $\mathcal{S}$, i.e. $[0, 1/2[\times[0, 1/2[= \mathcal{S}_1$, $[1/2, 1[\times[0, 1/2[= \mathcal{S}_2$, $[0, 1/2[\times[1/2, 1[= \mathcal{S}_3$, and $[1/2, 1[\times[1/2, 1[= \mathcal{S}_4$, over which each sub-image is a contracted version of the initial image with affinely transformed gray levels. Explicitly, transformation $T$ is defined by the union of the four sub-transformations:

$$\left|\begin{array}{rcl} \mathcal{S} \times \mathbb{R} & \longrightarrow & \mathcal{S}_1 \times \mathbb{R} \\ \big((x, y),\ s(x, y)\big) & \longmapsto & \big((\frac{x}{2}, \frac{y}{2}),\ a_1 s(x, y) + b_1\big)\ , \end{array}\right. \tag{1}$$

$$\left|\begin{array}{rcl} \mathcal{S} \times \mathbb{R} & \longrightarrow & \mathcal{S}_2 \times \mathbb{R} \\ \big((x, y),\ s(x, y)\big) & \longmapsto & \big((\frac{1}{2}+\frac{x}{2}, \frac{y}{2}),\ a_2 s(x, y) + b_2\big)\ , \end{array}\right. \tag{2}$$

$$\left|\begin{array}{rcl} \mathcal{S} \times \mathbb{R} & \longrightarrow & \mathcal{S}_3 \times \mathbb{R} \\ \big((x, y),\ s(x, y)\big) & \longmapsto & \big((\frac{x}{2}, \frac{1}{2}+\frac{y}{2}),\ a_3 s(x, y) + b_3\big) \end{array}\right. \tag{3}$$

and

$$\left|\begin{array}{rcl} \mathcal{S} \times \mathbb{R} & \longrightarrow & \mathcal{S}_4 \times \mathbb{R} \\ \big((x, y),\ s(x, y)\big) & \longmapsto & \big((\frac{1}{2}+\frac{x}{2}, \frac{1}{2}+\frac{y}{2}),\ a_4 s(x, y) + b_4\big)\ , \end{array}\right. \tag{4}$$

with real coefficients $a_j$ and $b_j$ verifying $0 < |a_j| < 1$, for $j = 1$ to 4, so as to have contractive mappings.

The transformation $T$ defined by Eqs. (1)–(4) implements on both the spatial coordinates $(x, y)$ and the gray level $s(x, y)$, contractive affine transforms. Consequently, the mapping $s(x, y) \longmapsto T[s(x, y)]$ is also a contractive affine transform. It results [1] that $s(x, y) \longmapsto T[s(x, y)]$ admits one single fixed point, i.e. an image $\sigma(x, y)$ verifying $T[\sigma(x, y)] = \sigma(x, y)$ also called the attractor of transformation $T$. Starting from any initial image $s_0(x, y) \in \mathcal{I}$, iterative application of the transformation $T$ defined by Eqs. (1)–(4) realizes an IFS. The process converges to a unique attractor $\sigma(x, y)$ that is completely determined by the set of 8 parameters $\{(a_j, b_j), j = 1 \ldots 4\}$. An important property of this correspondence [1] is that small smooth changes in $\{(a_j, b_j), j = 1 \ldots 4\}$ are associated to small smooth changes in $\sigma(x, y)$.

The attractor $\sigma(x, y)$ is defined as the solution to the fixed-point equation

$$\begin{cases} \sigma(x, y) = a_1\, \sigma(2x, 2y) + b_1 & , \forall (x, y) \in \mathcal{S}_1 \\ \sigma(x, y) = a_2\, \sigma(2x - 1, 2y) + b_2 & , \forall (x, y) \in \mathcal{S}_2 \\ \sigma(x, y) = a_3\, \sigma(2x, 2y - 1) + b_3 & , \forall (x, y) \in \mathcal{S}_3 \\ \sigma(x, y) = a_4\, \sigma(2x - 1, 2y - 1) + b_4 & , \forall (x, y) \in \mathcal{S}_4\ . \end{cases} \tag{5}$$

Such a functional equation expresses a self-transformability property of attractor $\sigma(x, y)$, which confers to it a self-affine or fractal character. This translates into complicated shapes for $\sigma(x, y)$, with structures or details occuring at all scales, as visible on the images of $\sigma(x, y)$ shown in Figs. 1 and 2.

Determining how to choose the parameters $\{(a_j, b_j)\}$ of the IFS in order to impose prescribed properties onto its attractor, is a key issue for image modelling from IFS. Yet, it is usually not possible to analytically solve Eq. (5) so as to obtain an explicit expression of $\sigma(x, y)$ as a function of the parameters $\{(a_j, b_j), j = 1 \ldots 4\}$. The ability of IFS of generating attractors with rich structures is exploited for fractal

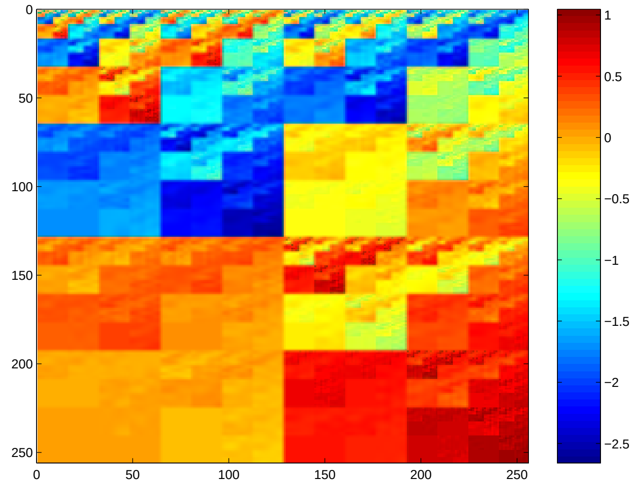


**Fig. 1**. An example of the attractor $\sigma(x,y)$ of the IFS of Eqs. (1)–(4).

image compression [1, 2], where the parameters of the IFS (similar to $\{(a_j, b_j)\}$) are usually determined by minimizing a mean-squared difference between the attractor and the target image to be coded. By contrast here, we address image modelling from IFS by means of exact matching of geometrical moments between the attractor and a target, instead of mean-squared distance minimization.

## 3. GEOMETRICAL MOMENTS

For any image $f(x,y)$ of $\mathcal{I}$ we define

$$\langle f(x,y)\rangle = \int_{x=0}^{1}\int_{y=0}^{1} f(x,y)\,dx\,dy\ . \tag{6}$$

From Eq. (5), we obtain

$$\begin{aligned}
&4\left\langle f(x,y)\sigma(x,y)\right\rangle = \\
&a_1\left\langle f\left(\frac{x}{2},\frac{y}{2}\right)\sigma(x,y)\right\rangle + b_1\left\langle f\left(\frac{x}{2},\frac{y}{2}\right)\right\rangle + \\
&a_2\left\langle f\left(\frac{1}{2}+\frac{x}{2},\frac{y}{2}\right)\sigma(x,y)\right\rangle + b_2\left\langle f\left(\frac{1}{2}+\frac{x}{2},\frac{y}{2}\right)\right\rangle + \\
&a_3\left\langle f\left(\frac{x}{2},\frac{1}{2}+\frac{y}{2}\right)\sigma(x,y)\right\rangle + b_3\left\langle f\left(\frac{x}{2},\frac{1}{2}+\frac{y}{2}\right)\right\rangle + \\
&a_4\left\langle f\left(\frac{1}{2}+\frac{x}{2},\frac{1}{2}+\frac{y}{2}\right)\sigma(x,y)\right\rangle + b_4\left\langle f\left(\frac{1}{2}+\frac{x}{2},\frac{1}{2}+\frac{y}{2}\right)\right\rangle .
\end{aligned} \tag{7}$$

For appropriate choices of $f(x,y)$, Eq. (7) provides useful linear equations relating various linear coefficients of $\sigma(x,y)$ to $\{(a_j,b_j), j = 1\dots 4\}$. For instance, when $f(x,y) = \exp[-i2\pi(nx+py)]$, the quantities $S_{np} = \langle \exp[-i2\pi(nx+py)]\sigma(x,y)\rangle$ are the Fourier coefficients of the attractor $\sigma(x,y)$. When $f(x,y) = x^n y^p$, the quantities $\mu_{np} = \langle x^n y^p \sigma(x,y)\rangle$ are the geometrical moments of the attractor $\sigma(x,y)$, which are found to verify, according to Eq. (7), with $(n,p)$ integers,

$$\begin{aligned}
&\left(2^{n+p+2} - a_1 - a_2 - a_3 - a_4\right)\mu_{np} = \\
&(a_2+a_4)\sum_{k=0}^{n-1}\binom{n}{k}\mu_{kp} + (a_3+a_4)\sum_{\ell=0}^{p-1}\binom{p}{\ell}\mu_{n\ell} + \\
&a_4\sum_{k=0}^{n-1}\sum_{\ell=0}^{p-1}\binom{n}{k}\binom{p}{\ell}\mu_{k\ell} + \\
&\frac{b_1+(2^{n+1}-1)b_2+(2^{p+1}-1)b_3+(2^{n+1}-1)(2^{p+1}-1)b_4}{(n+1)(p+1)}
\end{aligned} \tag{8}$$

with the binomial coefficients $\binom{k}{\ell}$. It is thus possible in principle, through Eq. (8), to determine the parameters $\{(a_j,b_j)$, $j = 1\dots 4\}$ of the IFS so as to impose up to 8 geometrical moments $\mu_{np}$ to its attractor $\sigma(x,y)$. For instance, for the 4 first moments one obtains from Eq. (8)

$$\begin{aligned}
\left(2^2 - a_1 - a_2 - a_3 - a_4\right)\mu_{00} &= \\
b_1+b_2+b_3+b_4 &
\end{aligned} \tag{9}$$

$$\begin{aligned}
\left(2^3 - a_1 - a_2 - a_3 - a_4\right)\mu_{10} - (a_2+a_4)\mu_{00} &= \\
\frac{1}{2}(b_1+3b_2+b_3+3b_4) &
\end{aligned} \tag{10}$$

$$\begin{aligned}
\left(2^3 - a_1 - a_2 - a_3 - a_4\right)\mu_{01} - (a_3+a_4)\mu_{00} &= \\
\frac{1}{2}(b_1+b_2+3b_3+3b_4) &
\end{aligned} \tag{11}$$

$$\begin{aligned}
\left(2^4 - a_1 - a_2 - a_3 - a_4\right)\mu_{11} - (a_2+a_4)\mu_{01} & \\
- (a_3+a_4)\mu_{10} - a_4\mu_{00} &= \\
\frac{1}{4}(b_1+3b_2+3b_3+9b_4) &
\end{aligned} \tag{12}$$

with an application shown in Fig. 2.

A similar approach could be used from Eq. (7) to impose Fourier coefficients of the attractor $\sigma(x,y)$.

## 4. CONTINUITY OF ATTRACTORS

The division into four sub-images, which is at the root of the IFS of Eqs. (1)–(4), gives rise to a block structure visible in the images of Figs. 1 and 2. This structure can be eliminated while preserving the self-similarity of the images, by ensuring that the IFS generate a continuous attractor. This is realized by imposing that on $\sigma(x,y)$ the sub-images over $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ are transformed by the IFS in a continuous way on both sides of the line boundaries that these sub-images have in common. Continuity along the $x$ axis thus is associated to

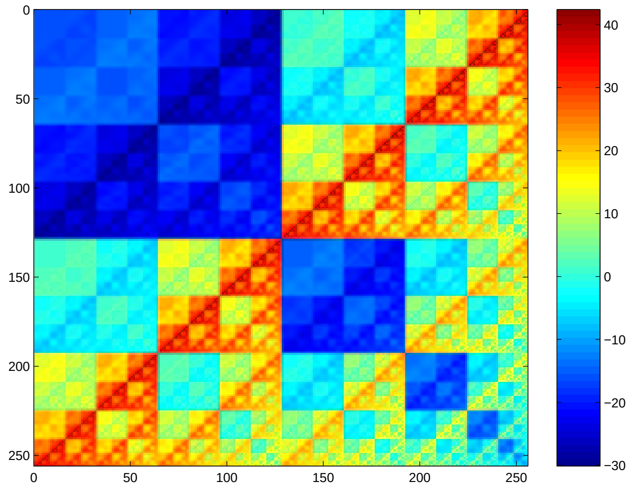


**Fig. 2**. Attractor $\sigma(x,y)$ of the IFS of Eqs. (1)–(4) with the parameters $b_j$'s selected from Eqs. (9)–(12) to impose the 4 geometrical moments $\mu_{00} = \mu_{11} = 1$ and $\mu_{10} = \mu_{01} = 2$.

the conditions

$$\left\{ \begin{array}{rcll} a_1\sigma(x,1) + b_1 &=& a_3\sigma(x,0) + b_3\ , & \forall x \in [0,1[ \\ a_2\sigma(x,1) + b_2 &=& a_4\sigma(x,0) + b_4\ , & \forall x \in [0,1[\ , \end{array} \right. \tag{13}$$

while continuity along the $y$ axis is associated to the conditions

$$\left\{ \begin{array}{rcll} a_1\sigma(1,y) + b_1 &=& a_2\sigma(0,y) + b_2\ , & \forall y \in [0,1[ \\ a_3\sigma(1,y) + b_3 &=& a_4\sigma(0,y) + b_4\ , & \forall y \in [0,1[\ . \end{array} \right. \tag{14}$$

If one sticks with only 8 scalar parameters $\{(a_j, b_j), j = 1\dots 4\}$ for the IFS, it is usually not possible to find an attractor $\sigma(x,y)$ that will fulfill the fixed-point equation (5) together with the continuity conditions (13)–(14). This can be easily understood if one considers a discretized version of $\sigma(x,y)$ over a spatial grid with size $N_x \times N_y$. In this case, Eq. (5) will translate into a set of $N_x \times N_y$ linear equations with $N_x \times N_y$ unknows (the values of $\sigma(x,y)$ at the grid points). This set of equations will generally have a unique solution defining a unique attractor $\sigma(x,y)$ parameterized by $\{(a_j, b_j), j = 1\dots 4\}$. Further, if continuity is to be imposed at the line boundaries where $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ contact, Eqs. (13)–(14) translate into $N_x + N_y$ additional conditions to be verified by $\sigma(x,y)$. This is usually not possible with only the 8 degrees of freedom $\{(a_j, b_j), j = 1\dots 4\}$ for $\sigma(x,y)$. However, if one-dimensional function parameters are introduced like for instance $b_1(x)$, $b_2(y)$, $b_3(y)$ and $b_4(x)$, this will introduce extra degrees of freedom in numbers $N_x$ and $N_y$, generally allowing to match the requirements of the continuity conditions (13)–(14) for $\sigma(x,y)$.

With the proposed solution, the fixed-point equation (5) transforms into

$$\left\{ \begin{array}{rcl} \sigma(x,y) &=& a_1\,\sigma(2x,2y) + b_1(2x) \\ && \qquad\qquad ,\forall (x,y) \in \mathcal{S}_1 \\ \sigma(x,y) &=& a_2\,\sigma(2x-1,2y) + b_2(2y) \\ && \qquad\qquad ,\forall (x,y) \in \mathcal{S}_2 \\ \sigma(x,y) &=& a_3\,\sigma(2x,2y-1) + b_3(2y-1) \\ && \qquad\qquad ,\forall (x,y) \in \mathcal{S}_3 \\ \sigma(x,y) &=& a_4\,\sigma(2x-1,2y-1) + b_4(2x-1) \\ && \qquad\qquad ,\forall (x,y) \in \mathcal{S}_4\ ; \end{array} \right. \tag{15}$$

and the continuity conditions Eq. (13) into

$$\left\{ \begin{array}{rcl} a_1\sigma(x,1)\ +\ b_1(x) &=& a_3\sigma(x,0) + b_3(y=0) \\ && ,\forall x \in [0,1[ \\ a_2\sigma(x,1)\ +\ b_2(y=1) &=& a_4\sigma(x,0) + b_4(x) \\ && ,\forall x \in [0,1[\ , \end{array} \right. \tag{16}$$

and Eq. (14) into

$$\left\{ \begin{array}{rcl} a_1\sigma(1,y)\ +\ b_1(x=1) &=& a_2\sigma(0,y) + b_2(y) \\ && ,\forall y \in [0,1[ \\ a_3\sigma(1,y)\ +\ b_3(y) &=& a_4\sigma(0,y) + b_4(x=0) \\ && ,\forall y \in [0,1[\ . \end{array} \right. \tag{17}$$

With a discretized version of $\sigma(x,y)$ over a spatial grid with size $N_x \times N_y$, Eq. (15) gives a set of $N_x \times N_y$ linear equations, whereas Eqs. (16) and (17) provide a set of additional linear equations in number (of order) $N_x + N_y$. The unknowns are the values of $\sigma(x,y)$ and of $b_1(x)$, $b_2(y)$, $b_3(y)$ and $b_4(x)$ over the discrete grid, amounting to a number of unknowns of order $N_x \times N_y + N_x + N_y$. The system of Eqs. (15)–(17) represents $N_x \times N_y + N_x + N_y$ linear equations for $N_x \times N_y + N_x + N_y$ unknows. In principle, it can be solved through inversion of the $(N_x \times N_y + N_x + N_y) \times (N_x \times N_y + N_x + N_y)$ matrix, a possibly very large matrix in practice for a sufficient spatial resolution of the fractal attractor $\sigma(x,y)$.

When continuity is not imposed upon the attractor $\sigma(x,y)$, the discretized scheme leads to a system of $N_x \times N_y$ linear equations provided by Eq. (5), for $N_x \times N_y$ unknows (the values of $\sigma(x,y)$ at the grid points), in the presence of fixed given parameters $(a_j, b_j)$. The iterative application of the transformation defined by Eqs. (1)–(4) converges to the attractor $\sigma(x,y)$ solution to Eq. (5). This can be seen as a (fast) iterative resolution of Eq. (5) avoiding the inversion of a $N_x \times N_y$ matrix.

When continuity is imposed upon $\sigma(x,y)$, we have not found a way of preserving this possibility of a fast iterative scheme to converge to the solution of Eqs. (15)–(17). This may be a serious drawback for the efficient construction of two-dimensional attractors $\sigma(x, y)$ when continuity is imposed onto them. However, if we limit ourselves to a sub-class of the class of continuous fractal attractors $\sigma(x, y)$, we may recover the possibility of a very efficient iterative scheme. This sub-class is restricted to two-dimensional attractors $\sigma(x, y)$ that are separable under the form

$$\sigma(x, y) = \sigma_x(x)\ \sigma_y(y)\ . \tag{18}$$

In Eq. (18), $\sigma_x(x)$ and $\sigma_y(y)$ are one-dimensional fractal attractors defined over the support $[0, 1[$, which can be generated by one-dimensional versions of the IFS of Section 2. Especially, imposing continuity to such one-dimensional attractors will amount to imposing a condition on a zero-dimensional frontier, i.e. one scalar condition, which can generally be fulfilled with the degrees of freedom afforded by the scalar parameters of one-dimensional IFS. This scalar condition can be explicitly solved for one parameter, reducing by one the number of degrees of freedom, for an IFS which can still be iterated to convergence towards its one-dimensional continuous attractor $\sigma_x(x)$ or $\sigma_y(y)$. This provides a fast iterative construction of one-dimensional continuous attractors, and subsequently, according to Eq. (18), of two-dimensional (separable) continuous attractors.

This process is illustrated by Fig. 3 which shows two continuous attractors $\sigma_x(x)$ and $\sigma_y(y)$ of one-dimensional IFS.

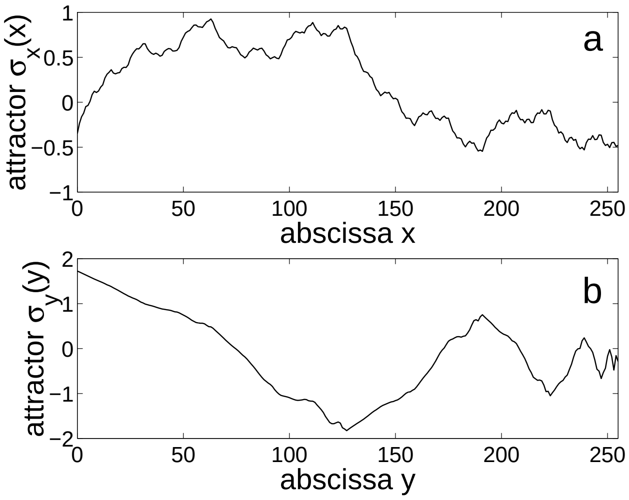


**Fig. 3**. Continuous fractal attractors $\sigma_x(x)$ and $\sigma_y(y)$ from one-dimensional IFS.

From the two continuous one-dimensional fractal attractors $\sigma_x(x)$ and $\sigma_y(y)$ of Fig. 3, it is possible to construct, according to Eq. (18), a continuous two-dimensional fractal attractor $\sigma(x, y)$ shown in Fig. 4.

Among other applications, fractal images such as that of Fig. 4 can be used as models for natural landscapes in image synthesis, as suggested by the relief rendering of Fig. 5.

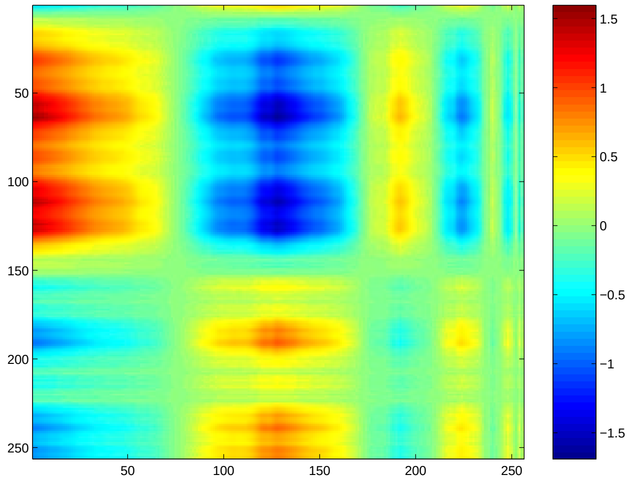

**Fig. 4**. Continuous two-dimensional fractal attractor $\sigma(x, y)$ constructed from Eq. (18) and $\sigma_x(x)$ and $\sigma_y(y)$ of Fig. 3.

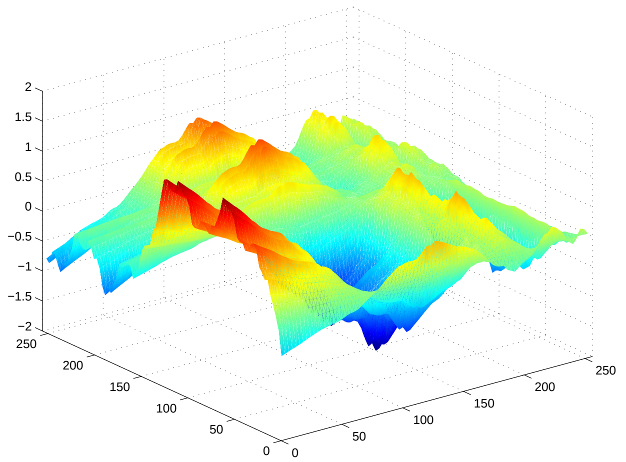

**Fig. 5**. Relief rendering of the continuous two-dimensional fractal attractor $\sigma(x, y)$ of Fig. 4.

## 5. CONCLUSION

Continuous separable two-dimensional attractors can also have their geometrical moments imposed, as products of the corresponding moments of the underlying one-dimensional attractors. Altogether, the approaches we have described offer simple models for fractal images with control over various of their properties, and which can serve different purposes, including image coding, compression and synthesis.

## 6. REFERENCES


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