2nd International Symposium on Physics in Signal & Image Processing, pp. 47–50,
Marseille, France, 23–24 Jan. 2001.

ESTIMATION AND FISHER INFORMATION ENHANCEMENT VIA NOISE ADDITION WITH NONLINEAR SENSORS

François Chapeau-Blondeau, Julio Rojas Varela

Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers
62 avenue Notre Dame du Lac, 49000 ANGERS, FRANCE
http://www.istia.univ-angers.fr/~chapeau/

Abstract : We analyze the parametric estimation that can be performed on a signal buried in noise by means of a quantized representation by a nonlinear sensor of the signal-plus-noise mixture. The Fisher information contained in the sensor output about the input parameter is used as the measure of performance in the estimation task. For given conditions on the nonlinear sensor, we establish the possibility of increasing the Fisher information as the noise level is raised. This behavior reveals a larger possible efficacy in the estimation when noise is added, with an optimal nonzero noise level which maximizes the estimation efficacy. These results demonstrate a new form of the nonlinear phenomenon of stochastic resonance where addition of noise may reveal beneficial to the signal.

1 Introduction

Certain nonlinear sensors in charge of the transmission of a signal corrupted by noise have recently been shown to lend themselves to an interesting effect known as stochastic resonance. This effect, in a counterintuitive way, establishes the possibility of exploiting the noise to improve the transmission of the signal, up to a point where adding noise may result in enhanced performance.

Standard stochastic resonance has been reported in the transmission, by various nonlinear sensors, of a periodic signal (of period T_s) in noise [1, 2, 3, 4]. The effect is usually measured by a signal-to-noise ratio at the sensor output in the frequency domain, defined as the power contained in the spectral line at $1/T_s$ divided by the power contained in the noise background in the region of $1/T_s$. When the level of the input noise is raised, nonlinear sensors exist that deliver an output signal-to-noise ratio experiencing a nonmonotonic evolution culminating at a maximum for an optimal nonzero noise level.

More recently, stochastic resonance has been extended to the noise-assisted transmission of other types (non-periodic, non-deterministic) of signal with other measures of performance receiving improvement from the noise [5, 6, 7].

Here we extend the effect to a situation interpretable as a signal estimation task assisted by noise. The performance is measured by the Fisher information, which sets a bound to the efficacy of any conceivable unbiased estimator, this applying especially to the asymptotic behavior of the maximum likelihood estimator [8]. We analyze situations where the Fisher information, for signal estimation from the output of certain nonlinear sensors, can be enhanced via noise addition.

2 The nonlinear sensor

We consider a white noise $\eta(t)$ with cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u) = dF_\eta/du$. As in standard stochastic resonance, we also consider a deterministic signal $s(t)$. The signal-plus-noise mixture $s(t)+\eta(t)$ is observed through some type of (nonlinear) sensor. In order to have an analytically tractable demonstration of an effect of noise-enhanced signal estimation, we consider here the simple nonlinear sensor which takes the form of a two-state threshold nonlinearity, delivering the output signal $y(t) = \mathrm{sign}[s(t) + \eta(t) - \theta]$, with the threshold θ. Such a sensor which quantizes the value of the analog input $s(t) + \eta(t)$ over one single bit $y(t) = \pm 1$, may be useful for low-cost low-complexity real-time processing in existing and future multisensor networks or distributed intelligent systems [9]. It also mimics the essential nonlinearity implemented by natural devices such as sensory neurons, which are interesting to consider since neural systems are recognized to perform very efficient signal processing.

Based on the sensor output signal $y(t)$, we want to make parametric estimation concerning the input signal $s(t) + \eta(t)$. Let us call a the parameter to be estimated. In general, a will be a parameter belonging to the definition of $s(t)$, for instance the value assumed by a constant signal $s(t) = a$ for all t, or the amplitude or frequency of a periodic $s(t)$, or any other parameter entering the specification of $s(t)$. Parametric estimation on $\eta(t)$ could also be considered and shown amenable to the same stochastic resonance effect, but here we will only address estimation on $s(t)$, yet in general conditions, as opposed to [10, 9, 11] which only consider noise-enhanced estimation of a constant signal. For the estimation of a from $y(t)$, a key quantity is the Fisher information [12] about a contained in $y(t)$, defined as

$$J[a; y(t)] = \sum_{x=-1,1} \frac{1}{\Pr\{y = x\}} \left(\frac{\partial}{\partial a} \Pr\{y = x\} \right)^2 . \tag{1}$$

At a given time t, this quantity can be explicitly evaluated under the form

$$J[a; y(t)] = \left[\frac{1}{q_1(t)} + \frac{1}{1 - q_1(t)} \right] \left[\frac{\partial q_1(t)}{\partial a} \right]^2 \tag{2}$$

where $q_1(t) = \Pr\{y(t) = 1\}$. We also have $q_1(t) = \Pr\{s(t) + \eta(t) > \theta\}$, which here with $s(t)$ deterministic gives $q_1(t) = \Pr\{\eta(t) > \theta - s(t)\}$, amounting to $q_1(t) = 1 - F_\eta[\theta - s(t)]$. Then, for the derivative we have

$$\frac{\partial q_1(t)}{\partial a} = f_\eta[\theta - s(t)] \, \frac{\partial s(t)}{\partial a} , \tag{3}$$

and Fisher information $J[a; y(t)]$ of Eq. (2) results as

$$J[a; y(t)] = \left[\frac{\partial s(t)}{\partial a} \right]^2 \frac{f_\eta^2[\theta - s(t)]}{F_\eta[\theta - s(t)]\{1 - F_\eta[\theta - s(t)]\}} . \tag{4}$$

We shall now demonstrate various conditions where the Fisher information $J[a; y(t)]$ at the sensor output can be increased when the level of the input noise $\eta(t)$ is raised.

For illustration of the effect, we consider the case of a constant signal $s(t) = a$ for all t, and $\eta(t)$ a zero-mean Gaussian noise with variance σ_η^2. Figure 1 represents the evolution of $J[a; y(t)]$ of Eq. (4) as the noise rms amplitude σ_η^2 is raised, for various values of the threshold θ of the nonlinear sensor. The evolutions of Fig. 1, at fixed θ and s, show conditions where $J[a; y(t)]$ increases as σ_η is raised, up to an optimal value of σ_η where $J[a; y(t)]$ is maximized. This behavior characterizes a larger possible efficacy in the estimation when noise is added, up to an optimal nonzero noise level where the estimation efficacy is maximized.

Qualitatively, this phenomenon of noise-enhanced estimation occurs when the signal $s(t)$ by itself is not well positioned relative to the threshold, and addition of noise can be viewed as a way of shifting the operating zone of the nonlinearity towards a region more favorable to the transmission of the signal $s(t)$.

This is the common feature in a stochastic resonance effect: a signal conveyed by a nonlinear sensor can draw benefit from addition of noise; this possibility can translate into improved transmission, or detection or estimation performance when noise is added. This nonlinear effect has now been reported in a large variety of nonlinearities and signals, and it can be quantified by various performance measures (signal-to-noise ratio, mutual information, cross-correlation, probability of detection, Fisher information, ...) which can be improved via noise addition.

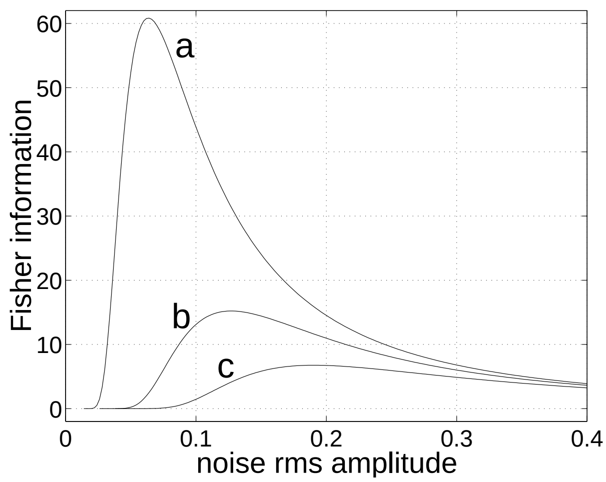

Figure 1: Fisher information $J[a; y(t)]$ from Eq. (4) with the constant signal $s(t) = a$, as a function of the rms amplitude σ_η of the noise η chosen zero-mean Gaussian, when $\theta - s = 0.1$ (a), $\theta - s = 0.2$ (b), and $\theta - s = 0.3$ (c).

In Fig. 1, the Gaussian noise selected to materialize a stochastic resonance effect is merely illustrative, and similar possibilities of noise-enhanced estimation are preserved in broad conditions for the noise distribution. For illustration, we consider the case where $\eta(t)$ is a zero-mean Laplacian noise of variance σ_η^2, with probability density function

$$f_\eta(u) = \frac{1}{\sqrt{2}\sigma_\eta} \exp\left(-\sqrt{2} \frac{|u|}{\sigma_\eta} \right) \tag{5}$$

and cumulative distribution function

$$F_\eta(u) = \begin{cases} \dfrac{1}{2} \exp\left(-\sqrt{2} \dfrac{|u|}{\sigma_\eta} \right) & \text{for} \quad u \leq 0 , \\ 1 - \dfrac{1}{2} \exp\left(-\sqrt{2} \dfrac{|u|}{\sigma_\eta} \right) & \text{for} \quad u \geq 0 . \end{cases} \tag{6}$$

Figure 2 then represents the evolution of Fisher information $J[a; y(t)]$ of Eq. (4). A similar possibility of noise-enhanced estimation is exhibited with Laplacian noise, with a Fisher information $J[a; y(t)]$ of Fig. 2 which peaks at a smaller value, but which decays more slowly when σ_η gets large, compared to the case with Gaussian noise.

The present noise-enhanced estimation task from the output of our two-state threshold nonlinearity displays an interesting property. In Eq. (4), if for instance $s(t)$ is a constant $s(t) = a$, then simply $\partial s(t)/\partial a = 1$; if $s(t)$ is a sinusoid of amplitude a, i.e. $s(t) = a \sin(t)$, then $\partial s(t)/\partial a = \sin(t)$. The general property is that the factor $\partial s(t)/\partial a$ in Eq. (4) depends only on the signal $s(t)$, and is independent of the noise $\eta(t)$. Thus, when one comes to studying the influence

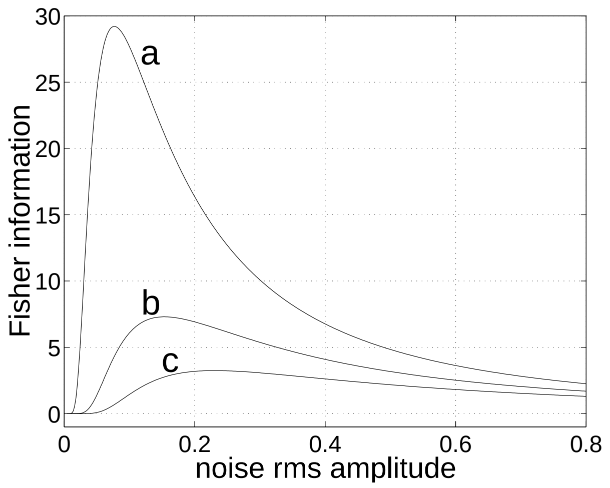

Figure 2: Fisher information $J[a; y(t)]$ from Eq. (4) with the constant signal $s(t) = a$, as a function of the rms amplitude σ_η of the noise η chosen zero-mean Laplacian, when $\theta - s = 0.1$ (a), $\theta - s = 0.2$ (b), and $\theta - s = 0.3$ (c).

of the noise properties on the Fisher information of Eq. (4), the prefactor $\partial s(t)/\partial a$ is a constant (at a fixed time t) which does not vary with the noise. As a consequence, any type of signal parameterized by any a with $\partial s(t)/\partial a \neq 0$, will benefit from the same possibility of noise enhancement when estimating a from the output of the two-state threshold nonlinearity. Therefore, in some sense, the evolutions obtained in Figs. 1 and 2 for a constant signal $s(t)$ are generic, and they apply in the same way to any type of parameter estimation on any type of signal $s(t)$.

3 Input–output characterization

Another relevant quantity interesting to consider is the Fisher information $J[a; s(t) + \eta(t)]$ about a contained in the input signal $s(t) + \eta(t)$. With $s(t)$ deterministic, the random signal $s(t) + \eta(t)$ is distributed with the probability density $f_\eta[u - s(t)]$, and $J[a; s(t) + \eta(t)]$ can be defined [12] as

$$J[a; s(t) + \eta(t)] = \int_{-\infty}^{+\infty} \frac{1}{f_\eta[u - s(t)]} \left\{ \frac{\partial}{\partial a} f_\eta[u - s(t)] \right\}^2 du \ , \quad (7)$$

which leads to

$$J[a; s(t) + \eta(t)] = \left[\frac{\partial s(t)}{\partial a} \right]^2 \int_{-\infty}^{+\infty} \frac{\{f'_\eta[u - s(t)]\}^2}{f_\eta[u - s(t)]} du \ . \quad (8)$$

The input–output ratio $J[a; y(t)]/J[a; s(t) + \eta(t)]$ characterizes the loss of Fisher information when, for the estimation, the analog input signal $s(t) + \eta(t)$ is replaced by the binary signal $y(t)$ by the sensor. It can be verified through the use of Eqs. (4) and (8), that in many conditions the ratio $J[a; y(t)]/J[a; s(t) + \eta(t)]$ can be increased as the noise level is raised, revealing that the Fisher information degrades more slowly at the output of the sensor than at its input when the noise grows. For illustration, we consider the case of a constant signal $s(t) = a$. First, we take $\eta(t)$ a Gaussian noise of mean zero and variance σ_η^2. In this case $J[a; s(t) + \eta(t)]$ of Eq. (8) reduces to $1/\sigma_\eta^2$ and the ratio $J[a; y(t)]/J[a; s(t) + \eta(t)]$ follows as shown in Fig. 3. Second, we take $\eta(t)$ a Laplacian noise of mean

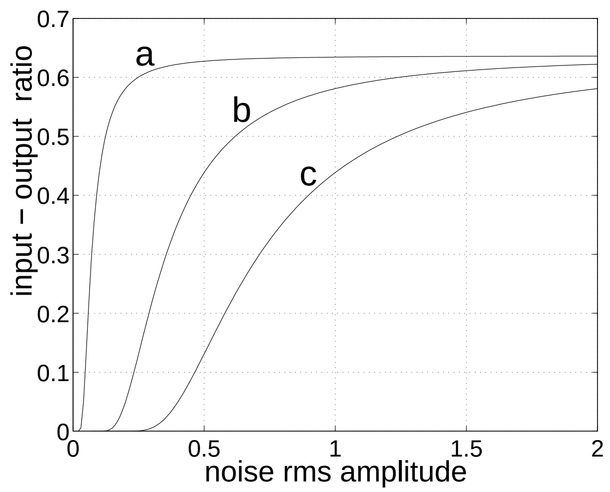

Figure 3: Input–output ratio of the Fisher information $J[a; y(t)]/J[a; s(t) + \eta(t)]$ with the constant signal $s(t) = a$, as a function of the rms amplitude σ_η of the noise η chosen zero-mean Gaussian, when $\theta - s = 0.1$ (a), $\theta - s = 0.5$ (b), and $\theta - s = 1$ (c).

zero and variance σ_η^2, as defined by Eqs. (5) and (6). In this case $J[a; s(t) + \eta(t)]$ of Eq. (8) reduces to $2/\sigma_\eta^2$ and the ratio $J[a; y(t)]/J[a; s(t) + \eta(t)]$ follows as shown in Fig. 4.

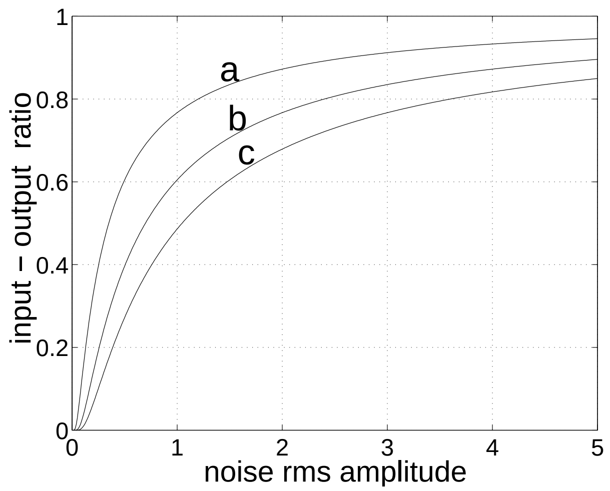

Figure 4: Input–output ratio of the Fisher information $J[a; y(t)]/J[a; s(t) + \eta(t)]$ with the constant signal $s(t) = a$, as a function of the rms amplitude σ_η of the noise η chosen zero-mean Laplacian, when $\theta - s = 0.1$ (a), $\theta - s = 0.5$ (b), and $\theta - s = 1$ (c).

The increasing evolution observed in Figs. 3 and 4 for $J[a; y(t)]/J[a; s(t) + \eta(t)]$ as σ_η is raised express that, as the noise level increases, the loss of information is less severe when the binary signal $y(t)$ instead of the analog one $s(t)+\eta(t)$ is used for estimation. Moreover, as before, thanks to the form of Eqs. (4) and (8), the evolution of the ratio $J[a; y(t)]/J[a; s(t) + \eta(t)]$ with the noise properties, will not be affected by the specific dependence of $s(t)$ upon a. Therefore, the evolutions of Figs. 3 and 4 are in some sense generic, and apply in the same way to any type of parameter estimation on any type of signal $s(t)$.

Another interesting property that can also be observed from Eqs. (4) and (8), is that the ratio $J[a; y(t)]/J[a; s(t) + \eta(t)]$ can come in many situations close to unity, as exemplified by Figs. 3 and 4. This reveals the possibility of a very small loss of Fisher information when the analog input is replaced by a single bit for estimation.

Such properties conveyed by Figs. 3 and 4 are specially meaningful if some notion of hardware parsimony is included in the assessment of the performance of the sensors for dealing with noisy signals. With standard hardware, the analog input $s(t)+\eta(t)$ will in practice be supported by an N-bit representation, typically with $N = 8$ or 12 or 16. This is to be contrasted with the single-bit representation of $y(t)$. As the bit budget is divided by N when $y(t)$ only is retained for estimation, it is very interesting to realize, as illustrated by Fig. 3 and 4, that the estimation efficacy is in general reduced by a much smaller factor. With this notion of limited hardware requirement included in the evaluation of the performance, the interest of simple nonlinear sensors [9, 13] as the one producing $y(t)$ gets even more manifest.

4 Conclusion

The present study analyzed a new instance of the nonlinear phenomenon of stochastic resonance, under the form of a noise-enhanced estimation process. It is to note that stochastic resonance is not limited to threshold nonlinearities as considered here. It can also occur with other types of nonlinear systems, under different forms, with various signals and noises, as it now appears in the light of the more recent developments on the phenomenon. Gradually, stochastic resonance is emerging as a nonlinear effect of general applicability. It expresses how in nonlinear contexts, the noise is not necessarily nuisance but may sometimes be turned into a benefit. This state of fact bears important conceptual significance for nonlinear signal processing, and it may lead to useful applications, especially in the presence of strong constraints imposed on sensors dealing with noisy signals.

References

[1] F. Moss, D. Pierson, and D. O'Gorman, "Stochastic resonance: Tutorial and update", *International Journal of Bifurcation and Chaos*, vol. 4, pp. 1383–1398, 1994.

[2] F. Chapeau-Blondeau and X. Godivier, "Theory of stochastic resonance in signal transmission by static nonlinear systems", *Physical Review E*, vol. 55, pp. 1478–1495, 1997.

[3] X. Godivier and F. Chapeau-Blondeau, "Noise-assisted signal transmission in a nonlinear electronic comparator: Experiment and theory", *Signal Processing*, vol. 56, pp. 293–303, 1997.

[4] F. Chapeau-Blondeau, "Stochastic resonance and optimal detection of pulse trains by threshold devices", *Digital Signal Processing*, vol. 9, pp. 162–177, 1999.

[5] J. J. Collins, C. C. Chow, A. C. Capela, and T. T. Imhoff, "Aperiodic stochastic resonance", *Physical Review E*, vol. 54, pp. 5575–5584, 1996.

[6] X. Godivier and F. Chapeau-Blondeau, "Stochastic resonance in the information capacity of a nonlinear dynamic system", *International Journal of Bifurcation and Chaos*, vol. 8, pp. 581–590, 1998.

[7] S. Mitaim and B. Kosko, "Adaptive stochastic resonance", *Proceedings of the IEEE*, vol. 86, pp. 2152–2183, 1998.

[8] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, New York, 1991.

[9] H. C. Papadopoulos, G. W. Wornell, and A. V. Oppenheim, "Low-complexity digital encoding strategies for wireless sensors networks", in *Proceedings IEEE International Conference on Acoustics, Speech and Signal Processing*, 1998, pp. 3273–3276.

[10] M. Stemmler, "A single spike suffices: The simplest form of stochastic resonance in model neurons", *Network: Computation in Neural Systems*, vol. 7, pp. 687–716, 1996.

[11] P. E. Greenwood, L. M. Ward, and W. Wefelmeyer, "Statistical analysis of stochastic resonance in a simple setting", *Physical Review E*, vol. 60, pp. 4687–4695, 1999.

[12] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, New York, 1991.

[13] F. Chapeau-Blondeau, "Nonlinear test statistic to improve signal detection in non-Gaussian noise", *IEEE Signal Processing Letters*, vol. 7, pp. 205–207, 2000.