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Information quantique :
depuis des basiques jusqu'à des problèmes ouverts, avec de l'algèbre et des probabilités.

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Quantum information

A definition (at large)

To exploit quantum properties and phenomena
for information processing and computation.

Motivations

1) When using elementary physical systems (photons, electrons, atoms, ions, nanodevices, . . .).

2) To benefit from purely quantum effects (parallelism, entanglement, . . .).

3) New field of research, rich of large potentialities.

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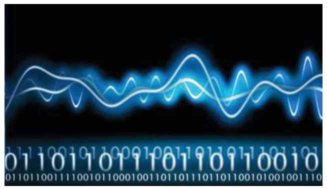
3 – Signal transmission

4 – Signal estimation



1 – Quantum basics

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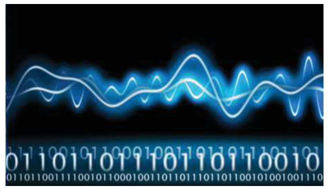


Quantum system

Represented by a state vector $|\psi\rangle$ in a complex Hilbert space $\mathcal{H}$, having unit norm $\langle\psi|\psi\rangle = ||\psi||^2 = 1$.

In dimension N (finite) (extensible to infinite & to continuous dimension)

State $|\psi\rangle = \sum_{n=1}^{N} \alpha_n |n\rangle$, in some orthonormal basis $\{|1\rangle, |2\rangle, \dots |N\rangle\}$ of $\mathcal{H}_N$, with coordinate $\alpha_n = \langle n|\psi\rangle \in \mathbb{C}$, and inner product $\langle\psi|\psi\rangle = \sum_{n=1}^{N} |\alpha_n|^2 = 1$.

$N = 2$ is the qubit (2 states of polarization for a photon, of spin for an electron, etc).

Measurement referred to a projective orthonormal basis $\{|n\rangle\}$, has a probabilistic outcome (Born rule) : $\Pr\{|n\rangle\} = |\alpha_n|^2 = |\langle n|\psi\rangle|^2$.

Quantum measurement : usually :
- a probabilistic process,
- as a destructive projection of the state $|\psi\rangle$ in an orthonormal basis,
- with statistics evaluable over repeated experiments with same preparation $|\psi\rangle$.

Density operator

Quantum system in (pure) state $|\psi_j\rangle \in \mathcal{H}$, measured in an orthonormal basis $\{|n\rangle\}$:

$\Longrightarrow$ probability $\Pr\{|n\rangle \big| |\psi_j\rangle\} = |\langle n|\psi_j\rangle|^2 = \langle n|\psi_j\rangle\langle\psi_j|n\rangle$.

Several possible states $|\psi_j\rangle$ with probabilities p_j (with $\sum_j p_j = 1$) :

$\Longrightarrow \Pr\{|n\rangle\} = \sum_j p_j \Pr\{|n\rangle \big| |\psi_j\rangle\} = \langle n|\Big(\sum_j p_j |\psi_j\rangle\langle\psi_j|\Big)|n\rangle = \langle n|\rho|n\rangle$,

with density operator $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| \in \mathcal{L}(\mathcal{H})$ Hermitian, positive, of unit trace.

and $\Pr\{|n\rangle\} = \langle n|\rho|n\rangle = \mathrm{tr}(\rho|n\rangle\langle n|) = \mathrm{tr}(\rho\,\Pi_n)$ with (orthogonal) projector $\Pi_n = |n\rangle\langle n|$.

The quantum system is in a **mixed** state, corresponding to the statistical ensemble $\{(p_j, |\psi_j\rangle)\}$, described by the density operator $\rho \in \mathcal{L}(\mathcal{H})$.

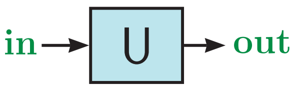
Lemma : For any operator A with trace $\mathrm{tr}(\mathsf{A}) = \sum_n \langle n|\mathsf{A}|n\rangle$, one has $\mathrm{tr}(\mathsf{A}|\psi\rangle\langle\phi|) = \sum_n \langle n|\mathsf{A}|\psi\rangle\langle\phi|n\rangle = \sum_n \langle\phi|n\rangle\langle n|\mathsf{A}|\psi\rangle = \langle\phi|\Big(\sum_n |n\rangle\langle n|\Big)\mathsf{A}|\psi\rangle = \langle\phi|\mathsf{A}|\psi\rangle$.

Evolution of a quantum system, when isolated :

Through a unitary operator U on $\mathcal{H}_N$ (an $N \times N$ matrix) : (i.e. $\mathsf{U}^{-1} = \mathsf{U}^\dagger$)

normalized vector $|\psi\rangle \in \mathcal{H}_N \longmapsto \mathsf{U}|\psi\rangle$ normalized vector $\in \mathcal{H}_N$,

density operator $\rho \in \mathcal{L}(\mathcal{H}_N) \longmapsto \mathsf{U}\rho\mathsf{U}^\dagger$ density operator $\in \mathcal{L}(\mathcal{H}_N)$.



The evolution operator U can be derived from Schrödinger equation :

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\mathsf{H}|\psi\rangle \Longrightarrow |\psi(t_2)\rangle = \underbrace{\exp\Big(-\frac{i}{\hbar}\int_{t_1}^{t_2}\mathsf{H}dt\Big)}_{\text{unitary } \mathsf{U}(t_1,t_2)}|\psi(t_1)\rangle = \mathsf{U}(t_1,t_2)|\psi(t_1)\rangle$$

Hermitian operator Hamiltonian H, or energy operator.

Ex. : A particle of mass m in potential $V(\vec{r},t)$ has Hamiltonian $\mathsf{H} = -\frac{\hbar^2}{2m}\Delta \cdot + V(\vec{r},t)$.

Evolution of a quantum system, when open :

A quantum system in state $\rho \in \mathcal{L}(\mathcal{H}_N)$ interacting with its environment represents an open quantum system. The state ρ usually undergoes a nonunitary evolution in $\mathcal{L}(\mathcal{H}_N)$.

With ρ_{env} the state of the environment at the onset of the interaction, the joint state $\rho \otimes \rho_{\mathrm{env}}$ can be considered as that of an isolated system, undergoing a unitary evolution by U as $\rho \otimes \rho_{\mathrm{env}} \longmapsto \mathsf{U}(\rho \otimes \rho_{\mathrm{env}})\mathsf{U}^\dagger$.

At the end of the interaction, the state of the quantum system of interest is obtained by the partial trace over the environment : $\rho \longmapsto \mathcal{N}(\rho) = \mathrm{tr}_{\mathrm{env}}\big[\mathsf{U}(\rho \otimes \rho_{\mathrm{env}})\mathsf{U}^\dagger\big]$. (1)

$\Big(\{\Pi_n\}$ measur$^{\mathrm{t}}$ for $A \Longrightarrow \{\Pi_n \otimes \mathrm{I}_B\}$ measur$^{\mathrm{t}}$ for AB. Then $\mathrm{tr}_{AB}[\rho_{AB}(\Pi_n \otimes \mathrm{I}_B)] = \mathrm{tr}_A(\rho_A\Pi_n)$ with $\rho_A = \mathrm{tr}_B(\rho_{AB})$.$\Big)$

Very often, the environment incorporates a huge number of degrees of freedom, and is largely uncontrolled ; it can be understood as quantum noise inducing decoherence.

A very nice feature is that, independently of the size of the environment, Eq. (1) can always be put in the form $\rho \longmapsto \mathcal{N}(\rho) = \sum_\ell \Lambda_\ell \rho \Lambda_\ell^\dagger$ operator-sum or Kraus representation, with the Kraus operators $\Lambda_\ell \in \mathcal{L}(\mathcal{H}_N)$, which need not be more than N^2, satisfying $\sum_\ell \Lambda_\ell^\dagger \Lambda_\ell = \mathrm{I}_N$, to ensure $\mathrm{tr}\big(\mathcal{N}(\rho)\big) = 1$, $\forall \rho$. [Hellwig & Kraus, *Commun. Math. Phys.* 1970]

Generalized measurement

In a Hilbert space $\mathcal{H}_N$ with dimension N, the state of a quantum system is specified by a Hermitian positive unit-trace density operator $\rho \in \mathcal{L}(\mathcal{H}_N)$.

• **Projective measurement :**

Defined by a set of orthogonal projectors $\Pi_n \in \mathcal{L}(\mathcal{H}_N)$, verifying $\sum_n \Pi_n = \mathrm{I}_N$,

and $\Pr\{\Pi_n\} = \mathrm{tr}(\rho\Pi_n)$. Moreover $\sum_n \Pr\{\Pi_n\} = 1 \,, \forall\rho \iff \sum_n \Pi_n = \mathrm{I}_N$.

• **Generalized measurement (POVM) :** (positive operator valued measure)

Equivalent to a projective measurement in a larger Hilbert space (Naimark th.).

Defined by a set of an arbitrary number M of positive operators $\mathsf{M}_m \in \mathcal{L}(\mathcal{H}_N)$,

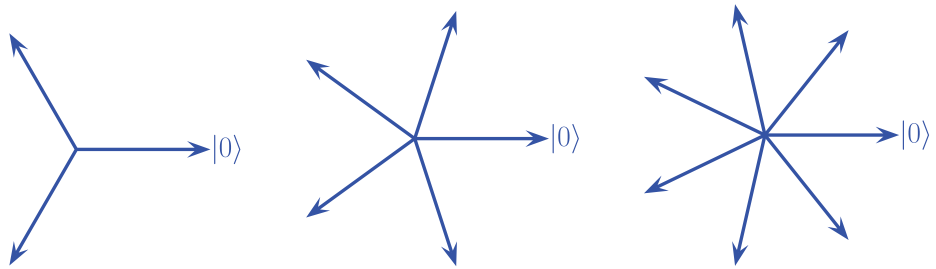
verifying $\sum_m \mathsf{M}_m = \mathrm{I}_N$,

and $\Pr\{\mathsf{M}_m\} = \mathrm{tr}(\rho\mathsf{M}_m)$. Moreover $\sum_m \Pr\{\mathsf{M}_m\} = 1 \,, \forall\rho \iff \sum_m \mathsf{M}_m = \mathrm{I}_N$.

A generalized measurement (POVM) for the qubit in $\mathcal{H}_2$

POVM $\left\{\mathsf{M}_m = \dfrac{2}{M}\,|e_m\rangle\langle e_m|\right\}$, for $m = 0, 1, \ldots M-1$, and $M > 2$,

with $|e_m\rangle = \cos\left(\dfrac{2\pi m}{M}\right)|0\rangle + \sin\left(\dfrac{2\pi m}{M}\right)|1\rangle \in \mathcal{H}_2$.



$M = 3$ $M = 5$ $M = 7$

Summary of quantum basics

A quantum system has a state represented by a normalized vector $|\psi\rangle \in \mathcal{H}_N$, or more generally by a (positive unit-trace) density operator $\rho \in \mathcal{L}(\mathcal{H}_N)$.

Its evolution is described by
$\rho \longmapsto \mathsf{U}\rho\mathsf{U}^\dagger$ when isolated, with unitary $\mathsf{U} \in \mathcal{L}(\mathcal{H}_N)$,
or more generally $\rho \longmapsto \mathcal{N}(\rho) = \sum_\ell \Lambda_\ell\rho\Lambda_\ell^\dagger$ with $\sum_\ell \Lambda_\ell^\dagger\Lambda_\ell = \mathrm{I}_N$ in $\mathcal{L}(\mathcal{H}_N)$.

Its measurement can be performed
with a set of an arbitrary number of positive operators M_m of $\mathcal{L}(\mathcal{H}_N)$
verifying $\sum_m \mathsf{M}_m = \mathrm{I}_N$,
yielding the probabilistic outcome $\Pr\{\mathsf{M}_m\} = \mathrm{tr}(\rho\mathsf{M}_m)$.

Entangled states

Two quantum systems A with Hilbert space $\mathcal{H}(A)$, and B with $\mathcal{H}(B)$, form a composite quantum system AB with joint state in the **tensor-product space** $\mathcal{H}(A)\otimes\mathcal{H}(B)$.

Any state of the tensor-product space $\mathcal{H}(A)\otimes\mathcal{H}(B)$ which is not factorizable as the product of a state of $\mathcal{H}(A)$ and a state of $\mathcal{H}(B)$ is an **entangled state** .

Ex. : A qubit A in state $|A\rangle = \big(|0\rangle + |1\rangle\big)/\sqrt{2} = |+\rangle \in \mathcal{H}(A) = \mathcal{H}_2$,
another qubit B in state $|B\rangle = \big(|0\rangle - |1\rangle\big)/\sqrt{2} = |-\rangle \in \mathcal{H}(B) = \mathcal{H}_2$,
with canonical orthonormal basis $\big\{|0\rangle\,, |1\rangle\big\}$ of $\mathcal{H}_2$.

The qubit pair AB is in $\mathcal{H}_2\otimes\mathcal{H}_2$ referred to the canonical orthonormal basis
$\big\{|0\rangle\otimes|0\rangle = |00\rangle\,,\ |0\rangle\otimes|1\rangle = |01\rangle\,,\ |1\rangle\otimes|0\rangle = |10\rangle\,,\ |1\rangle\otimes|1\rangle = |11\rangle\big\}$,
with state $|AB\rangle = |+\rangle\otimes|-\rangle = \big(|00\rangle - |01\rangle + |10\rangle - |11\rangle\big)/2$ which is a **separable** (factorizable) state.

Meanwhile, $|AB\rangle = \big(|01\rangle + |10\rangle\big)/\sqrt{2}$ is an **entangled** (non factorizable) state of the pair.

Physically an entangled state behaves as a nonlocal whole : what is done on one part may influence the other part instantly, no matter how distant they are. (And more.)

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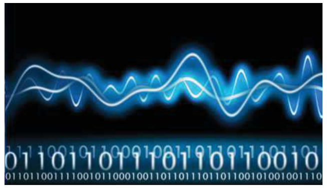


Quantum state detection or discrimination

A quantum system can be in one of two alternative states ρ_0 or $\rho_1 \in \mathcal{L}(\mathcal{H}_N)$ with prior probabilities P_0 and $P_1 = 1 - P_0$.

Question : What is the best pair of measurement operators $\{\mathsf{M}_0, \mathsf{M}_1\}$ in $\mathcal{L}(\mathcal{H}_N)$ to decide with a maximal probability of success $P_{\rm suc}$?

Answer : One has $P_{\rm suc} = P_0\,{\rm tr}(\rho_0\mathsf{M}_0) + P_1\,{\rm tr}(\rho_1\mathsf{M}_1) = P_0 + {\rm tr}(\mathsf{T}\mathsf{M}_1)$,
with the test operator $\mathsf{T} = P_1\rho_1 - P_0\rho_0$.
Then $P_{\rm suc}$ is maximized by the optimal operator $\mathsf{M}_1^{\rm opt} = \sum_{\lambda_n>0} |\lambda_n\rangle\langle\lambda_n|$,
which is the projector on the eigensubspace of T with positive eigenvalues λ_n.

The optimal measurement $\left\{\mathsf{M}_1^{\rm opt},\ \mathsf{M}_0^{\rm opt} = \mathrm{I}_N - \mathsf{M}_1^{\rm opt}\right\}$

achieves the maximum $P_{\rm suc}^{\max} = \dfrac{1}{2}\left(1 + \sum_{n=1}^{N} |\lambda_n|\right) = \dfrac{1}{2}\left(1 + {\rm tr}(|\mathsf{T}|)\right)$.

C. W. Helstrom, "Quantum Detection & Estimation Theory", Academic Press 1976.

Discrimination among $M > 2$ quantum states

A quantum system can be in one of M alternative states $\rho_m \in \mathcal{L}(\mathcal{H}_N)$,
for $m = 1$ to M, with prior probabilities P_m with $\sum_{m=1}^{M} P_m = 1$.

Problem : What is the best measurement $\{\mathsf{M}_m\}$ with M outcomes to decide with a maximal probability of success $P_{\rm suc}$?

$\Longrightarrow$ Maximize $P_{\rm suc} = \sum_{m=1}^{M} P_m\,{\rm tr}(\rho_m\mathsf{M}_m)$ according to the M operators M_m,

subject to $0 \le \mathsf{M}_m \le \mathrm{I}_N$ and $\sum_{m=1}^{M} \mathsf{M}_m = \mathrm{I}_N$.

For $M > 2$ this problem is only partially solved, in some special cases.

S. M. Barnett, S. Croke; "Quantum state discrimination"; *Advances in Optics and Photonics*, vol. 1, pp. 238–278, 2009.

Numerical solution

Y. C. Eldar, A. Megretski, G. C. Verghese; "Designing optimal quantum detectors via semidefinite programming"; *IEEE Transactions on Information Theory*, vol. 49, pp. 1007–1012, 2003.

For distinguishing among a collection of density operators,
find the optimal measurement maximizing the probability of success.

For the problem, no closed-form analytical solutions are known in general.

But it is a convex optimization problem
that can be solved numerically by a semidefinite program
converging to the global optimum in polynomial time within any desired accuracy.

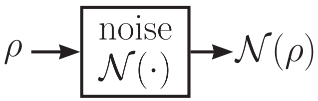
On Matlab using the linear matrix inequality (LMI) Toolbox.

Other numerical solutions ? interval calculus ? machine learning ? . . . .

Discrimination from $M = 2$ noisy qubits

Quantum noise on qubit states : $\rho \longmapsto \mathcal{N}(\rho)$.

Discrimination from the noisy qubit states $\mathcal{N}(\rho_0)$ and $\mathcal{N}(\rho_1)$.



- For given noise $\mathcal{N}(\cdot)$, what are the best input states (ρ_0, ρ_1) ?

  F. Chapeau-Blondeau, "Optimization of quantum states for signaling across an arbitrary qubit noise channel with minimum-error detection"; *IEEE Transactions on Information Theory* 61 (2015) 4500–4510.

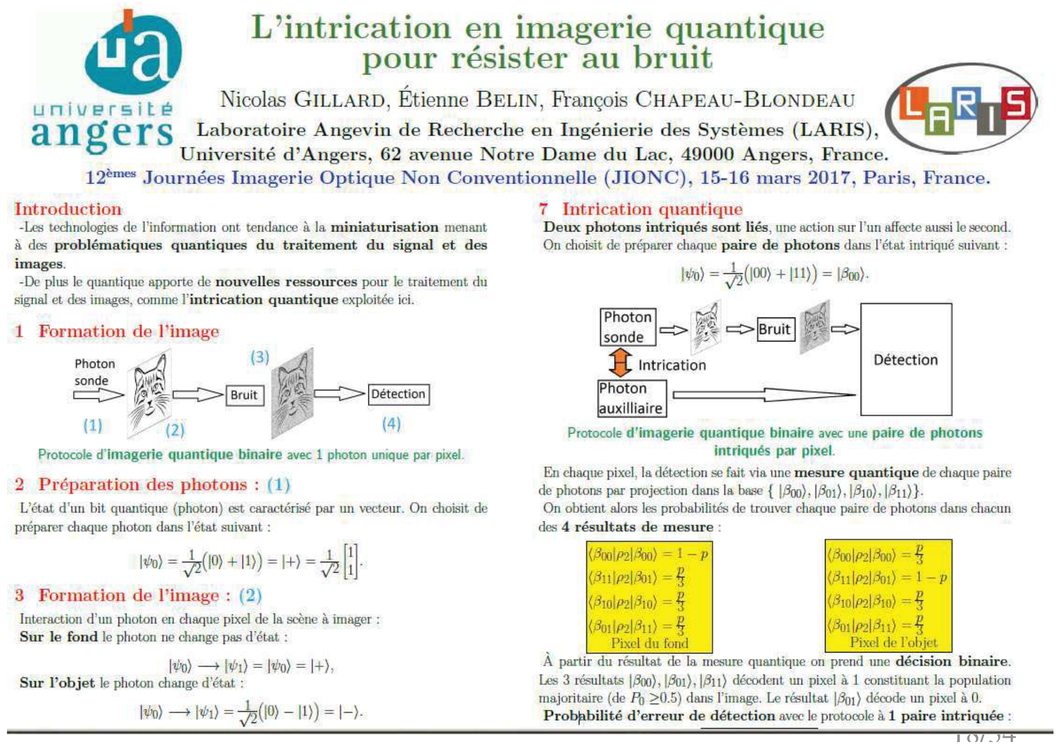
  F. Chapeau-Blondeau, "Détection quantique optimale sur un qubit bruité", *25ème Colloque GRETSI sur le Traitement du Signal et des Images*, Lyon, France, 8–11 sept. 2015.

- As the noise level increases, possibility of nonmonotonic evolution of the performance $P_{\rm suc}$ (stochastic resonance).

  F. Chapeau-Blondeau; "Quantum state discrimination and enhancement by noise"; *Physics Letters A* 378 (2014) 2128–2136.

  N. Gillard, E. Belin, F. Chapeau-Blondeau; "Qubit state detection and enhancement by quantum thermal noise"; *Electronics Letters* 54 (2018) 38–39.

The case $M > 2$, or in dimension higher than that of the qubit, remain largely unsolved / unexplored for noisy quantum systems.



L'intrication en imagerie quantique pour résister au bruit

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12èmes Journées Imagerie Optique Non Conventionnelle (JIONC), 15-16 mars 2017, Paris, France.

Introduction

-Les technologies de l'information ont tendance à la **miniaturisation** menant à des **problématiques quantiques du traitement du signal et des images**.

-De plus le quantique apporte de **nouvelles ressources** pour le traitement du signal et des images, comme l'**intrication quantique** exploitée ici.

1 Formation de l'image

Protocole d'**imagerie quantique binaire** avec 1 photon unique par pixel.

2 Préparation des photons : (1)

L'état d'un bit quantique (photon) est caractérisé par un vecteur. On choisit de préparer chaque photon dans l'état suivant :

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}.$$

3 Formation de l'image : (2)

Interaction d'un photon en chaque pixel de la scène à imager :

Sur le fond le photon ne change pas d'état :

$$|\psi_0\rangle \longrightarrow |\psi_1\rangle = |\psi_0\rangle = |+\rangle,$$

Sur l'objet le photon change d'état :

$$|\psi_0\rangle \longrightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = |-\rangle.$$

7 Intrication quantique

Deux photons intriqués sont liés, une action sur l'un affecte aussi le second. On choisit de préparer chaque **paire de photons** dans l'état intriqué suivant :

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = |\beta_{00}\rangle.$$

Protocole **d'imagerie quantique binaire** avec une **paire de photons intriqués par pixel**.

En chaque pixel, la détection se fait via une **mesure quantique** de chaque paire de photons par projection dans la base $\{\, |\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle \}$.

On obtient alors les probabilités de trouver chaque paire de photons dans chacun des **4 résultats de mesure** :

Pixel du fond	Pixel de l'objet
$\langle\beta_{00}\vert\rho_2\vert\beta_{00}\rangle = 1 - p$	$\langle\beta_{00}\vert\rho_2\vert\beta_{00}\rangle = \frac{p}{3}$
$\langle\beta_{11}\vert\rho_2\vert\beta_{01}\rangle = \frac{p}{3}$	$\langle\beta_{11}\vert\rho_2\vert\beta_{01}\rangle = 1 - p$
$\langle\beta_{10}\vert\rho_2\vert\beta_{10}\rangle = \frac{p}{3}$	$\langle\beta_{10}\vert\rho_2\vert\beta_{10}\rangle = \frac{p}{3}$
$\langle\beta_{01}\vert\rho_2\vert\beta_{11}\rangle = \frac{p}{3}$	$\langle\beta_{01}\vert\rho_2\vert\beta_{11}\rangle = \frac{p}{3}$

À partir du résultat de la mesure quantique on prend une **décision binaire**. Les 3 résultats $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{11}\rangle$ décodent un pixel à 1 constituant la population majoritaire (de $P_0 \geq 0.5$) dans l'image. Le résultat $|\beta_{01}\rangle$ décode un pixel à 0.

Probabilité d'erreur de détection avec le protocole à **1 paire intriquée** :

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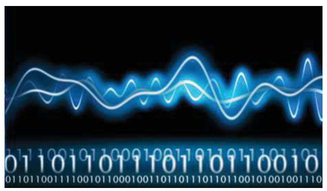


Communication over a noisy quantum channel (1/4)

$$(X = x_j, p_j) \longrightarrow \rho_j \longrightarrow \boxed{\mathcal{N}} \longrightarrow \mathcal{N}(\rho_j) = \rho'_j \longrightarrow \boxed{K\text{-element POVM}} \longrightarrow Y = y_k$$

At input, each classical symbol x_j is coded by a quantum state $|\psi_j\rangle \in \mathcal{H}_N$ or $\rho_j \in \mathcal{L}(\mathcal{H}_N)$, for $j = 1, 2, \dots J$.

Noisy quantum channel $\rho_j \longmapsto \mathcal{N}(\rho_j) = \rho'_j$ produced as outputs.

A generalized measurement by the POVM with K elements M_k, for $k = 1, 2, \dots K$, generates measurement outcome Y with K possible values y_k, for $k = 1, 2, \dots K$, of conditional probabilities $\Pr\{Y = y_k | X = x_j\} = \mathrm{tr}(\rho'_j \mathsf{M}_k)$, and total probabilities $\Pr\{Y = y_k\} = \sum_{j=1}^{J} \Pr\{Y = y_k | X = x_j\} p_j = \mathrm{tr}(\rho' \mathsf{M}_k)$, with $\rho' = \sum_{j=1}^{J} p_j \rho'_j$ the average output state.

$\Longrightarrow$ Input–output mutual information $I(X;Y) = H(Y) - H(Y|X) = H(X) - H(X|Y)$, with Shannon entropy $H(X) = -\sum_{j=1}^{J} p_j \log(p_j)$.

Question : Which POVM to maximize $I(X;Y)$ and at which level $I_{\max}(X;Y)$?

Communication over a noisy quantum channel (2/4)

One has the majorization $I(X;Y) \le \chi(\rho'_j, p_j)$

by the Holevo information $\chi(\rho'_j, p_j) = S(\rho') - \sum_{j=1}^{J} p_j S(\rho'_j)$

with von Neumann entropy $S(\rho') = -\mathrm{tr}\big[\rho' \log(\rho')\big]$.

$\chi(\rho'_j, p_j)$ characterizes the maximum transmission rate of the source $\{(p_j, \rho_j)\}$, without the need to refer to any definite POVM.

$\forall \{(p_j, \rho_j)\}$ and $\mathcal{N}(\cdot)$, there always exists a POVM to achieve $I(X;Y) = \chi(\rho'_j, p_j)$,
(by measuring blocks of length $L \to \infty$ from successive independent input symbols X),
i.e. $\chi(\rho'_j, p_j)$ is an achievable maximum rate for error-free communication, with a given statistical ensemble $\{(p_j, \rho_j)\}$ of input signaling states.

B. Schumacher, M. D. Westmoreland; "Sending classical information via noisy quantum channels"; *Physical Review A* 56 (1997) 131–138.

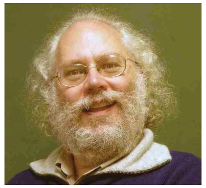
A. S. Holevo; "The capacity of the quantum channel with general signal states"; *IEEE Transactions on Information Theory* 44 (1998) 269–273.

Communication over a noisy quantum channel (3/4)

For a given noisy channel $\mathcal{N}(\cdot)$ therefore $\chi_{\max} = \max_{\{p_j, \rho_j\}} \chi\big(\mathcal{N}(\rho_j), p_j\big)$

is the overall maximum and achievable rate for error-free communication of classical information over a given noisy quantum channel, or the classical information capacity of the quantum channel, for product states or successive independent uses of the channel.

NB : The maximum $\chi_{\max}$ can be achieved by no more than N^2 *pure* input states $\rho_j = |\psi_j\rangle\langle\psi_j|$ with $|\psi_j\rangle \in \mathcal{H}_N$ (Not necessarily easy to characterize).
[Shor, *J. Math. Phys.* 43 (2002) 4334. Shor, *Com. Math. Phys.* 246 (2004) 453].

Peter W. SHOR   1998.

Communication over a noisy quantum channel (4/4)

For product states or successive independent uses of the channel (with given dimensionality), the Holevo information is additive $\chi_{\max}(\mathcal{N}_1 \otimes \mathcal{N}_2) = \chi_{\max}(\mathcal{N}_1) + \chi_{\max}(\mathcal{N}_2)$.

For non-product states or successive non-independent but entangled uses of the channel, due to a convexity property, the Holevo information is always superadditive
$\chi_{\max}(\mathcal{N}_1 \otimes \mathcal{N}_2) \ge \chi_{\max}(\mathcal{N}_1) + \chi_{\max}(\mathcal{N}_2)$. [Wilde 2016 Eq. (20.126)]

For many quantum channels it is found additive, $\chi_{\max}(\mathcal{N}_1 \otimes \mathcal{N}_2) = \chi_{\max}(\mathcal{N}_1) + \chi_{\max}(\mathcal{N}_2)$ so that entanglement does not improve over the product-state capacity.
(Like for classical channels where the max of $I(\cdot\,;\cdot)$ always occurs with independent product laws.)

Yet for some quantum channels it has been found strictly superadditive,
$\chi_{\max}(\mathcal{N}_1 \otimes \mathcal{N}_2) > \chi_{\max}(\mathcal{N}_1) + \chi_{\max}(\mathcal{N}_2)$ meaning that entanglement does improve over the product-state capacity.
M. B. Hastings; "Superadditivity of communication capacity using entangled inputs"; *Nature Physics* 5 (2009) 255–257.

Then, which channels ? which entanglements ? which improvement ? which capacity ? . . . (largely, these are open issues).

Additive quantum channels

For the Holevo information, additivity $\chi_{\max}(\mathcal{N}_1 \otimes \mathcal{N}_2) = \chi_{\max}(\mathcal{N}_1) + \chi_{\max}(\mathcal{N}_2)$ has been proved for a number of channels.

Additivity has been proved when one channel is the identity, or a unital qubit channel, or a c-q or a q-c channel, or an entanglement-breaking channel.
P. W. Shor; "Additivity of the classical capacity of entanglement-breaking quantum channels"; *Journal of Mathematical Physics*, vol. 43, pp. 4334–4340, 2002.

Additivity has been proved for unital qubit channels, the depolarizing channel, the erasure channel, the purely lossy bosonic channel, the whole class of entanglement-breaking channels.
A. S. Holevo, V. Giovannetti; "Quantum channels and their entropic characteristics"; *Reports on Progress in Physics* vol. 75, pp. 046001,1–30, 2012.

A superadditive quantum channel

A counterexample where the Holevo information is strictly superadditive $\mathcal{X}_{\max}(\mathcal{N}_1 \otimes \mathcal{N}_2) > \mathcal{X}_{\max}(\mathcal{N}_1) + \mathcal{X}_{\max}(\mathcal{N}_2)$, has been reported in

M. B. Hastings; "Superadditivity of communication capacity using entangled inputs"; *Nature Physics* 5 (2009) 255–257.

with channels of the form $\mathcal{N}(\rho) = \sum_{\ell=1}^{L} P_\ell \mathsf{U}_\ell \rho \mathsf{U}_\ell^\dagger$, with L random unitary operators U_ℓ on $\mathcal{H}_N$, random probabilities P_ℓ, and $1 \ll L \ll N$, requiring an (high) output dimension $N \geq 183$ [Belinschi, *Com. Math. Phys.* 341 (2016) 885].

Based on equivalence of additivity of Holevo information with additivity of minimal output entropy $S_{\min}(\mathcal{N}_1 \otimes \mathcal{N}_2) = S_{\min}(\mathcal{N}_1) + S_{\min}(\mathcal{N}_2)$, with $S_{\min}(\mathcal{N}) = \min_\rho S\big(\mathcal{N}(\rho)\big)$ this min being achievable over *pure* input states $\rho = |\psi\rangle\langle\psi|$ on $\mathcal{H}_N$, as proved in

P. W. Shor; "Equivalence of additivity questions in quantum information theory"; *Communications in Mathematical Physics* 246 (2004) 453–472.

Any other ? simpler ? more generic ? physically motivated ? . . .

Stochastic resonance with quantum informational measures

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Enhancing qubit information with quantum thermal noise

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HIGHLIGHTS

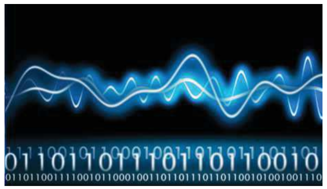
- Several generic informational quantities characterizing the qubit are analyzed.
- Qubit decoherence is represented by a quantum thermal noise at arbitrary temperature.
- Nontrivial regimes of variation are reported for the informational quantities.
- They do not always degrade but can show nonmonotonic variation at increasing temperature.
- Higher noise temperatures or increased decoherence may prove beneficial informationally.

1 – Quantum basics

2 – Signal detection

3 – Signal transmission

4 – Signal estimation



Parametric estimation from a quantum signal state

A quantum system has its state $\rho_\xi \in \mathcal{L}(\mathcal{H}_D)$ dependent on an unknown parameter ξ.

A generalized measurement by the POVM with K elements M_k, for $k = 1, 2, \dots K$, generates measurement outcome X with K possible values x_k, for $k = 1, 2, \dots K$, of probabilities $\Pr\{X = x_k\,; \xi\} = \mathrm{tr}(\rho_\xi \mathsf{M}_k)$.

From X an estimator $\widehat{\xi} = \widehat{\xi}(X)$ is devised for ξ, and its performance is assessed by the mean-squared error $\langle(\widehat{\xi} - \xi)^2\rangle$.

Question : What is the best (optimal) estimation strategy, leading to the minimal achievable mean-squared error ?

C. W. Helstrom, "Quantum Detection & Estimation Theory", Academic Press 1976.

M. G. A. Paris; "Quantum estimation for quantum technology"; *International Journal of Quantum Information* 7 (2009) 125–137.

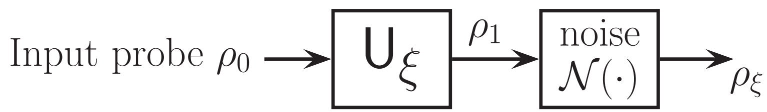
• Any estimator $\widehat{\xi} = \widehat{\xi}(X)$ verifies $\langle(\widehat{\xi} - \xi)^2\rangle \geq$ Cramér-Rao bound $\sim \dfrac{1}{F_c(\xi)}$
with classical Fisher information $F_c(\xi) = \left\langle \left[\partial_\xi \ln \Pr(X\,;\xi)\right]^2 \right\rangle$.

• The maximum likelihood estimator $\widehat{\xi}(X) = \arg\max_\xi \Pr(X\,;\xi)$ can reach the bound
by achieving $\langle(\widehat{\xi} - \xi)^2\rangle = \dfrac{1}{F_c(\xi)}$, the minimal error.

• In turn, $F_c(\xi)$ is upper-bounded by the quantum Fisher information $F_q(\xi)$,
i.e. $F_c(\xi) \leq F_q(\xi) = \left\langle \left[\mathcal{D}_\xi \rho_\xi\right]^2 \right\rangle$, with $\mathcal{D}_\xi$ symmetric logarithmic derivative.
From eigendecomposition of ρ_ξ in its orthonormal eigenbasis $\rho_\xi = \sum_{j=1}^{D} \lambda_j |\lambda_j\rangle\langle\lambda_j|$,
one has $F_q(\xi) = 2 \sum_{j,k} \dfrac{|\langle\lambda_j|\partial_\xi \rho_\xi|\lambda_k\rangle|^2}{\lambda_j + \lambda_k}$, (summing on all eigenvalues $\lambda_j + \lambda_k \neq 0$).

S. L. Braunstein, C. M. Caves; "Statistical distance and the geometry of quantum states"; *Physical Review Letters* 72 (1994) 3439–3443.

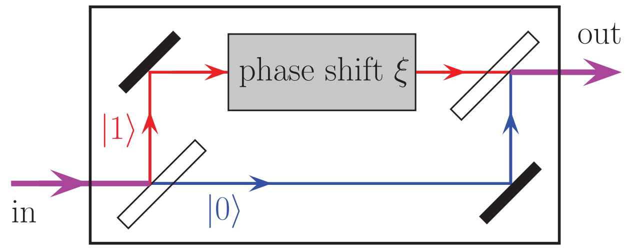
$\Longrightarrow$ Which POVM to achieve $F_c(\xi) = F_q(\xi)$?
What is the maximum achievable $F_q(\xi)$?



ξ-dependent unitary U_ξ delivers $\rho_1(\xi) = \mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger$
providing access to the noisy observation $\rho_\xi = \mathcal{N}\big(\rho_1(\xi)\big)$.

A photon (qubit) in an interferometer undergoing the unitary transformation

$$\mathsf{U}_\xi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\xi} \end{bmatrix} = |0\rangle\langle 0| + e^{i\xi}|1\rangle\langle 1| \,.$$



For a separable probe $\rho_0 \leftarrow \rho_0^{\otimes N}$ over N successive independent experiments, the problem is solved in

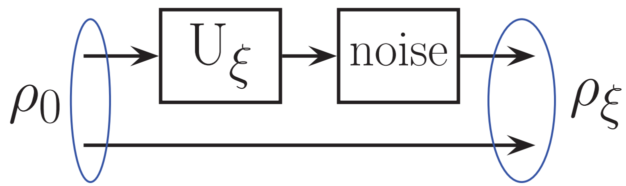
F. Chapeau-Blondeau; "Optimizing qubit phase estimation"; *Physical Review A* 94 (2016) 022334.

characterizing • the optimal input probe ρ_0 maximizing $F_q(\xi)$,
• the optimal POVM reaching the maximum $F_c(\xi) = F_q(\xi)$,
• the optimal estimator $\widehat{\xi}$ achieving the minimum $\langle(\widehat{\xi} - \xi)^2\rangle = \dfrac{1}{F_c(\xi)}$.

For an N-qubit entangled probe ρ_0, the optimal estimation strategy largely remains open.
Which entangled probe ρ_0 ? which size N ? which maximal $F_q(\xi)$ achievable ? …

The problem is addressed in (among others, and references there in)
F. Chapeau-Blondeau; "Entanglement-assisted quantum parameter estimation from a noisy qubit pair: A Fisher information analysis"; *Physics Letters A* 381 (2017) 1369–1378.
showing that $N = 2$ properly entangled qubits can improve over
$N = 2$ independent qubits in optimal configuration.

Already, entangling the active qubit with one inactive qubit



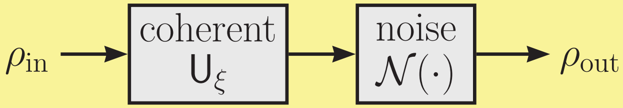
provides a net improvement of the estimation performance (although the inactive qubit never interacts with the ξ-dependent process to be estimated !).

F. Chapeau-Blondeau; "Entanglement-assisted quantum parameter estimation from a noisy qubit pair: A Fisher information analysis"; *Physics Letters A* 381 (2017) 1369–1378.

N. Gillard, E. Belin, F. Chapeau-Blondeau ;
"Estimation quantique en présence de bruit améliorée par l'intrication" ;
Actes du 26ème Colloque GRETSI sur le Traitement du Signal et des Images, 5–8 sept. 2017.

F. Chapeau-Blondeau ; "Qubit state estimation and enhancement by quantum thermal noise" ;
Electronics Letters 51 (2015) 1673–1675 .

Summary and outlook



Performance measures with informational significance :

Probability of successful detection $P_{suc} = \sum_{m=1}^{M} P_m \operatorname{tr}(\rho_m \mathsf{M}_m)$,

Holevo information $\chi(\rho'_j, p_j) = S(\rho') - \sum_{j=1}^{J} p_j S(\rho'_j)$,

Quantum Fisher information $F_q(\xi) = 2 \sum_{j,k} \frac{|\langle \lambda_j | \partial_\xi \rho_\xi | \lambda_k \rangle|^2}{\lambda_j + \lambda_k}$,

for maximization, superadditivity, improvement by noise, . . .

Or else, quantum computation and algorithms, optimization, quantum image processing, quantum automatic control, physical implementations, . . .

Merci de votre attention.

Si vous avez compris . . .
c’est que je me suis mal exprimé !

“Nobody really understands quantum mechanics.”
R. P. Feynman