Available online at www.sciencedirect.com
SCIENCE DIRECT®
Digital Signal Processing 15 (2005) 19–32
Digital Signal Processing
www.elsevier.com/locate/dsp

Stochastic resonance and improvement by noise in optimal detection strategies

David Rousseau, François Chapeau-Blondeau *

Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers, 62 avenue Notre Dame du Lac, 49000 Angers, France

Available online 5 October 2004

Abstract

A stochastic resonance effect, under the form of a noise-improved performance, is shown feasible for a whole range of optimal detection strategies, including Bayesian, minimum error-probability, Neyman–Pearson, and minimax detectors. In each case, situations are demonstrated where the performance of the optimal detector can be improved (locally) by raising the level of the noise. This is obtained with a nonlinear signal-noise mixture where a non-Gaussian noise acts on the phase of a periodic signal.

Keywords: Optimal detectors; Noise; Stochastic resonance

1. Introduction

Stochastic resonance is a phenomenon in which some processing done on a signal can be improved by the action of the noise [1–3]. The feasibility of stochastic resonance has now been reported in a large variety of processes, under many different forms [4–9]. Yet, until very recently, stochastic resonance as an improvement of the performance by noise, was limited to suboptimal devices or processors. In the context of detection problems, various aspects of stochastic resonance have been investigated [10–18], yet with improvement by noise limited to suboptimal detection strategies. Very recently [19,20], stochastic resonance has been shown feasible also in optimal processing, in a Bayesian detection problem

* Corresponding author.
E-mail address: chapeau@univ-angers.fr (F. Chapeau-Blondeau).

1051-2004/$ – see front matter
doi:10.1016/j.dsp.2004.09.006

on a nonlinear signal-noise mixture with non-Gaussian noise. In the present paper we consider the same type of detection problem, and we extend the demonstration of feasibility of stochastic resonance, to a whole range of standard optimal detection strategies, in a coherent perspective.

2. Strategies for optimal detection

In this section, we briefly review standard strategies for optimal detection. Our aim here is to exhibit the definition of the optimal detectors we will be considering, and the way their performance is assessed. Classical proofs and developments can be found in [21,22]. Later on, our point will be to show that situations can be found where the performance of each one of these optimal detectors can be improved by operating them at higher noise levels, over some ranges of the noise.

In a standard detection problem, one among two known signals $s_0(t)$ or $s_1(t)$ may be mixed to a noise $\eta(t)$, the resulting mixture forming the observable signal $x(t)$. Observation of $x(t)$ at N distinct times t_k, for $k = 1$ to N, provides N data points $x_k = x(t_k)$. From the data $\boldsymbol{x} = (x_1, \ldots, x_N)$, it is to be decided whether $x(t)$ is formed by $\eta(t)$ mixed to $s_0(t)$ (hypothesis H_0) or to $s_1(t)$ (hypothesis H_1). Any conceivable detector is equivalent to a partition of $\mathbb{R}^N$ into two disjoint complementary subsets $\mathcal{R}_0$ and $\mathcal{R}_1$, such that when $\boldsymbol{x}$ falls in $\mathcal{R}_i$ then the detector decides H_i, for $i \in \{0, 1\}$.

2.1. Bayesian detection

If the prior probabilities are known, P_0 for hypothesis H_0, and $P_1 = 1 - P_0$ for H_1, it is possible to assess the performance of a given detector by means of a Bayesian cost. One introduces four elementary costs C_{ij} of deciding H_i when H_j holds, $i, j \in \{0, 1\}$, with necessarily $C_{10} > C_{00}$ and $C_{01} > C_{11}$ to penalize wrong decisions. The average Bayesian cost is then

$$C = P_0 C_{00} \int_{\mathcal{R}_0} p(\boldsymbol{x}|\mathrm{H}_0)\, d\boldsymbol{x} + P_1 C_{01} \int_{\mathcal{R}_0} p(\boldsymbol{x}|\mathrm{H}_1)\, d\boldsymbol{x} + P_0 C_{10} \int_{\mathcal{R}_1} p(\boldsymbol{x}|\mathrm{H}_0)\, d\boldsymbol{x} + P_1 C_{11} \int_{\mathcal{R}_1} p(\boldsymbol{x}|\mathrm{H}_1)\, d\boldsymbol{x}, \tag{1}$$

where $p(\boldsymbol{x}|\mathrm{H}_i)$ is the probability density for observing $\boldsymbol{x}$ when H_i holds, and $\int d\boldsymbol{x}$ stands for the N-dimensional integral $\int \ldots \int dx_1 \ldots dx_N$. The cost C of Eq. (1) is minimized by the optimal Bayesian detector that uses the likelihood ratio

$$L(\boldsymbol{x}) = \frac{p(\boldsymbol{x}|\mathrm{H}_1)}{p(\boldsymbol{x}|\mathrm{H}_0)} = \frac{\Pr\{\boldsymbol{x}|\mathrm{H}_1\}}{\Pr\{\boldsymbol{x}|\mathrm{H}_0\}} \tag{2}$$

to implement the test

$$L(\boldsymbol{x}) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \frac{P_0}{P_1} \frac{C_{10} - C_{00}}{C_{01} - C_{11}}, \tag{3}$$

and by doing so achieves the minimal cost

$$C_{\min} = \frac{1}{2}\big[P_1(C_{01}+C_{11}) + P_0(C_{10}+C_{00})\big] - \frac{1}{2}\int_{\mathbb{R}^N} \big|P_1(C_{01}-C_{11})p(\boldsymbol{x}|\mathrm{H}_1) - P_0(C_{10}-C_{00})p(\boldsymbol{x}|\mathrm{H}_0)\big|\,d\boldsymbol{x}. \tag{4}$$

2.2. Minimum error-probability detection

In the special case where $C_{00} = C_{11} = 0$ and $C_{10} = C_{01} = 1$, the cost C of Eq. (1) represents the overall probability of detection error P_{er}. The optimal Bayesian detector of Eq. (4) then represents the detector minimizing P_{er}, also known as the maximum a posteriori probability (MAP) detector since test (4) becomes equivalent to a test comparing $\Pr\{\mathrm{H}_1|\boldsymbol{x}\}/\Pr\{\mathrm{H}_0|\boldsymbol{x}\}$ to 1.

2.3. Neyman–Pearson detection

When P_0 and P_1 are unknown, a strategy to implement an optimal detection is to seek to maximize the probability of detection

$$P_{\mathrm{d}} = \int_{\mathcal{R}_1} p(\boldsymbol{x}|\mathrm{H}_1)\,d\boldsymbol{x}, \tag{5}$$

while keeping the probability of false alarm

$$P_{\mathrm{f}} = \int_{\mathcal{R}_1} p(\boldsymbol{x}|\mathrm{H}_0)\,d\boldsymbol{x} \tag{6}$$

no larger than a prescribed level $P_{\mathrm{f,sup}}$.

This constrained maximization is achieved by the optimal Neyman–Pearson detector, which also implements a likelihood-ratio test

$$L(\boldsymbol{x}) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \mu(P_{\mathrm{f,sup}}), \tag{7}$$

with a threshold $\mu(P_{\mathrm{f,sup}})$, a function of $P_{\mathrm{f,sup}}$, which is found from Eq. (6) by imposing $P_{\mathrm{f}} \le P_{\mathrm{f,sup}}$, with the subset $\mathcal{R}_1$ defined as those points $\boldsymbol{x}$ for which $p(\boldsymbol{x}|\mathrm{H}_1) > \mu p(\boldsymbol{x}|\mathrm{H}_0)$.

2.4. Minimax detection

In the absence of known P_0 and P_1, another strategy for an optimal detection, which does not require to specify a $P_{\mathrm{f,sup}}$, is to look for the value P_0^* of P_0 that maximizes the minimal cost $C_{\min}$ of Eq. (4). The optimal minimax detector then implements the likelihood-ratio test of Eq. (4) with P_0 set to P_0^*. For any detector, the Bayesian cost C in Eq. (1) is a function of P_0, which goes through a maximum for some $P_0 \in [0, 1]$. This maximum turns out to be minimized by the optimal minimax detector[1] defined above, and the minimal value achieved for this maximum is expressable as

$$C_{\text{minimax}} = C_{10} + (C_{00} - C_{10}) \int_{\mathcal{R}_0} p(\boldsymbol{x}|\mathrm{H}_0)\, d\boldsymbol{x}$$
$$= C_{01} + (C_{11} - C_{01}) \int_{\mathcal{R}_1} p(\boldsymbol{x}|\mathrm{H}_1)\, d\boldsymbol{x}. \tag{8}$$

So for an unknown P_0, adopting the optimal minimax detector minimizes the highest cost we would incur if nature were to impose us the least favorable P_0.

3. Detection with phase noise

We now apply the strategies of Section 2 to a specific detection problem. This is the same detection problem that was used in [19,20] for the first report of a noise-improved performance in an optimal Bayesian detector, and that we adopt here to test a whole family of optimal detectors. We consider a periodic "mother" wave $w(t)$ of period unity. A possibility could be $w(t) = \sin(2\pi t)$, but $w(t)$ will be further specified later. One of the two signals to be detected is the wave $w(t)$ with frequency ν_0, i.e., $s_0(t) = w(\nu_0 t)$; the other signal is the same wave $w(t)$ with frequency $\nu_1 \neq \nu_0$, i.e., $s_1(t) = w(\nu_1 t)$. A noise $\eta(t)$ acts on signals $s_0(t)$ and $s_1(t)$ as a phase noise, so as to form the observable signal

$$x(t) = w\big[\nu_0 t + \eta(t)\big] \quad (\text{hypothesis H}_0) \quad \text{or} \tag{9}$$
$$x(t) = w\big[\nu_1 t + \eta(t)\big] \quad (\text{hypothesis H}_1). \tag{10}$$

Such periodic signals corrupted by a phase noise arise, for instance, when a periodic wave propagates in a fluctuating medium or traverses a fluctuating interface. Phase noise is present in oscillators, phase-locked loops, coherent imaging. A simple concretization of the present setting is realized by a plane wave radiated or received by a transducer subjected to a random motion producing the phase noise.

Based on the data $\boldsymbol{x} = (x_1, \ldots, x_N)$ it is to be decided whether the wave corrupted by the phase noise has frequency ν_0 or ν_1. In order to allow a complete analytical treatment of the optimal detection strategies presented in Section 2, we consider, as in [19,20], the case where $w(t)$ is a square wave of period 1 with $w(t) = 1$ when $t \in [0, 1/2)$ and $w(t) = -1$ when $t \in [1/2, 1)$. In such a case, the possible values of the observations x_k reduce to ± 1, and quantities such $p(\boldsymbol{x}|\mathrm{H}_i)\, d\boldsymbol{x}$ define the finite probabilities $\Pr\{\boldsymbol{x}|\mathrm{H}_i\}$ which are nonzero only at the locations of the 2^N states $\boldsymbol{x} = (\pm 1, \ldots, \pm 1)$.

We assume the noise samples $\eta(t_k)$ statistically independent for distinct t_k's, so that the conditional probabilities factorize as $\Pr\{\boldsymbol{x}|\mathrm{H}_i\} = \prod_{k=1}^{N} \Pr\{x_k|\mathrm{H}_i\}$. Also, the samples $\eta(t_k)$ are identically distributed, with cumulative distribution function $F_\eta(u)$ and probabil-

[1] Also, it can be shown that the Bayesian cost C in Eq. (1) attached to the optimal minimax detector turns out to be equal to the constant C_{minimax} of Eq. (8) for any P_0.

ity density function $f_\eta(u) = dF_\eta/du$. We then have

$$\Pr\{x_k = 1|\mathrm{H}_1\} = \Pr\big\{w\big[\nu_1 t_k + \eta(t_k)\big] = 1\big\}, \tag{11}$$

$$= \Pr\Big\{\nu_1 t_k + \eta(t_k) \in \bigcup_\ell [\ell, \ell + 1/2)\Big\}, \tag{12}$$

$$= \Pr\Big\{\eta(t_k) \in \bigcup_\ell [\ell - \nu_1 t_k, \ell - \nu_1 t_k + 1/2)\Big\}, \tag{13}$$

$$= \sum_{\ell=-\infty}^{+\infty} \int_{\ell-\nu_1 t_k}^{\ell-\nu_1 t_k+1/2} f_\eta(u)\, du, \tag{14}$$

$$= \sum_{\ell=-\infty}^{+\infty} \big[F_\eta(\ell - \nu_1 t_k + 1/2) - F_\eta(\ell - \nu_1 t_k)\big], \tag{15}$$

ℓ integer, and

$$\Pr\{x_k = -1|\mathrm{H}_1\} = 1 - \Pr\{x_k = 1|\mathrm{H}_1\}. \tag{16}$$

In the same way, we have

$$\Pr\{x_k = 1|\mathrm{H}_0\} = \sum_{\ell=-\infty}^{+\infty} \big[F_\eta(\ell - \nu_0 t_k + 1/2) - F_\eta(\ell - \nu_0 t_k)\big] \tag{17}$$

and

$$\Pr\{x_k = -1|\mathrm{H}_0\} = 1 - \Pr\{x_k = 1|\mathrm{H}_0\}. \tag{18}$$

The probabilities $\Pr\{\boldsymbol{x}|\mathrm{H}_i\}$ that follow from Eqs. (15)–(18), allow an explicit implementation of the optimal detectors of Section 2 along with the assessment of their performance, as a function of the properties of the noise conveyed by $F_\eta(u)$.

4. Improvement by noise

The optimal Bayesian detector of Eq. (4) achieves the minimal cost $C_{\min}$ of Eq. (4) which here is computable as

$$C_{\min} = \frac{1}{2}\big[P_1(C_{01} + C_{11}) + P_0(C_{10} + C_{00})\big] - \frac{1}{2} \sum_{x_1 \in \{-1,1\}} \cdots \sum_{x_N \in \{-1,1\}} \big|P_1(C_{01} - C_{11}) \Pr\{x_1|\nu_1\} \ldots \Pr\{x_N|\nu_1\} - P_0(C_{10} - C_{00}) \Pr\{x_1|\nu_0\} \ldots \Pr\{x_N|\nu_0\}\big|, \tag{19}$$

the multiple sum running over the 2^N states accessible to the data $\boldsymbol{x}$.

For illustration of the possibility of a stochastic resonance, Fig. 1 shows the performance of the optimal Bayesian detector measured by the cost $C_{\min}$ of Eq. (19), as a function of

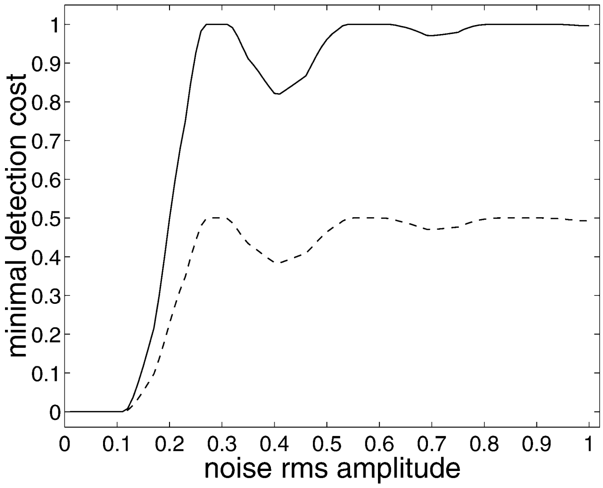

Fig. 1. Minimal detection cost $C_{\min}$ from Eq. (19) of the optimal Bayesian detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$. Also $C_{00} = C_{11} = 0$, $P_0 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 13$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{13} = 2.4$. Solid line: $C_{10} = 2$ and $C_{01} = 5$, dashed line: $C_{10} = 1$ and $C_{01} = 2$.

the rms amplitude σ_η of the phase noise $\eta(t)$ which has been chosen zero-mean uniform. In the representative conditions of Fig. 1, the evolutions of the cost $C_{\min}$ are clearly nonmonotonic as the noise level σ_η increases. This demonstrates the possibility, over some ranges of the noise level σ_η, of reducing the cost of the optimal detection by operating at higher noise levels. It is this very possibility that we want to emphasize, and that we interpret as a form of stochastic resonance, the nonmonotonic evolution of the optimal performance, when the noise level increases, instead of a monotonic degradation as commonly observed in standard optimal detection processes.

For another illustration, we choose for the phase noise $\eta(t)$, the family of Gaussian-mixture densities $f_\eta(u) = f_{\mathrm{gm}}(u/\sigma_\eta)/\sigma_\eta$ defined through the standardized probability density

$$f_{\mathrm{gm}}(u) = \frac{1}{2\sqrt{2\pi}\sqrt{1-m^2}}\left\{\exp\left[-\frac{(u+m)^2}{2(1-m^2)}\right] + \exp\left[-\frac{(u-m)^2}{2(1-m^2)}\right]\right\}, \tag{20}$$

with $0 < m < 1$, and cumulative distribution function

$$F_{\mathrm{gm}}(u) = \frac{1}{2} + \frac{1}{4}\left[\operatorname{erf}\left(\frac{u+m}{\sqrt{2}\sqrt{1-m^2}}\right) + \operatorname{erf}\left(\frac{u-m}{\sqrt{2}\sqrt{1-m^2}}\right)\right]. \tag{21}$$

Figure 2 shows the performance of the optimal Bayesian detector measured by the cost $C_{\min}$ of Eq. (19), as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$, in two representative sets of conditions. Again, the evolutions of Fig. 2 are clearly nonmonotonic as the noise level σ_η increases, demonstrating the same type of stochastic resonance with the optimal Bayesian detector. Figure 2 also presents numerical valida-

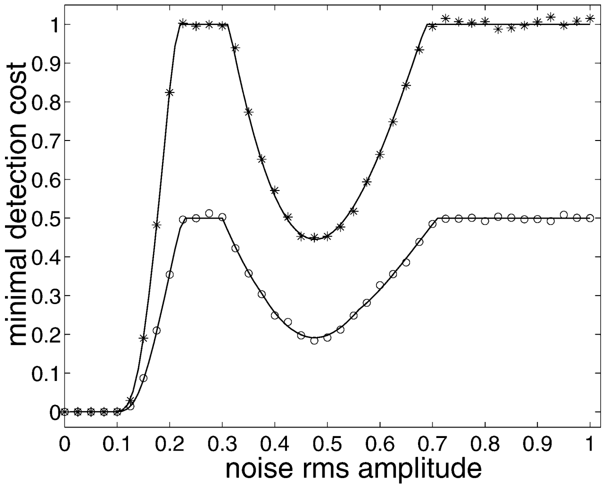

Fig. 2. Minimal detection cost $C_{\min}$ of the optimal Bayesian detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with $m = 0.95$. The solid lines are $C_{\min}$ of Eq. (19); the discrete points are $C_{\min}$ numerically evaluated with 10^4 Monte Carlo trials of the test (4) for each σ_η; with $C_{10} = 2$ and $C_{01} = 5$ ($*$), $C_{10} = 1$ and $C_{01} = 2$ ($\circ$). Also $C_{00} = C_{11} = 0$, $P_0 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 6$ data samples equispaced with time step 0.3 from $t_1 = 0$ to $t_6 = 1.5$.

tions of the theoretical performance, through a Monte Carlo implementation of the optimal detector of Eq. (4).

The same possibility of improvement by noise also exists for the performance of the optimal MAP detector, or minimum error-probability detector, measured by the (minimum) probability of detection error P_{er}. Figures 3 and 4 illustrate the possibility of reducing P_{er}, by increasing the level of noise σ_η, over some ranges, successively for the zero-mean uniform noise and for the Gaussian-mixture noise from Eq. (20).

For the optimal Neyman–Pearson detector, with the nonlinear signal-noise mixture with phase noise, we did not find it possible to obtain a general explicit analytical characterization of the threshold $\mu(P_{\text{f,sup}})$ of Eq. (8), for an arbitrary noise density $f_\eta(u)$. Nevertheless here, for a reasonable number of data points N leading to $2^N = N_S$ different states accessible to the data $\boldsymbol{x}$, it is feasible to exhaustively test all the 2^{N_S} possible partitions $(\mathcal{R}_0, \mathcal{R}_1)$, select those associated to a P_{f} in Eq. (6) no larger than $P_{\text{f,sup}}$, and among them retain the one with maximal P_{d} in Eq. (5). This was done so at each σ_η in various configurations, and the results are shown in Figs. 5 and 6, successively for the zero-mean uniform noise and for the Gaussian-mixture noise from Eq. (20). The evolutions of Figs. 5 and 6 demonstrate, for the optimal Neyman–Pearson detector, the possibility of increasing its probability of detection P_{d}, by raising the level of noise σ_η, over some ranges. This is again a nonmonotonic evolution of the performance of the optimal detector when the noise level is raised, that we interpret as a form of stochastic resonance.

Finally, it is possible to find P_0^* that maximizes $C_{\min}$ of Eq. (19), opening the way to the optimal minimax detector and its performance measured by the cost C_{minimax} of Eq. (8). Evolutions of C_{minimax} are shown in Figs. 7 and 8, successively for the zero-mean uniform noise and for the Gaussian-mixture noise from Eq. (20). The results of Figs. 7

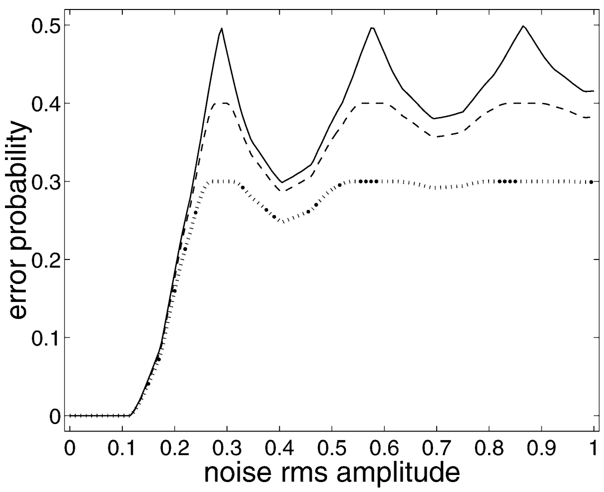

Fig. 3. Probability of detection error P_{er} from Eq. (19) of the optimal MAP detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$. Also $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 11$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{11} = 2$. Solid line: $P_0 = 0.5$, dashed line: $P_0 = 0.4$, dotted line: $P_0 = 0.3$.

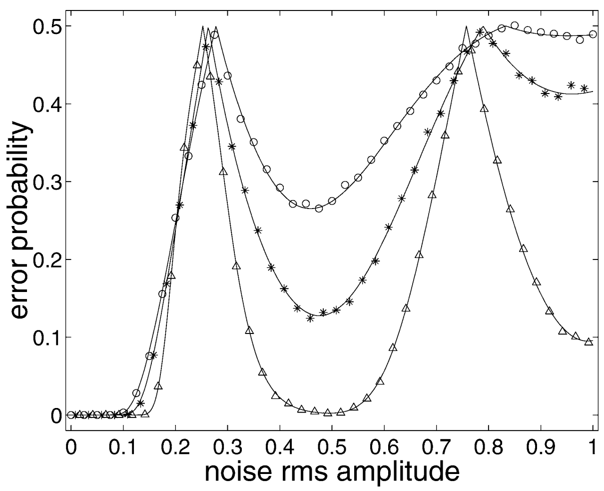

Fig. 4. Probability of detection error P_{er} of the optimal MAP detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$. The solid lines are P_{er} from Eq. (19); the discrete points are P_{er} numerically evaluated from 10^4 Monte Carlo trials of the MAP test (4) for each σ_η; with $m = 0.9$ ($\circ$), $m = 0.95$ ($*$), $m = 0.99$ ($\triangle$). Also $P_1 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 6$ data samples equispaced with time step 0.3 from $t_1 = 0$ to $t_6 = 1.5$.

and 8 establish the possibility of decreasing the cost C_{minimax}, by increasing the level of noise σ_η, over some ranges. This again reveals the same type of nonmonotonic evolution of the performance of the optimal detector when the noise level is raised, instead of a monotonic degradation.

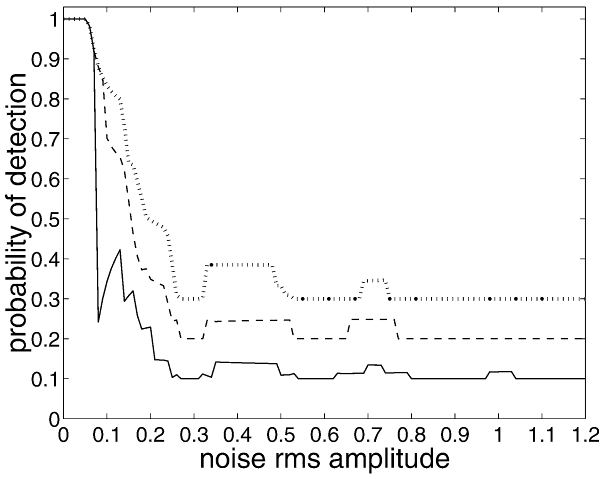

Fig. 5. Probability of detection P_d of Eq. (5) of the optimal Neyman–Pearson detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, with $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 3$ data samples measured at $t_1 = 0.1$, $t_2 = 0.4$, and $t_3 = 0.6$. Also $P_{f,sup} = 0.1$ (solid line), $P_{f,sup} = 0.2$ (dashed line), $P_{f,sup} = 0.3$ (dotted line).

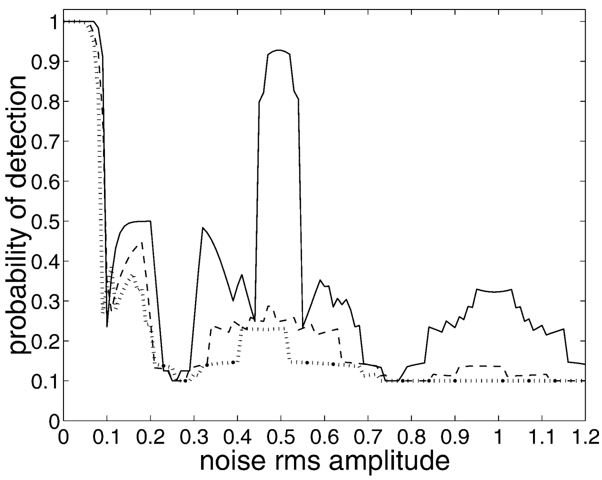

Fig. 6. Probability of detection P_d of Eq. (5) of the optimal Neyman–Pearson detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with $m = 0.9$ (dotted line), $m = 0.95$ (dashed line), $m = 0.99$ (solid line). Also $P_{f,sup} = 0.1$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 3$ data samples measured at $t_1 = 0.1$, $t_2 = 0.4$, and $t_3 = 0.6$.

5. Discussion

The present results essentially stand for a proof of feasibility in principle, by direct examination, of a stochastic resonance effect under the form of a noise-improved performance in a whole range of standard optimal detection strategies. Detailed analyses of the

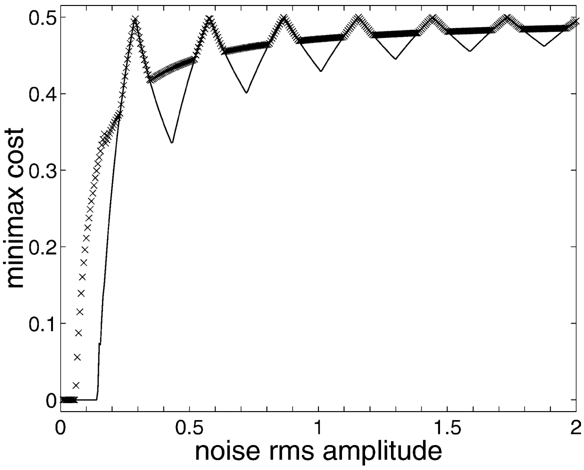

Fig. 7. Minimax cost C_{minimax} of Eq. (8) of the optimal minimax detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$. Also $C_{00} = C_{11} = 0$, $C_{01} = C_{10} = 1$. Solid line: $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 6$ data samples equispaced with time step 0.25 from $t_1 = 0$ to $t_6 = 1.5$. Crosses ($\times$): $\nu_0 = 1$, $\nu_1 = 1/3$, with $N = 4$ data samples equispaced with time step 0.6 from $t_1 = 0$ to $t_4 = 1.8$.

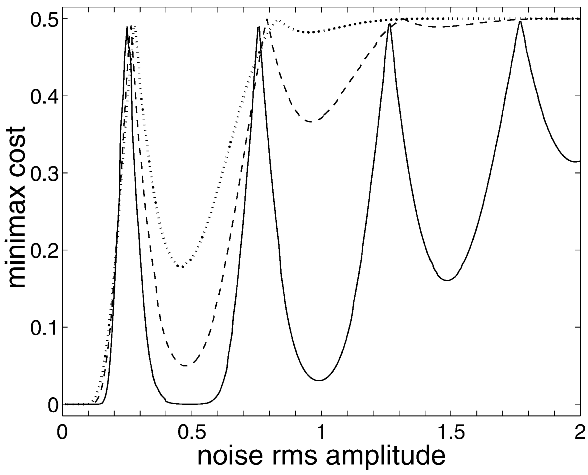

Fig. 8. Minimax cost C_{minimax} of Eq. (8) of the optimal minimax detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with $m = 0.9$ (dotted line), $m = 0.95$ (dashed line), $m = 0.99$ (solid line). Also $C_{00} = C_{11} = 0$, $C_{01} = C_{10} = 1$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 11$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{11} = 2$.

influence of the distinct parameters playing a role in the detection process are not accomplished in the present study, and remain open for subsequent investigations.

Among the important parameters is the waveform of the periodic input the frequency of which we seek to detect. We have chosen here a square wave, essentially because this case is simple enough to lend itself to a complete analytical treatment, which was at the same time verified by Monte Carlo simulations of the optimal detectors, as shown in Figs. 2 and 4. These conditions allowed our proof of feasibility in principle, with the double check of an analytical treatment and a numerical simulation. The case of detection on a sine wave, for instance, is a more complicated one, that remains open for future study.

It is to note that in all the optimal detection strategies we have tested, the performance is always at its best when the detectors are operated with no noise, at $\sigma_\eta = 0$. This is a common behavior which can reasonably be expected from any optimal detector, to achieve its absolute best performance when no noise is present. Then, when a small amount of noise is added above $\sigma_\eta = 0$, the performance of the optimal detectors naturally starts to degrade compared to the absolute best performance at $\sigma_\eta = 0$. Our main finding is that this evolution of the performance of the optimal detectors does not continue as a monotonic degradation: at higher levels of noise, ranges of σ_η exist where the performance improves, at least locally, when the noise is further raised. A pre-existing amount of noise has to be present, in order to have access to improvement by noise via the stochastic resonance effect. But a pre-existing noise is usually the rule in a detection problem, compared to the situation of no noise.

A qualitative explanation can be proposed for the observed effect of improvement by noise in optimal detection. For the observable signal-noise mixture $x(t)$ of Eqs. (9)–(10), let us assume that the corrupting phase noise $\eta(t)$ is a binary or dichotomous noise, randomly switching between the two levels $-\sigma_\eta$ and $+\sigma_\eta$. When $\sigma_\eta = 1$, because of the periodicity 1 of the wave $w(t)$, the binary phase noise $\eta(t)$ has actually no action on the signal $x(t)$, which is preserved exactly as if no noise were present. Therefore, at the noise level $\sigma_\eta = 1$, the performance of any detector will return to the performance of this detector at $\sigma_\eta = 0$. It can be expected that the performance will first degrade as σ_η is raised above zero, but later on, the performance will improve as σ_η approaches one, to return to its initial value at $\sigma_\eta = 0$. A similar effect can be expected when the binary phase noise $\eta(t)$ switches between $\pm\sigma_\eta = \pm 0.5$, or any integer multiple of such configurations. The Gaussian-mixture noise from Eq. (20) that we have tested in Figs. 2, 4, 6, and 8, precisely tends to the binary noise at $\pm\sigma_\eta$ when $m \to 1$, and it reflects this qualitative behavior we have explained for the binary noise. In this respect, our present report can be seen as a quantitative analysis, extended to any type of noise, of the qualitative behavior anticipated for the binary noise. The important and nonobvious point is that the behavior anticipated for the binary noise and which leads to a nonmonotonic evolution of the performance, is in actuality preserved for non-binary noises, noises with a bimodal structure like the Gaussian-mixture noise from Eq. (20), but also unimodal noises like the uniform noise tested in Figs. 1, 3, 5, and 7. The nonmonotonic behavior of the performance is essentially caused by the action of the nonlinearity itself, as in Eqs. (9)–(10), which mixes the information signal and the noise, through a phase coupling which introduces a sensitivity to the amplitude of the noise in relation to the periodicity[2] of $w(t)$. The noise, somehow,

[2] It is to be noted that the level of the phase noise $\eta(t)$ ought not be measured modulo 1. The noise level is given, in principle at an arbitrary value, prior to the nonlinear coupling with the information signal through Eqs. (9)–(10). It is for instance the arbitrary level at which the transducer evoked at the end of the paragraph after Eq. (10), is randomly shaken. Subsequently, it is the action of the nonlinear coupling via Eqs. (9)–(10), which brings the (arbitrary) amplitude of the noise to have an action on the phase between 0 and 1.

is able to bring some phase shift which places the nonlinear coupling in a better configuration for the detection task, by making the waves at ν_0 and at ν_1 more distinguishable. And this mechanism can take place for various types of noise, not necessarily binary noise, not necessarily bimodal noise, especially unimodal noise. The outcome, which is the main point of this report, is the possibility of (local) improvement of the performance of *optimal* detectors as the noise level increases.

In the tested configurations, with a square wave $w(t)$, the stochastic resonance effect was not observed with Gaussian noise ($m = 0$), but it appears with Gaussian-mixture noise, and in a more pronounced way as we depart more from the Gaussian, with $m \to 1$ (evolution to binary noise), and also with uniform noise. The uniform noise is a generalized Gaussian noise [14] with exponent $+\infty$, while the standard Gaussian noise corresponds to an exponent of 2. We have observed, with a few other instances of generalized Gaussian noises, that the stochastic resonance effect tends to vanish smoothly (the nonmonotonic evolution of the performance gradually gives way to a monotonic degradation) as the exponent tends to 2 from above. The effect disappears for an exponent of 2 or below. Further studies will be useful for a more detailed appreciation of the influence of the type of the noise on the stochastic resonance. Many other settings and conditions have been reported where a form of stochastic resonance (in suboptimal processing) takes place with Gaussian noise [1–3,11,23]. In the future, conditions may well be found allowing stochastic resonance in optimal processing with Gaussian noise. This issue remains open for future investigation. In this perspective, studies are currently under way to investigate the possibility of extending stochastic resonance to optimal estimation [24].

Important nonlinear processes that have been shown to lend themselves to many interesting forms of stochastic resonance are the neural processes [4,13,25–31]. Neurons are intrinsically nonlinear devices, they naturally operate in noisy conditions (of external or internal origins), and they achieve high efficiency for signal and information processing. In addition, neurons handle the signals essentially under the form of trains of stereotyped pulses (action potentials) mostly invariant in shape, but with their coding capability supported by the temporal sequencing of the pulses along the trains. This can be viewed as a coding in phase, and the conditions of the present study, with phase coupling and phase noise, could carry some relevance for neural information processing, this perspective also remaining open for detailed investigation.

The present results contribute a new step in the inventory and analysis of the properties and potentialities of stochastic resonance, at large, understood as an effect of noise-improved information processing. Beyond the present proof of feasibility in principle, of a stochastic resonance in optimal detection strategies, further studies can proceed in many directions to extend our knowledge of noise-enhanced information processing.

References

[1] S. Mitaim, B. Kosko, Adaptive stochastic resonance, Proc. IEEE 86 (1998) 2152–2183.

[2] B. Andò, S. Graziani, Stochastic Resonance: Theory and Applications, Kluwer Academic, Boston, 2000.

[3] G.P. Harmer, B.R. Davis, D. Abbott, A review of stochastic resonance: Circuits and measurement, IEEE Trans. Instrum. Meas. 51 (2002) 299–309.

[4] A. Bulsara, E.W. Jacobs, T. Zhou, F. Moss, L. Kiss, Stochastic resonance in a single neuron model: Theory and analog simulation, J. Theor. Biol. 152 (1991) 531–555.

[5] X. Godivier, F. Chapeau-Blondeau, Noise-assisted signal transmission in a nonlinear electronic comparator: Experiment and theory, Signal Process. 56 (1997) 293–303.

[6] F. Vaudelle, J. Gazengel, G. Rivoire, X. Godivier, F. Chapeau-Blondeau, Stochastic resonance and noise-enhanced transmission of spatial signals in optics: The case of scattering, J. Opt. Soc. Amer. B 15 (1998) 2674–2680.

[7] S. Zozor, P.O. Amblard, Stochastic resonance in discrete time nonlinear AR(1) models, IEEE Trans. Signal Process. 47 (1999) 108–122.

[8] D.G. Luchinsky, R. Mannella, P.V.E. McClintock, N.G. Stocks, Stochastic resonance in electrical circuits. II: Nonconventional stochastic resonance, IEEE Trans. Circuits Syst. II 46 (1999) 1215–1224.

[9] F. Chapeau-Blondeau, Noise-assisted propagation over a nonlinear line of threshold elements, Electron. Lett. 35 (1999) 1055–1056.

[10] M.E. Inchiosa, A.R. Bulsara, Signal detection statistics of stochastic resonators, Phys. Rev. E 53 (1996) R2021–R2024.

[11] V. Galdi, V. Pierro, I.M. Pinto, Evaluation of stochastic-resonance-based detectors of weak harmonic signals in additive white Gaussian noise, Phys. Rev. E 57 (1998) 6470–6479.

[12] F. Chapeau-Blondeau, Stochastic resonance and optimal detection of pulse trains by threshold devices, Digital Signal Process. 9 (1999) 162–177.

[13] J. Tougaard, Stochastic resonance and signal detection in an energy detector—implications for biological receptor systems, Biol. Cybern. 83 (2000) 471–480.

[14] F. Chapeau-Blondeau, Nonlinear test statistic to improve signal detection in non-Gaussian noise, IEEE Signal Process. Lett. 7 (2000) 205–207.

[15] S. Kay, Can detectability be improved by adding noise? IEEE Signal Process. Lett. 7 (2000) 8–10.

[16] S. Zozor, P.O. Amblard, On the use of stochastic resonance in sine detection, Signal Process. 82 (2002) 353–367.

[17] J. Tougaard, Signal detection theory, detectability and stochastic resonance effects, Biol. Cybern. 87 (2002) 79–90.

[18] A.A. Saha, G.V. Anand, Design of detectors based on stochastic resonance, Signal Process. 83 (2003) 1193–1212.

[19] F. Chapeau-Blondeau, D. Rousseau, Noise improvements in stochastic resonance: From signal amplification to optimal detection, Fluct. Noise Lett. 2 (2002) L221–L233.

[20] F. Chapeau-Blondeau, Stochastic resonance for an optimal detector with phase noise, Signal Process. 83 (2003) 665–670.

[21] R.N. McDonough, A.D. Whalen, Detection of Signals in Noise, Academic Press, New York, 1995.

[22] S.M. Kay, Fundamentals of Statistical Signal Processing: Detection Theory, Prentice Hall International, Englewood Cliffs, NJ, 1998.

[23] M.A. Fuentes, R. Toral, H.S. Wio, Enhancement of stochastic resonance: The role of non Gaussian noise, Physica A 295 (2001) 114–122.

[24] F. Chapeau-Blondeau, D. Rousseau, Noise-enhanced performance for an optimal Bayesian estimator, IEEE Trans. Signal Process. 52 (2004) 1327–1334.

[25] A. Longtin, Stochastic resonance in neuron models, J. Statist. Phys. 70 (1993) 309–327.

[26] X. Pei, K. Bachmann, F. Moss, The detection threshold, noise and stochastic resonance in the Fitzhugh-Nagumo neuron model, Phys. Lett. A 206 (1995) 61–65.

[27] F. Chapeau-Blondeau, X. Godivier, N. Chambet, Stochastic resonance in a neuron model that transmits spike trains, Phys. Rev. E 53 (1996) 1273–1275.

[28] B.J. Gluckman, T.I. Netoff, E.J. Neel, W.L. Ditto, M.L. Spano, S.J. Schiff, Stochastic resonance in a neuronal network from mammalian brain, Phys. Rev. Lett. 77 (1996) 4098–4101.

[29] X. Godivier, F. Chapeau-Blondeau, Noise-enhanced transmission of spike trains in the neuron, Europhys. Lett. 35 (1996) 473–477.

[30] G. Deco, B. Schürmann, Stochastic resonance in the mutual information between input and output spike trains of noisy central neurons, Physica D 117 (1998) 276–282.

[31] N.G. Stocks, R. Mannella, Generic noise-enhanced coding in neuronal arrays, Phys. Rev. E 64 (2001) 1–4, 030902.

David Rousseau was born in 1973 in Le Mans, France. He obtained a Master degree in acoustics and signal processing at the Institut de Recherche Coordination Acoustique et Musique (IRCAM), Paris, France, in 1996. He is currently a Professeur agrégé in physics at the University of Angers, France. He is also preparing a PhD in nonlinear signal processing and stochastic resonance, at the Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), University of Angers.

François Chapeau-Blondeau was born in France in 1959. He received the Engineer Diploma from ESEO, Angers, France, in 1982, the PhD degree in electrical engineering from University Paris 6, Paris, France, in 1987, and the Habilitation degree from the University of Angers, France, in 1994. In 1988, he was a research associate in the Department of Biophysics at the Mayo Clinic, Rochester, Minnesota, USA, working on biomedical ultrasonics. Since 1990, he has been with the University of Angers, France, where he is currently a professor of electronic and information sciences. His research interests include nonlinear systems and signal processing, and the interface between physics and information sciences.