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Digital Signal Processing 18 (2008) 434–443

Digital Signal Processing

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Fisher information and noise-aided power estimation from one-bit quantizers

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Available online 13 May 2007

Abstract

We investigate power estimation on a random noise from measurements taken by one-bit quantizers, with an efficacy assessed by the Fisher information. In isolated quantizers, an optimal tuning of the quantization threshold exists to maximize the estimation efficacy. When the quantizers are assembled in parallel arrays, no specific tuning of the quantization threshold is any longer required. Instead, addition of noise in the array can be employed as a means of enhancing the estimation efficacy. This is interpreted as a form of stochastic resonance or improvement by noise, applied to parametric estimation on a noise, which is shown improvable by adding more noise.

Keywords: Power estimation; Fisher information; Quantizer; Stochastic resonance; Noise

1. Introduction

For signal and information processing, it is progressively realized that noise is not necessarily a nuisance, but can sometimes play a beneficial role. This possibility of a constructive action of noise can be identified under the generic term of "stochastic resonance." Various forms of stochastic resonance or improvement by noise have been reported for signal processing, mainly in nonlinear systems or processes. Stochastic resonance effects have been demonstrated with the amplification by noise of the signal amplitude, or the signal-to-noise ratio, at the output of various nonlinear systems [1–3]. Improvements by noise have also been obtained in nonlinear detection [4–8], and in nonlinear estimation [9].

When isolated nonlinear systems are assembled into parallel arrays, it has been shown that new mechanisms of improvement by noise can come into play [10–12]. Arrays of threshold comparators or one-bit quantizers are among the simplest arrays of this type. With these arrays of comparators, enhancement by noise has been reported for the input–output mutual information [11], for the input–output signal correlation [13], and for the output signal-to-noise ratio [14].

For estimation purposes as we address in the present paper, arrays of one-bit quantizers have been studied in [15]. In the study of [15], parametric estimation is considered on a deterministic input signal contaminated by some input noise. When measurements for estimation are performed by an array of quantizers, it is shown in [15] that a nonzero

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doi:10.1016/j.dsp.2007.04.012

amount of added noise injected into the array is able to improve the performance for estimation on the deterministic input signal. The estimation performance in [15] is assessed by the Fisher information, and an enhancement of this Fisher information is shown possible thanks to the action of noise added into the array. Another type of nonlinear arrays, made with saturating devices, has also been studied recently by [16] in this context of array stochastic resonance and noise-aided estimation. In [16] also, parametric estimation is considered on a deterministic input signal, with a performance assessed by the Fisher information.

In stochastic resonance effects assessed with the Fisher information, essentially a deterministic signal has been considered for estimation, be it in arrays as in the recent studies of [15,16] or in isolated nonlinearities as in the earlier studies of [17–20]; and often the examples are worked out for a constant deterministic signal. This is consistent with a classic picture for stochastic resonance, where an information-carrying signal coupled to a noise benefits from the presence (or increase) of the noise. By contrast here, we consider estimation of a statistical parameter (power) of a noise. There is no deterministic signal. This somehow enlarges the scope of stochastic resonance to estimation on a noise, which is shown improvable by adding more noise. We study in the context of stochastic resonance, how to handle power estimation on an input noise, when the measurements are taken by one-bit quantizers. These very common nonlinear devices offer a very parsimonious signal representation. This can be specially relevant for fast real-time processing, or for systems with limited hardware resources or energy supply. We demonstrate a constructive impact of added noise, for power estimation using one-bit quantizers assembled in arrays.

2. Estimation from a single quantizer

2.1. Output Fisher information

A random signal $x(t)$ has probability density function $f_x(u)$ and power or mean-squared value $E[x^2(t)] = \sigma^2$. Without loss of generality, we consider throughout that signal $x(t)$ is zero mean. This signal $x(t)$ is observed by means of a one-bit quantizer with threshold θ delivering the binary signal

$$y(t) = \Gamma\big[x(t) - \theta\big] = 0 \quad \text{or } 1, \tag{1}$$

with the Heaviside function $\Gamma(u) = 0$ for $u \leqslant 0$ and $\Gamma(u) = 1$ otherwise. From the quantizer output $y(t)$, one wants to estimate the noise power σ^2. To this aim, at M distinct times t_j one collects M observations $y(t_j) = y_j$, for $j = 1$ to M. For estimation purposes from the data $(y_1, \dots, y_M) = \boldsymbol{y}$, a fundamental quantity is the Fisher information $J(\boldsymbol{y}, \sigma^2)$ which provides a measure of the information contained in the data $\boldsymbol{y}$ about the unknown parameter σ^2 to be estimated [21]. $J(\boldsymbol{y}, \sigma^2)$ serves to define a bound fixing a limit to the performance of any conceivable estimator of σ^2 from $\boldsymbol{y}$: the mean squared estimation error is always lower bounded by a bound (the Cramér-Rao lower bound) directly related to the reciprocal Fisher information. Moreover, the well known maximum likelihood estimator, in the asymptotic regime of a large number of measurements M, achieves estimation with minimum mean squared error. And this minimum, fixing the best achievable performance, is exactly given by the reciprocal Fisher information. From the probability $\Pr\{\boldsymbol{y}\}$ of a given data vector $\boldsymbol{y}$, Fisher information $J(\boldsymbol{y}, \sigma^2)$ is defined as [21]

$$J(\boldsymbol{y}, \sigma^2) = \sum_{\boldsymbol{y}} \frac{1}{\Pr\{\boldsymbol{y}\}} \left[\frac{\partial}{\partial \sigma^2} \Pr\{\boldsymbol{y}\}\right]^2, \tag{2}$$

where the sum runs over the 2^M feasible configurations of the data vector $\boldsymbol{y}$.

We further assume a common condition in metrology where the data y_j are mutually statistically independent; this is achieved by choosing the observation times t_j sufficiently separated in relation to the correlation duration of $x(t)$. In this common case of independent measurements, Fisher information is additive and yields

$$J\big(\boldsymbol{y}, \sigma^2\big) = \sum_{j=1}^{M} J\big(y_j, \sigma^2\big), \tag{3}$$

where

$$J\big(y_j, \sigma^2\big) = \sum_{y_j \in \{0,1\}} \frac{1}{\Pr\{y_j\}} \left[\frac{\partial}{\partial \sigma^2} \Pr\{y_j\}\right]^2 \tag{4}$$

is the Fisher information contained in measurement y_j about σ^2. Moreover, when the measurements y_j are identically distributed, $J(y_j,\sigma^2)$ is the same for every j and $J(\boldsymbol{y},\sigma^2)=MJ(y_j,\sigma^2)$.

From the observed binary signal of Eq. (1), one has

$$\Pr\{y_j=0\}=\Pr\{x(t_j)\leqslant\theta\}=F_x(\theta), \tag{5}$$

and $\Pr\{y_j=1\}=1-\Pr\{y_j=0\}=1-F_x(\theta)$, where $F_x(u)=\int_{-\infty}^{u}f_x(v)\,\mathrm{d}v$ is the cumulative distribution function of the random signal $x(t)$. In general for a zero-mean signal $x(t)$ with power σ^2, one always has for the density $f_x(u)=f_{\text{stand}}(u/\sigma)/\sigma$ and for the cumulative distribution $F_x(u)=F_{\text{stand}}(u/\sigma)$ with the σ-independent standardized functions $f_{\text{stand}}(u)$ and $F_{\text{stand}}(u)$ of zero mean and unit power. This allows one to express the derivative

$$\frac{\partial}{\partial\sigma^2}F_x(u)=-\frac{u}{2\sigma^2}f_x(u), \tag{6}$$

leading for Eq. (4), via Eq. (5), to

$$J(y_j,\sigma^2)=\frac{f_x^2(\theta)}{[1-F_x(\theta)]F_x(\theta)}\frac{\theta^2}{4\sigma^4}. \tag{7}$$

Equation (7) conveys important properties on the possibility of estimating σ^2 from the y_j's. In particular, from Eq. (7) one finds that in general $J(y_j,\sigma^2)=0$ when $\theta=0$. This means that there is no possibility of estimating the power σ^2 when the zero-mean signal $x(t)$ is quantized at the threshold $\theta=0$. In many situations, $\theta=0$ is the optimal threshold for the quantization of a zero-mean signal $x(t)$, allowing for instance to minimize the mean-squared distortion between $x(t)$ and its quantized version $y(t)$. Yet, for power estimation, $\theta=0$ is a completely inefficient setting of the quantizer, making the quantized signal $y(t)$ flip between 0 and 1 in a way completely independent of the power σ^2.

In addition Fisher information $J(y_j,\sigma^2)$ of Eq. (7) can also be expressed as

$$J(y_j,\sigma^2)=\frac{1}{4\sigma^4}\frac{f_{\text{stand}}^2(\theta/\sigma)}{[1-F_{\text{stand}}(\theta/\sigma)]F_{\text{stand}}(\theta/\sigma)}\left(\frac{\theta}{\sigma}\right)^2=\frac{1}{\sigma^4}\rho(\theta/\sigma). \tag{8}$$

Beyond the case $J(y_j,\sigma^2)=0$ at $\theta=0$, function $\rho(\theta/\sigma)$ defined by Eq. (8) expresses that the dependence of $J(y_j,\sigma^2)$ with θ is universal in θ/σ, materializing a scale invariance of the process with respect to signal amplitudes, which is conveyed to the performance in estimation. When the quantization threshold θ varies in $]-\infty,\infty[$, in Eq. (8) function $\rho(\theta/\sigma)$ remains positive, and for any $\sigma>0$ in general $\rho(\theta=0)=0$ and $\rho(\theta\to\infty)=0$. Then, depending on the specific shape of the probability density of $x(t)$, $\rho(\theta/\sigma)$ is expected to reach a maximum $\rho_{\max}>0$ at a universal reduced threshold $u_{\text{opt}}=\theta_{\text{opt}}/\sigma$, with $\rho_{\max}$ and u_{opt} independent of σ, in such a way that at any given σ, Fisher information $J(y_j,\sigma^2)$ is maximized for a quantization threshold $\theta_{\text{opt}}=u_{\text{opt}}\sigma$ where it culminates at $\rho_{\max}/\sigma^4$. This establishes the optimal setting of the quantization threshold θ to maximize the performance in the power estimation.

For illustration, Fig. 1 presents the evolution of Fisher information $J(y_j,\sigma^2)$ from Eq. (8), as a function of the quantization threshold θ, in two cases, when $x(t)$ is Gaussian with density

$$f_x(u)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{u^2}{2\sigma^2}\right) \tag{9}$$

and cumulative distribution

$$F_x(u)=\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{u}{\sqrt{2}\sigma}\right)\right], \tag{10}$$

and when $x(t)$ is Laplacian with density

$$f_x(u)=\frac{1}{\sigma\sqrt{2}}\exp\left(-\sqrt{2}\frac{|u|}{\sigma}\right) \tag{11}$$

and cumulative distribution

$$F_x(u)=\begin{cases}\dfrac{1}{2}\exp\left(-\sqrt{2}\dfrac{|u|}{\sigma}\right) & \text{for } u\leqslant 0,\\[2ex] 1-\dfrac{1}{2}\exp\left(-\sqrt{2}\dfrac{|u|}{\sigma}\right) & \text{for } u\geqslant 0.\end{cases} \tag{12}$$

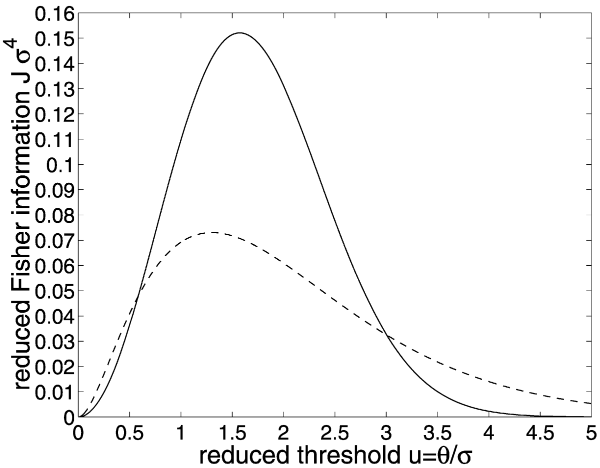

Fig. 1. Reduced Fisher information $J(y_j,\sigma^2)\times\sigma^4=\rho(\theta/\sigma)$ from Eq. (8), as a function of the reduced quantization threshold $u=\theta/\sigma$, when the random signal $x(t)$ is Gaussian (solid line), or Laplacian (dashed line).

The evolutions of Fig. 1 show, as announced, that Fisher information $J(y_j,\sigma^2)$ is always zero when the quantization threshold is $\theta=0$, and that $J(y_j,\sigma^2)$ culminates at a maximum $\rho_{\max}/\sigma^4$ at an optimal threshold $\theta_{\mathrm{opt}}=u_{\mathrm{opt}}\sigma$. For the Gaussian case one finds $(u_{\mathrm{opt}},\rho_{\max})\approx(1.575, 0.1521)$, and for the Laplacian case $(u_{\mathrm{opt}},\rho_{\max})\approx(1.302, 0.0730)$. The superior value of the Fisher information means that estimation of the power σ^2 from the quantized signal $y(t)$ will be more efficient (more accurate for a fixed number of data points) for a Gaussian signal $x(t)$ than for a Laplacian signal $x(t)$.

This analysis of Fisher information $J(y_j,\sigma^2)$ reveals a practical difficulty that the optimal quantization threshold $\theta_{\mathrm{opt}}=u_{\mathrm{opt}}\sigma$ for estimation is dependent upon the unknown parameter σ one is seeking to estimate. This difficulty can be resolved if one is able to a priori specify an initial range over which the power σ^2 or standard deviation σ can possibly take its value. In a Bayesian framework, one introduces a prior probability density $p_0(u)$ describing these possible values. One can then consider the average Fisher information over the prior $p_0(u)$, i.e.

$$\bar{J}(y_j)=\int_0^\infty J\big(y_j,\sigma^2\big)p_0(\sigma)\,\mathrm{d}\sigma. \tag{13}$$

Then selecting θ that maximizes $\bar{J}(y_j)$ from Eqs. (13) and (8) will ensure good performance on average, with a fixed quantizer for estimating many power values drawn from the prior $p_0(u)$. The performance of this fixed quantizer will certainly be made strictly larger than that of the inoperative quantizer tuned at $\theta=0$. At the same time, the performance of Eq. (13), as an average value, will be somewhat less than the maximum performance deduced from Eq. (8), depending on the prior density. For illustration, Fig. 2 shows the average Fisher information $\bar{J}(y_j)$ from Eq. (13), for estimation of the power σ^2 on a Gaussian and on a Laplacian signal $x(t)$, with a prior $p_0(\sigma)$ uniform over [0.5, 1.5]. Fisher information $\bar{J}(y_j)$ in Fig. 2, from the location of its maximum, suggests to place the quantization threshold at $\theta=0.97$ for Gaussian signals, and at $\theta=0.83$ for Laplacian signals, for optimal average performance in power estimation.

2.2. Input–output efficacy of Fisher information

For further appreciation of the efficacy of estimation from the quantized signal $y(t)$, a useful reference is to compare $J(y_j,\sigma^2)$ to $J(x_j,\sigma^2)$, where $J(x_j,\sigma^2)$ is the Fisher information about the power σ^2 contained in a sample $x_j=x(t_j)$ which would be directly observed from the analog signal $x(t)$ instead of its quantized version $y(t)$. This Fisher information $J(x_j,\sigma^2)$ is computable as [21]

$$J\big(x_j,\sigma^2\big)=\int_{-\infty}^{\infty}\frac{1}{f_x(u)}\left[\frac{\partial}{\partial\sigma^2}f_x(u)\right]^2\mathrm{d}u, \tag{14}$$

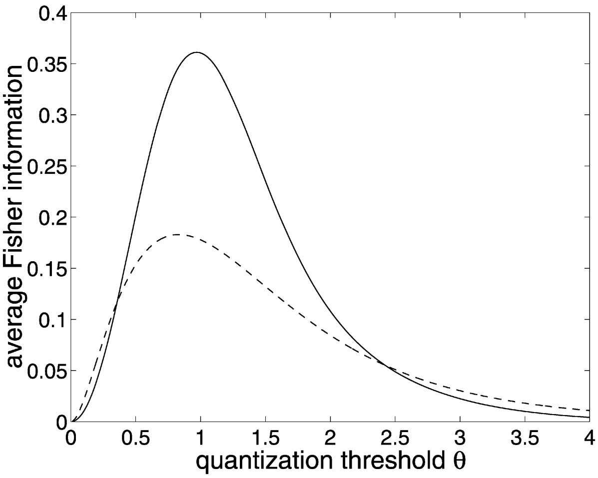

Fig. 2. Average Fisher information $\bar{J}(y_j)$ from Eq. (13), as a function of the quantization threshold θ, when the random signal $x(t)$ is Gaussian (solid line), or Laplacian (dashed line), with a prior $p_0(\sigma)$ uniform over [0.5, 1.5].

or equivalently

$$J\left(x_j,\sigma^2\right)=\frac{1}{4\sigma^4}\int_{-\infty}^{\infty}\frac{1}{f_{\mathrm{stand}}(u)}\left[f_{\mathrm{stand}}(u)+uf'_{\mathrm{stand}}(u)\right]^2\mathrm{d}u=\frac{1}{\sigma^4}\rho_{\mathrm{in}},\tag{15}$$

where ρ_{in} defined by Eq. (15) is a constant, independent of the power σ^2, and only function of the standardized shape of the probability density of input signal $x(t)$.

With a Gaussian $x(t)$, from Eq. (9) one finds $J(x_j,\sigma^2)=1/(2\sigma^4)=0.5/\sigma^4$ to be compared with $J(y_j,\sigma^2)\approx 0.1521/\sigma^4$ after quantization at the optimal threshold θ_{opt}. And with a Laplacian $x(t)$, from Eq. (11) one finds $J(x_j,\sigma^2)=1/(4\sigma^4)=0.25/\sigma^4$ to be compared with $J(y_j,\sigma^2)\approx 0.0730/\sigma^4$ after quantization at the optimal threshold. The superior value of $J(x_j,\sigma^2)$ shows that power estimation directly from the analog signal $x(t)$ (with a fixed number of data points) is also more efficient for a Gaussian $x(t)$ than for a Laplacian $x(t)$. The ratio $J(y_j,\sigma^2)/J(x_j,\sigma^2)$ provides a measure of the loss of efficiency entailed by the use of the quantized signal $y(t)$ in place of the analog signal $x(t)$ for estimation of the power σ^2. The most favorable condition is at quantization with the optimal threshold, which sets the maximum value of this ratio at $\rho_{\max}(u_{\mathrm{opt}})/\rho_{\mathrm{in}}$. This maximum ratio is independent of the power σ^2, and only function of the standardized shape of the probability density of input signal $x(t)$. One finds for the Gaussian case $\rho_{\max}(u_{\mathrm{opt}})/\rho_{\mathrm{in}}\approx 0.304$, and for the Laplacian case $\rho_{\max}(u_{\mathrm{opt}})/\rho_{\mathrm{in}}\approx 0.292$. The slightly superior value of the ratio means that there will also be less loss of efficacy for power estimation on a Gaussian signal $x(t)$ than on a Laplacian signal $x(t)$.

3. Estimation from an array of quantizers

3.1. Array Fisher information

We now investigate for power estimation, the capabilities of parallel arrays of one-bit quantizers in the framework of stochastic resonance. We consider N identical quantizers as in Eq. (1) with a common quantization threshold θ, assembled into an uncoupled parallel array as in [11,15]. Each quantizer i receives the same input signal $x(t)$ with in addition the possibility of an independent noise input $\eta_i(t)$. So quantizer i delivers the output binary signal $y_i(t)$ according to

$$y_i(t)=\Gamma\left[x(t)+\eta_i(t)-\theta\right]=0 \text{ or } 1,\tag{16}$$

for $i=1$ to N. The N added noises $\eta_i(t)$ are white, independent of $x(t)$, mutually independent and identically distributed with cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u)=\mathrm{d}F_\eta(u)/\mathrm{d}u$. The N output signals $y_i(t)$ are then averaged to produce the array output $Y(t)$ as

$$Y(t)=\frac{1}{N}\sum_{i=1}^{N} y_i(t). \tag{17}$$

As before, at M distinct times t_j one collects M observations $Y(t_j)=Y_j$, and from the data $(Y_1,\dots,Y_M)=\boldsymbol{Y}$ one seeks to estimate the power σ^2 of input signal $x(t)$. Estimation efficacy is assessed by the Fisher information $J(\boldsymbol{Y},\sigma^2)$ contained in the data $\boldsymbol{Y}$ about the unknown power σ^2. Since the observations Y_j are independent, one has $J(\boldsymbol{Y},\sigma^2)=\sum_{j=1}^{M} J(Y_j,\sigma^2)$. Any given observation Y_j can assume $N+1$ distinct values, namely $Y_j=0/N, 1/N, 2/N, \dots, N/N$ since $\sum_{i=1}^{N} y_i(t_j)$ in Eq. (17) counts the number of quantizer outputs at 1, ranging from 0 to N. It results that the Fisher information $J(Y_j,\sigma^2)$ contained in one measurement Y_j is

$$J\big(Y_j,\sigma^2\big)=\sum_{n=0}^{N}\frac{1}{\Pr\{Y_j=n/N\}}\left[\frac{\partial}{\partial\sigma^2}\Pr\left\{Y_j=\frac{n}{N}\right\}\right]^2. \tag{18}$$

For any given binary output $y_i(t)$, at a fixed value x for the input signal $x(t)$, one has from Eq. (16) the conditional probability

$$\Pr\big\{y_i(t)=0|x\big\}=\Pr\big\{x+\eta_i(t)\leqslant\theta\big\}=F_\eta(\theta-x), \tag{19}$$

and $\Pr\{y_i(t)=1|x\}=1-\Pr\{y_i(t)=0|x\}=1-F_\eta(\theta-x)$. From the binomial distribution [22], we deduce for the array output Y_j the conditional probability

$$\Pr\{Y_j=n/N|x\}=C_n^N\big[1-F_\eta(\theta-x)\big]^n F_\eta(\theta-x)^{N-n}, \tag{20}$$

where C_n^N is the binomial coefficient. Since x is distributed according to the density $f_x(x)$, we obtain the total probability

$$\Pr\left\{Y_j=\frac{n}{N}\right\}=\int_{-\infty}^{\infty} C_n^N\big[1-F_\eta(\theta-x)\big]^n F_\eta(\theta-x)^{N-n} f_x(x)\,\mathrm{d}x, \tag{21}$$

as also found in [15] for estimation on a deterministic input signal with similar arrays of quantizers. By contrast to [15], we seek here to estimate the power σ^2 of the random signal $x(t)$ with no deterministic component at the input, and so we now need to evaluate the derivative $\partial\Pr\{Y_j=n/N\}/\partial\sigma^2$. Through the use of the standardized density $f_{\mathrm{stand}}(\cdot)$, one deduces that

$$\frac{\partial}{\partial\sigma^2}f_x(x)=-\frac{1}{2\sigma^2}\big[f_x(x)+xf_x'(x)\big], \tag{22}$$

with $f_x'(x)=\mathrm{d}f_x(x)/\mathrm{d}x$, so that

$$\frac{\partial}{\partial\sigma^2}\Pr\left\{Y_j=\frac{n}{N}\right\}=\frac{-C_n^N}{2\sigma^2}\int_{-\infty}^{\infty}\big[1-F_\eta(\theta-x)\big]^n F_\eta(\theta-x)^{N-n}\big[f_x(x)+xf_x'(x)\big]\,\mathrm{d}x. \tag{23}$$

Equations (21) and (23) now provide access, possibly through numerical integration, to Fisher information $J(Y_j,\sigma^2)$ of Eq. (18).

For estimation with the array of quantizers, we can now study the impact of the added noises $\eta_i(t)$ in the framework of stochastic resonance. Figure 3 presents evolutions of Fisher information $J(Y_j,\sigma^2)$ of Eq. (18) in an array of fixed size $N=15$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$ chosen zero-mean Gaussian.

It is clearly visible in Fig. 3 that addition of the array noises $\eta_i(t)$ leads to an improvement of the Fisher information. With no added array noises $\eta_i(t)$, i.e. at $\sigma_\eta=0$ in Fig. 3, all the quantizers switch in unison as a single one, and Fisher information $J(Y_j,\sigma^2)$ is at the level corresponding to a single quantizer as shown in Fig. 1. Then, as the level σ_η of the array noises grows above zero in Fig. 3, Fisher information $J(Y_j,\sigma^2)$ starts to increase, to culminate at a

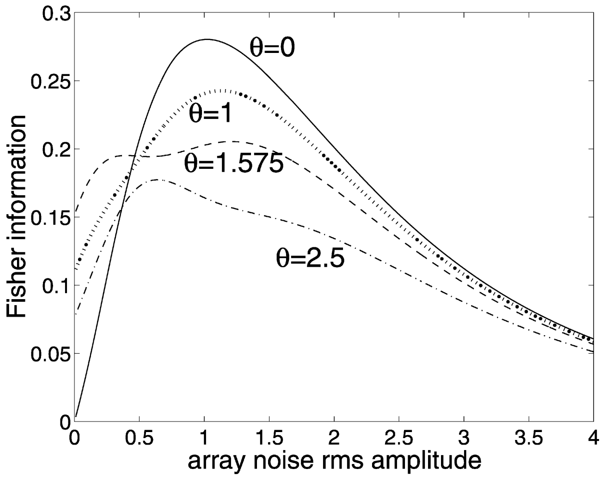

Fig. 3. Fisher information $J(Y_j,\sigma^2)$ of Eq. (18) in an array of size $N=15$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$ chosen zero-mean Gaussian, and for different values of the quantization threshold θ. The input signal $x(t)$ is Gaussian with rms amplitude $\sigma=1$.

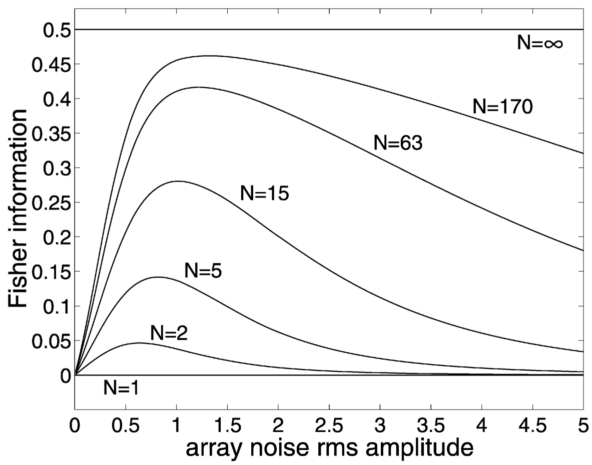

Fig. 4. Fisher information $J(Y_j,\sigma^2)$ of Eq. (18), as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$ chosen zero-mean Gaussian, for different array sizes N, with quantization threshold $\theta=0$. The input signal $x(t)$ is Gaussian with rms amplitude $\sigma=1$.

maximum level for a nonzero value of σ_η. Moreover, the highest maximum in Fig. 3 is obtained with the quantization threshold $\theta=0$. This is in contrast to the case of the single quantizer, as in Fig. 1, where Fisher information $J(y_j,\sigma^2)$ is strictly zero at $\theta=0$, and maximization of $J(y_j,\sigma^2)$ requires a specific tuning of the quantization threshold θ. For the Gaussian signal, the optimal threshold $\theta_{\text{opt}}=1.575$ deduced from Fig. 1 for an isolated quantizer, is also tested for the array in Fig. 3. It is visible in Fig. 3 that $\theta=1.575$ is no longer the optimal threshold for the array, where $\theta=0$ is the best threshold. In the array, for any θ, an improvement of the Fisher information is always possible thanks to the added noises $\eta_i(t)$ compared to the situation with no added noises $\eta_i(t)$, and the best improvement is achieved at threshold $\theta=0$.

Another point of view on the improvement by noise in the array is provided by Fig. 4, which presents Fisher information $J(Y_j,\sigma^2)$ at the quantization threshold $\theta=0$ deduced from Fig. 3, and for various values of the array size N.

Figure 4 shows that enhancement by noise of Fisher information $J(Y_j,\sigma^2)$ gets more and more pronounced as the size N of the array increases. For each array size $N>1$ in Fig. 4, there is an optimal level of the array noises $\eta_i(t)$ where Fisher information $J(Y_j,\sigma^2)$ is maximized. Also, the array structure, through the use of moderate array sizes N sufficiently above 1, always offers the possibility of achieving thanks to the added noises, a maximum $J(Y_j,\sigma^2)$ superior to the largest Fisher information $J(y_j,\sigma^2)$ achievable by a single quantizer tuned at its optimal threshold

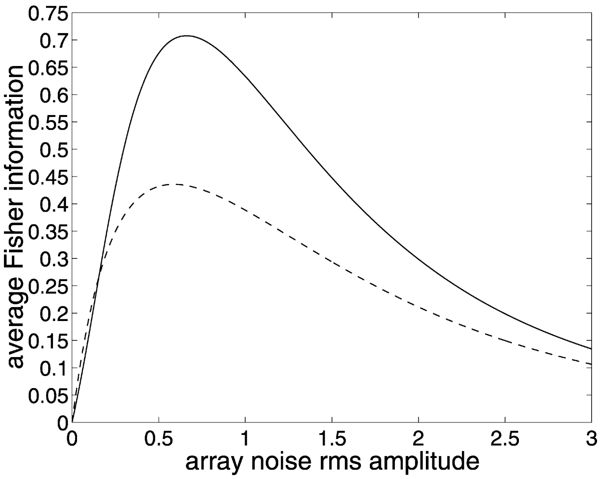

Fig. 5. Average Fisher information $\bar{J}(Y_j)$ from Eq. (24) in an array of size $N = 15$ and quantization threshold $\theta = 0$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$ chosen zero-mean Gaussian. Random signal $x(t)$ is Gaussian (solid line), or Laplacian (dashed line), with a prior $p_0(\sigma)$ uniform over [0.5, 1.5].

$\theta_{\text{opt}} \neq 0$. Therefore, compared to isolated quantizers, arrays with added noises bring two benefits: (i) no specific tuning other than $\theta = 0$ is needed for the quantization threshold; (ii) the maximum achievable Fisher information can always be enhanced. For a given configuration of the array, benefit (ii) is maximized with an optimal level of the added noises $\eta_i(t)$, as seen in Fig. 4. This optimal noise level, like the optimal threshold θ_{opt} of the isolated quantizer, is usually dependent upon the unknown parameter σ one is seeking to estimate. This difficulty is resolved, as before, in a Bayesian context where one specifies the feasible range for σ with a prior probability density $p_0(u)$. The level σ_η of the added array noises $\eta_i(t)$, can then be tuned to maximize the average array Fisher information

$$\bar{J}(Y_j) = \int_0^\infty J\big(Y_j, \sigma^2\big) p_0(\sigma)\, d\sigma, \tag{24}$$

to ensure good estimation performance on average. An illustration is presented in Fig. 5 for the array, which parallels the illustration of Fig. 2 for the isolated quantizer. Figure 5 shows the average Fisher information $\bar{J}(Y_j)$ from Eq. (24), for estimation of the power σ^2 on a Gaussian and on a Laplacian signal $x(t)$, with a prior $p_0(\sigma)$ uniform over [0.5, 1.5].

Fisher information $\bar{J}(Y_j)$ in Fig. 5, from the location of its maximum, suggests to adjust the rms amplitude of the added array noises at $\sigma_\eta = 0.66$ for Gaussian signals, and at $\sigma_\eta = 0.59$ for Laplacian signals, for optimal average performance in power estimation.

3.2. Input–output efficacy of array Fisher information

As seen in Section 2.2, for the Gaussian case of Fig. 4 the Fisher information contained in an analog measurement x_j at the input is $J(x_j, \sigma^2) = 1/(2\sigma^4) = 0.5/\sigma^4$. As the array size $N \to \infty$ in Fig. 4, the maximum Fisher information $J(Y_j, \sigma^2)$ from the quantized data tends to the complete Fisher information $J(x_j, \sigma^2)$ contained in the analog data. This is true not only for a Gaussian density of random signal $x(t)$ as in Fig. 4, but for any density of $x(t)$. The reason is that as $N \to \infty$, the array output of Eq. (17), by the law of large numbers, tends to exactly match the statistical expectation $E_\eta[y_i(t)]$, with departure vanishing as $1/\sqrt{N}$, where expectation $E_\eta[\cdot]$ is according to the density $f_\eta(u)$ of the array noises $\eta_i(t)$. At this limit $N \to \infty$, the array with added noises $\eta_i(t)$, is therefore turned into a deterministic device with input–output characteristic $Y(t) = E_\eta[y_i(t)] = 1 - F_\eta[\theta - x(t)]$. Since the cumulative distribution $F_\eta(u)$ is generally an increasing and consequently invertible function, the input–output characteristic $Y(t) = 1 - F_\eta[\theta - x(t)]$ is in general a smooth invertible characteristic, replacing the hard noninvertible characteristic $y(t) = \Gamma[x(t) - \theta]$ of Eq. (1). An invertible transformation leaves unchanged the Fisher information, leading to $J(Y_j, \sigma^2) \to J(x_j, \sigma^2)$ in large arrays as announced. The noninvertible threshold characteristic $y(t) = \Gamma[x(t) - \theta]$

of an isolated quantizer entails a significant loss of information, as discussed in Section 2.2. Meanwhile, the invertible characteristic $Y(t) = 1 - F_\eta[\theta - x(t)]$ realized by large arrays of quantizers with added noises, is able to recover the complete Fisher information contained in the input analog data. This outcome carries over to the average output Fisher $\bar{J}(Y_j)$ of Eq. (24), which in large arrays will tend to the average input Fisher information $\bar{J}(x_j)$ contained in the input analog data x_j.

4. Conclusion

We have addressed estimation of the power of a noise from measurements taken by one-bit quantizers offering a useful condensed representation of the data. With an isolated quantizer, we have shown that there is usually a need for a specific tuning of the quantization threshold, in order to allow an efficient estimation. There is also an inherent reduction of efficacy when, for estimation, the quantized data are used in place of the original analog data. This was measured here through the analysis of the Fisher information in the data. Next, we have shown that no specific tuning of the quantization threshold is needed when the quantizers are assembled in parallel arrays. Efficient estimation can be obtained, through a stochastic resonance effect, by injection of independent extra noises into the array. Maximum estimation efficacy is obtained at a nonzero level of added array noises. Furthermore, for estimation, arrays with added noises can always be made more efficient than isolated quantizers. Also, in large arrays of quantizers, the same estimation efficacy that would be afforded by the analog data is recovered.

These results, in some sense, extend the scope of stochastic resonance for noise-aided signal processing. The initial form of suprathreshold stochastic resonance in arrays similar to Eqs. (16)–(17), as introduced in [11,23], was concerned with a random input signal with no input noise. The measure of performance in [11,23] was an input–output mutual information assessing the similarity between the input information-carrying signal and the array output. For estimation tasks, stochastic resonance effects have been reported only for estimation on deterministic signals, and very often constant deterministic signals. By contrast here, stochastic resonance for estimation on a noise, with no deterministic signal, is examined for the first time. An intrinsic statistical parameter of the noise, its power, is estimated with enhanced efficacy when more noise is added. This demonstrates that enhancement by noise in estimation is a general possibility, accessible for parametric estimation on noise as well as on deterministic signals.

The present results can bear significance for sensors and measurement, data acquisition and processing, with low-cost devices such as quantizers, or smart arrays, or nanodevices with threshold all-or-nothing response. The results can also carry significance for neural signal processing. Many previous forms of stochastic resonance have been shown relevant in this area [24–29]. Threshold nonlinearities and parallel arrays with internal noise are common features found in neural processes. In this context, the present findings suggest for instance that sensory neural systems could constructively exploit their internal noise to improve estimation of the level of a background activity from the environment. These form open perspectives to be further examined, for nonlinear signal and information processing aided by noise.

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