Nonlinear SNR amplification of harmonic signal in noise

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The SNR of a harmonic signal in additive white noise is computed after transformation by an arbitrary memoryless nonlinearity. With a simple saturating nonlinearity having direct electronic implementation, an amplification of the SNR can be obtained, an outcome which is inaccessible with linear devices.

Assessing the presence of a harmonic signal hidden in additive noise is a very common problem in many areas of experimental sciences and technologies. This type of signal–noise mixture has a very characteristic signature in the frequency domain: its power spectrum is formed by a sharp spectral line at the harmonic frequency ν_s emerging out of a broadband background contributed by the noise. A signal-to-noise ratio (SNR) $\mathcal{R}$ is conveniently defined as the ratio of the power contained in the spectral line at ν_s divided by the power contained in the noise background in a small reference frequency band ΔB around ν_s. This SNR quantifies how well the spectral line at ν_s emerges out of the noise background. A narrowband filter at ν_s used to extract the harmonic component, will have an efficacy directly increasing with this SNR [1]. As a pre-processing, it is known that no linear filter is able to improve (increase) such an SNR $\mathcal{R}$. This is because a linear filter multiplies both the spectral line and the noise background at ν_s by the same factor (the squared modulus of its transfer function at ν_s), and therefore leaves the SNR $\mathcal{R}$ unchanged [1]. On the contrary, we will show that very simple nonlinear devices can act as an SNR amplifier providing an enhancement of $\mathcal{R}$.

We consider the signal–noise mixture $x(t)=s(t)+\xi(t)$, with the harmonic component $s(t)=A\cos(2\pi\nu_s t+\varphi)$, and $\xi(t)$ a stationary white noise with probability density function $f_\xi(u)$. This signal $x(t)$ is fed into a memoryless (nonlinear) system [2] with input–output characteristic $g(.)$ producing the output

$$y(t)=g[s(t)+\xi(t)] \tag{1}$$

In this case, both $x(t)$ and $y(t)$ are cyclostationary random signals with period $T_s=1/\nu_s$, both showing a power spectrum with a sharp spectral line at ν_s emerging out of a broadband noise background. The SNR, as defined above, for the output $y(t)$ can be expressed as [3]

$$\mathcal{R}_{\text{out}}=\frac{|\langle \mathrm{E}[y(t)]\exp(-i2\pi t/T_s)\rangle|^2}{\langle \mathrm{var}[y(t)]\rangle \Delta t\Delta B} \tag{2}$$

In (2), a time average is defined as

$$\langle \dots \rangle = \frac{1}{T_s}\int_0^{T_s} \dots dt \tag{3}$$

$\mathrm{E}[y(t)]$ and $\mathrm{var}[y(t)]=\mathrm{E}[y^2(t)]-\mathrm{E}^2[y(t)]$ represent the expectation and variance of $y(t)$ at a fixed time t; and Δt is the time resolution of the measurement (i.e. the signal sampling period in a discrete time implementation). The white noise assumption here models a broadband physical noise with a correlation duration much shorter than the other relevant time scales, i.e. T_s and Δt, and finite variance σ_ξ^2 [3].

From (1), one has

$$\mathrm{E}[y(t)]=\int_{-\infty}^{+\infty} g(u)f_\xi[u-s(t)]du \tag{4}$$

and

$$\mathrm{E}[y^2(t)]=\int_{-\infty}^{+\infty} g^2(u)f_\xi[u-s(t)]du \tag{5}$$

In a similar way, the SNR for the input $x(t)$ is

$$\mathcal{R}_{\text{in}}=\frac{A^2/4}{\sigma_\xi^2\Delta t\Delta B} \tag{6}$$

We then consider for $g(.)$ a very simple nonlinearity, easily implementable with an operational amplifier, the linear-limiting saturation

$$g(u)=\begin{cases}-\lambda & \text{for } u\le-\lambda\\ u & \text{for } -\lambda<u<\lambda\\ \lambda & \text{for } u\ge\lambda\end{cases} \tag{7}$$

with the 'clipping' parameter $\lambda>0$. With $f_\xi(u)$ a zero-mean Gaussian density associated to the cumulative distribution function $F_\xi(u)$, (4) and (5) give

$$\mathrm{E}[y(t)]=\lambda+(-\lambda-s(t))F_\xi(-\lambda-s(t))-(\lambda-s(t))\times F_\xi(\lambda-s(t))+\sigma_\xi^2[f_\xi(-\lambda-s(t))-f_\xi(\lambda-s(t))] \tag{8}$$

and

$$\mathrm{E}[y^2(t)]=\lambda^2+(\lambda^2-s^2(t)-\sigma_\xi^2)[F_\xi(-\lambda-s(t))-F_\xi(\lambda-s(t))]+\sigma_\xi^2[(-\lambda-s(t))f_\xi(\lambda-s(t))-(\lambda-s(t))f_\xi(-\lambda-s(t))] \tag{9}$$

In these conditions, we can analyse the behaviour of the input–ouput SNR gain $G=\mathcal{R}_{\text{out}}/\mathcal{R}_{\text{in}}$. It turns out that there is a broad range of values for λ, both $>A$ or $<A$ depending on the noise level σ_ξ, where the SNR gain G is above unity. Qualitatively, the clipping device (7) on the signal–noise mixture $x(t)=s(t)+\xi(t)$, is able to reduce the noise $\xi(t)$ more than the harmonic signal $s(t)$, this resulting in an improved SNR. Furthermore, at each noise level σ_ξ, it is possible to find the optimal clipping λ_{opt} that maximises the SNR gain G, as presented in Fig. 1. Fig. 1 shows that the optimal clipping λ_{opt} is not necessarily at the signal amplitude A; depending on the noise level, λ_{opt} can be below or above A. Also, for any noise level σ_ξ, at the optimal clipping λ_{opt}, the SNR gain G is always above unity, although it returns (from above) to unity at large noise when $\lambda_{\text{opt}}\to\infty$ (linearity of $g(.)$ is recovered as the optimal processor).

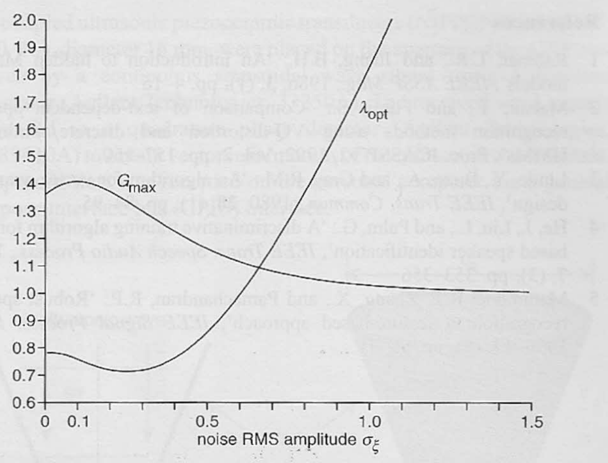

Fig. 1 *Optimal clipping λ_{opt} in (7) and maximum input–ouput SNR gain G_{max} at λ_{opt} against function of RMS amplitude σ_ξ (in units of $A = 1$) of zero-mean Gaussian noise $\xi(t)$*

The present analysis establishes that simple nonlinear devices can be used as SNR amplifiers for a harmonic signal in noise, an outcome which is inaccessible with linear devices. The present treatment is general in $g(.)$ (and also in the noise density f_ξ); we have tested here the simple $g(.)$ of (7), the electronic implementation of which is easy; but other nonlinearities $g(.)$ can be tested for an SNR amplification $G > 1$. Power-law nonlinearities tested in [4] exhibit a similar property of $G > 1$ but with a more complex physical implementation. Additionally, application of the present treatment shows that hard-threshold nonlinearities, like signum or Heaviside functions for $g(.)$, do not allow $G > 1$ with Gaussian noise. Other common nonlinearities encountered, for instance in semiconductor devices, could also be tested for SNR amplification. Such simple nonlinear operators offer a useful complement to linear techniques for signal processing and sensors.

23 March 2005
Electronics Letters online no: 20051065
doi: 10.1049/el:20051065

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