Injecting noise to improve performance of optimal detector

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A report is presented on the feasibility in principle of a new property for stochastic resonance or improvement by noise in optimal processing. An optimal detector is shown, the overall best performance of which is not at zero noise, but at a finite nonzero level of noise.

Studies have shown that an increase in the amount of noise can improve the performance in various signal-processing tasks. This constructive action of noise, under its various possible forms, is often referred to as 'stochastic resonance', and is currently under intense investigation [1]. Stochastic resonance (SR) when it exists very often takes place with non-optimal processors. With a fixed non-optimally tuned processor, injection of noise can sometimes move the process closer to its optimal conditions of operation. In such configurations, injection of noise has been shown capable, for instance, of improving the signal-to-noise ratio delivered by a nonlinear system, or increasing the capacity of an information channel, or enhancing the performance of suboptimal detectors or estimators [2]. In such SR with suboptimal processors, usually the overall best performance occurs at a nonzero level of noise, while at zero noise the performance is strictly poorer.

In addition, SR has recently been shown possible also for optimal processors [3, 4]. In these conditions, the SR instances that have been reported are identified by the performance of an optimal processor which can be enhanced by operating the process at a higher level of noise. This type of SR in optimal processors has been shown essentially as a local improvement by noise of the performance. This means that there can be some ranges where a local increase in the level of noise induces an improvement of the performance. These studies on SR in optimal processes have in fact pointed out that, contrary maybe to intuition, the performance of an optimal processor does not necessarily monotonically degrade as the level of noise increases. However, in the SR instances reported with optimal processes, the overall best performance was always found at zero noise. This is especially always true for optimal detectors as addressed here, with one exception for an optimal Bayesian estimator from the Monte Carlo study of [4]. Usually, if the process can be operated with no noise, no benefit was found in injecting more noise. It is only when a certain initial amount of noise pre-exists and fixes a given initial performance for the optimal processor that injection of more noise may sometimes improve the initial performance.

In this Letter, we go one step further in the inventory of the feasible properties of SR. We show that, for SR in optimal detection, situations can be found where the overall best performance is not a zero noise, but at a finite nonzero level of noise.

We consider as in [3] the optimal detection of a periodic wave $x(t) = w[\nu t + \eta(t)]$ corrupted by phase noise $\eta(t)$. This is a nonlinear signal-noise mixture which lends itself to SR in optimal processing. The frequency of the wave can be $\nu = \nu_0$ with prior probability P_0 or $\nu = \nu_1$ with prior probability $P_1 = 1 - P_0$. At N distinct times t_j which are given, N observations are collected, $x(t_j) = x_j$, for $j = 1$ to N. From the N data points $(x_1, \ldots, x_N) = \mathbf{x}$, one has to decide whether the observed wave has frequency ν_0 or ν_1. The optimal detector is the likelihood ratio test [3]

$$L(\mathbf{x}) = \frac{P(\mathbf{x}|\nu = \nu_1)}{P(\mathbf{x}|\nu = \nu_0)} \overset{\nu_1}{\underset{\nu_0}{\gtrless}} \frac{P_0}{P_1} \qquad (1)$$

which achieves the minimum probability of detection error

$$P_{er} = \frac{1}{2} - \frac{1}{2}\int_{\mathbb{R}^N} \left| P_1 P(\mathbf{x}|\nu = \nu_1) - P_0 P(\mathbf{x}|\nu = \nu_0) \right| d\mathbf{x} \qquad (2)$$

When $\eta(t)$ is white, the probabilities factorise as $P(\mathbf{x}|\nu) = \Pi_{j=1}^{N} P(x_j|\nu)$. The case where $w(\cdot)$ is a square wave oscillating between ± 1 as in [3] is completely tractable analytically, through

$$P(x_j = 1|\nu) = 1 - P(x_j = -1|\nu) = \sum_{k=-\infty}^{+\infty} [F_\eta(k - \nu t_j + \frac{1}{2}) - F_\eta(k - \nu t_j)] \qquad (3)$$

with $F_\eta(\cdot)$ the cumulative distribution function of $\eta(t)$.

Fig. 1 shows the probability of error P_{er} from (2)–(3) of the optimal detector: P_{er} starts to improve (decrease) when the level of noise σ_η rises above zero, and P_{er} reaches it overall minimum for a nonzero level of the phase noise $\eta(t)$. For certain configurations of the N observation times $\mathbf{t} = (t_1, \ldots, t_N)$ as in Fig. 1, the two waves $w(\nu_0 t)$ and $w(\nu_1 t)$ at those times assume the same values, so that the two waves cannot be distinguished without noise. The injected phase noise $\eta(t)$ then has the ability to make the two waves distinct, on average. This leads in Fig. 1 to P_{er} which improves when σ_η rises above zero when the two waves become distinguishable thanks to the added noise.

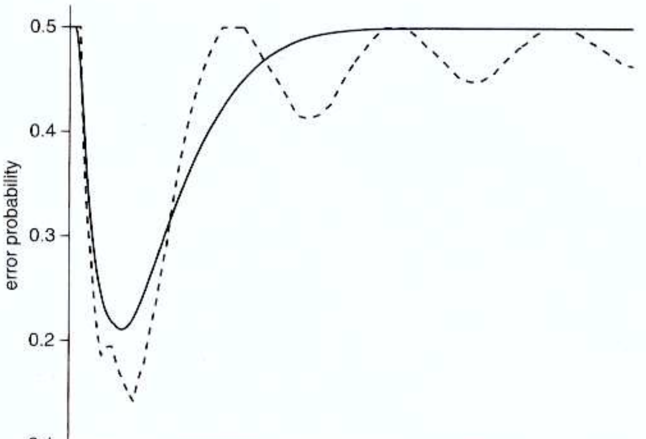

Fig. 1 *Probability of detection error P_{er} of optimal detector against RMS amplitude σ_η of zero-mean noise $\eta(t)$ chosen Gaussian (solid line), uniform (dashed line) (also $P_0 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, $N = 6$ data points taken at times $\mathbf{t} = (0.1, 0.4, 0.8, 2.2, 2.6, 2.8)$)*

Other configurations are shown in Fig. 2 where the two waves are already distinguishable at zero noise where P_{er} is at its best. But as the noise level σ_η increases, there is no monotonic degradation of P_{er}, but ranges of σ_η where again P_{er} can be locally improved when the noise grows. Also, Fig. 2 reveals a complex influence of the set of observation times $\mathbf{t}$, with larger improvement which can be obtained by sets $\boldsymbol{t}$ containing more times t_j where the two waves differ at zero noise.

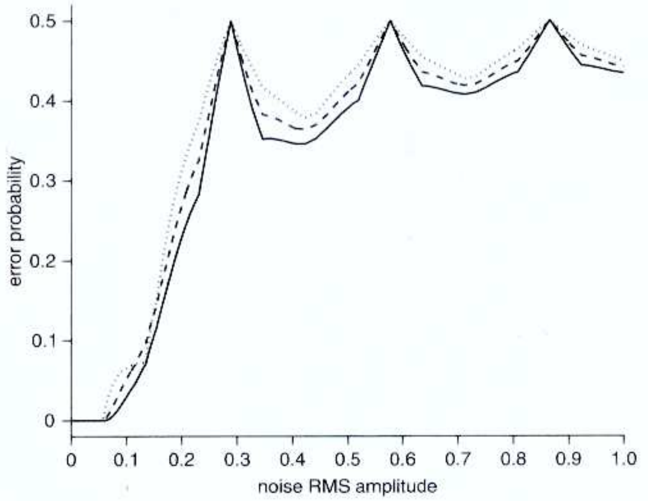

Fig. 2 *Same as Fig. 1 with uniform noise $\eta(t)$ and $N = 6$ data points taken at times $\boldsymbol{t} = (0.1, 0.4, \underline{1.1}, \underline{1.6}, 2.6, 2.8)$ (dotted line), $\boldsymbol{t} = (0.1, \underline{0.7}, \underline{1.1}, \underline{1.6}, \underline{2.4}, 2.8)$ (dashed line), $\boldsymbol{t} = (\underline{0.6}, \underline{0.7}, \underline{1.1}, \underline{1.6}, \underline{2.4}, \underline{3.6})$ (solid line) (underlined are times t_j where two waves differ at zero noise)*

Figs. 1 and 2 illustrate two aspects of the specific mechanism of improvement by noise in optimal detection here, where configurations with noise can be more easily distinguished than without noise. This complements other mechanisms reported for SR.

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