Qubit state estimation and enhancement by quantum thermal noise

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From a qubit in a noisy state affected by an uncontrolled decohering environment represented by a thermal bath at temperature T, we consider estimation of the norm of its Bloch vector, equivalent to estimation of its purity or of its linear entropy. The performance in estimation is assessed by the quantum Fisher information. When the qubit can be prepared with an optimal orientation precisely matched to the thermal noise, the estimation performance always degrades with an increasing temperature of the bath. On the contrary, for a nonoptimal orientation of the qubit, we demonstrate the possibility of an enhancement of the estimation performance with an increasing temperature of the bath. Such behavior shows that increased decoherence is not necessarily associated with poorer informational performance, and can be compared to stochastic resonance or useful noise effects previously reported in classical (non-quantum) estimation.

Introduction: Measurement of quantum states is an indispensable step in the exploitation of quantum systems, especially for applications in quantum information and quantum computation. The theory of quantum state estimation allows one to control and optimize the conditions and performance for efficient measurements. We consider here a qubit in a state represented by a density operator ρ_ξ dependent on an unknown real scalar parameter ξ that we want to estimate from measurement of the qubit. From estimation theory [1], it is known that any estimator for ξ comes with a mean-squared error lower bounded by the Cramér-Rao bound proportional to the reciprocal of the classical Fisher information $F_c(\xi)$. Estimators are known, such as the maximum likelihood estimator, which are optimal in the sense that they allow to reach the Cramér-Rao bound in the limit of a large number of independent measurements on identical copies of ρ_ξ. In turn, $F_c(\xi)$ is upper bounded by the quantum Fisher information $F_q(\xi)$, i.e. $F_c(\xi) \le F_q(\xi)$. Adaptive strategies with feedback exist [2, 3] to devise measurement protocols allowing to reach $F_c(\xi) = F_q(\xi)$. These properties put together confer an essential role to the quantum Fisher information $F_q(\xi)$ in controlling the ultimate performance accessible in quantum estimation.

The action of quantum noise on the qubit usually hinders estimation and provokes a reduction of $F_q(\xi)$. However, we are going to show here the possibility of some circumstances where an increase in the level of noise can lead to an enhancement of the estimation performance assessed by $F_q(\xi)$. In the classical (non-quantum) context, such effects of noise-enhanced performance have been reported in numerous processes, especially for signal estimation [4, 5], and related to the phenomenon of stochastic resonance [6]. In the context of quantum information, several processes with noise-enhanced performance have been reported [7, 8, 9]. Yet it is the first time to our knowledge that noise-enhanced performance is shown feasible for quantum state estimation as considered here.

Quantum Fisher information for qubit state estimation: We consider a qubit in a state ρ_ξ characterized by the Bloch vector $\vec{r} = \vec{r}(\xi)$ in the Bloch ball of $\mathbb{R}^3$ defined by $\|\vec{r}\| \le 1$. Prior to measurement for estimation of ξ, the qubit is affected by some quantum noise. The action of such noise can always be described as an affine transformation of the Bloch vector

$$\vec{r} \to \mathrm{A}\vec{r} + \vec{c}\,, \tag{1}$$

with A a 3×3 real matrix and $\vec{c}$ a real vector in $\mathbb{R}^3$ characterizing the quantum noise, and mapping the Bloch ball onto itself.

According to [10], its Eq. (47), the quantum Fisher information for estimating ξ from the noisy qubit, is expressible as

$$F_q(\xi) = \frac{\left[(\mathrm{A}\vec{r} + \vec{c})\mathrm{A}\partial_\xi \vec{r}\right]^2}{1 - (\mathrm{A}\vec{r} + \vec{c})^2} + \left(\mathrm{A}\partial_\xi \vec{r}\right)^2 . \tag{2}$$

For definiteness of the estimation task, we consider the scalar parameter $\xi \equiv \|\vec{r}\| \equiv r$. We write $\vec{r} = r\vec{r}^{\,\mathrm{un}}$ with $\vec{r}^{\,\mathrm{un}}$ the unit vector fixing the direction of $\vec{r}$ in $\mathbb{R}^3$, so that the derivative $\partial_\xi \vec{r} = \partial\vec{r}/\partial r = \vec{r}^{\,\mathrm{un}}$, leading for Eq. (2) to

$$F_q(\xi \equiv r) = \frac{\left[(r\mathrm{A}\vec{r}^{\,\mathrm{un}} + \vec{c})\mathrm{A}\vec{r}^{\,\mathrm{un}}\right]^2}{1 - (r\mathrm{A}\vec{r}^{\,\mathrm{un}} + \vec{c})^2} + \left(\mathrm{A}\vec{r}^{\,\mathrm{un}}\right)^2 . \tag{3}$$

This relation then provides the possibility to assess the impact of the quantum noise, via $(\mathrm{A}, \vec{c})$, on the quantum Fisher information F_q of Eq. (3). The quantum state with Bloch vector $\vec{r}$ has purity $u = (1 + r^2)/2 = 1 - S_L$ with S_L the linear entropy of the state. Upon changing of parameter ξ, we therefore have $F_q(\xi \equiv u) = F_q(\xi \equiv S_L) = F_q(\xi \equiv r)/r^2$. In this way, the evolution with $(\mathrm{A}, \vec{c})$ of $F_q(\xi \equiv r)$ of Eq. (3) characterizes as well the estimation of the purity $\xi \equiv u$ or of the linear entropy $\xi \equiv S_L$, which are all important scalars attached to a quantum state.

An important quantum noise relevant to the qubit is the generalized amplitude damping (GAD) noise [11, 10], characterized in Eq. (1) by

$$\mathrm{A}\vec{r} + \vec{c} = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \vec{r} + \begin{bmatrix} 0 \\ 0 \\ (2p-1)\gamma \end{bmatrix} , \tag{4}$$

describing the interaction of the qubit with a thermal bath at temperature T. The damping coefficient $\gamma \in [0, 1]$ characterizes the coupling of the qubit with the bath. At long interaction times, $\gamma \to 1$, and the noisy qubit relaxes to the equilibrium mixed state $\rho_\infty = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$. At equilibrium, the qubit has probabilities p of being measured in state $|0\rangle$ and $1-p$ of being measured in state $|1\rangle$. With the energies E_0 and $E_1 \ge E_0$ respectively for the states $|0\rangle$ and $|1\rangle$, the equilibrium probabilities are governed by the Boltzmann distribution $p = \exp[-E_0/(k_BT)]/Z$ and $1-p = \exp[-E_1/(k_BT)]/Z$ with $Z = \exp[-E_0/(k_BT)] + \exp[-E_1/(k_BT)]$. In this way, in the GAD quantum thermal noise of Eq. (4), p is determined by the temperature T of the bath.

Optimal orientation of the qubit: From Eq. (3), the quantum Fisher information F_q is influenced by the noise properties $(\mathrm{A}, \vec{c})$, and also by the direction $\vec{r}^{\,\mathrm{un}}$ of the qubit state in $\mathbb{R}^3$ which can be parameterized by a coelevation angle $\theta \in [0, \pi]$ with the axis Oz and an azimuth $\varphi \in [0, 2\pi)$ around Oz. Because of the symmetry of the GAD noise around Oz, the Fisher information F_q of Eq. (3) is invariant with the azimuth φ, and it is enough to parameterize $\vec{r}^{\,\mathrm{un}} = [\sin(\theta), 0, \cos(\theta)]^\top$ in the plane (Ox, Oz) at $\varphi = 0$ with no loss of generality.

Then Eq. (3) allows one to analyze the evolution of the Fisher information F_q with the orientation θ, and reveals that in general F_q is maximized at a nontrivial optimal angle θ_{opt} dependent on the noise properties $(\mathrm{A}, \vec{c})$. An illustration is given in Fig. 1, for different temperatures T of the thermal GAD noise.

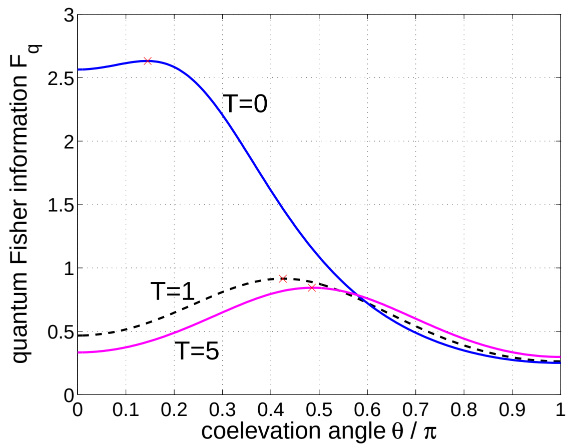

Fig. 1 *Quantum Fisher information F_q from Eq. (3) as a function of the coelevation angle θ characterizing, in the plane (Ox, Oz) of $\mathbb{R}^3$, the direction $\vec{r}^{\,\mathrm{un}}$ of the Bloch vector $\vec{r}$ whose norm r is to be estimated. The qubit is affected by the GAD noise of Eq. (4), with damping $\gamma = 0.5$, at three temperatures of the thermal bath, $T = 0$, 1 and 5, in units where $k_B = 1$ and with the two energy levels $E_0 = 0$ and $E_1 = 1$. Evaluation is at $r = 0.9$. The crosses ($\times$) locate the maximum of F_q defining the optimal angle θ_{opt}.*

For each temperature T of the noise, Fig. 1 shows the existence of a specific angle $\theta = \theta_{\mathrm{opt}}$, i.e. a specific orientation $\vec{r}^{\,\mathrm{un}}$ of $\vec{r}$, maximizing the Fisher information F_q. Also, Fig. 1 shows that the maximum of the Fisher information $F_q(\theta_{\mathrm{opt}})$ decreases as the temperature T increases. This corresponds to a performance in estimation which degrades as the temperature T of the thermal bath increases, manifesting an expected detrimental action of a rising temperature of the thermal GAD noise.

Beneficial action of the quantum noise: When the qubit is prepared with a direction θ which is not optimal, our essential observation is that the quantum noise is not necessarily detrimental to the estimation task. This is illustrated in Fig. 2, with an example at $\theta = 0.7\pi \neq \theta_{\rm opt}$.

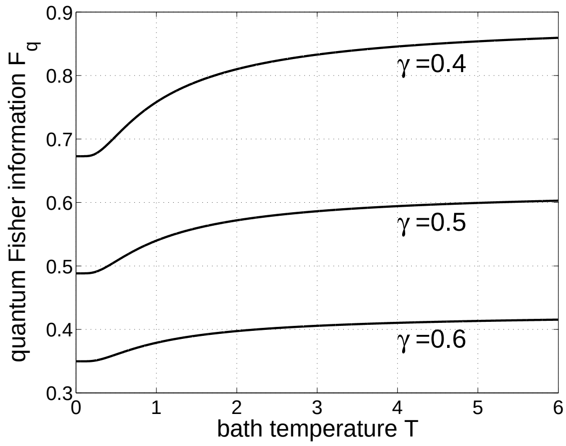

Fig. 2 *Quantum Fisher information F_q from Eq. (3) as a function of the temperature T of the thermal bath, for a nonoptimal orientation $\theta = 0.7\pi$ of $\vec{r}^{\,\rm un}$, and three values of the damping γ. Other conditions are similar to Fig. 1.*

In Fig. 2, it is observed that the quantum Fisher information F_q increases as the temperature T of the thermal bath increases. This beneficial action of an increasing temperature is preserved as the damping coefficient γ is varied, with an overall lower F_q at larger damping γ in Fig. 2. The beneficial action of the temperature is also preserved when the norm r to be estimated is varied, as witnessed by Fig. 3.

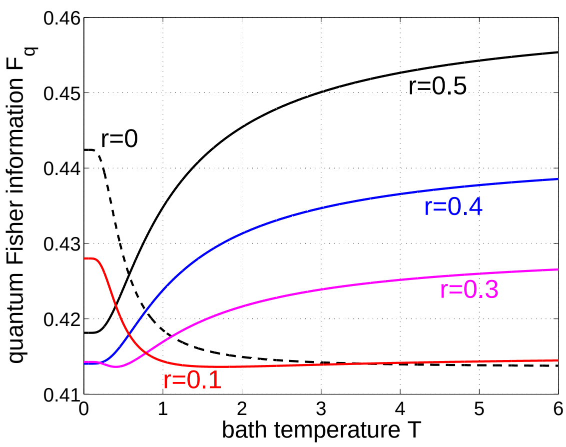

Fig. 3 *Quantum Fisher information F_q from Eq. (3) as a function of the temperature T of the thermal bath, for a nonoptimal orientation $\theta = 0.7\pi$ of $\vec{r}^{\,\rm un}$, and five values of the norm r of Bloch vector $\vec{r}$. Other conditions are similar to Fig. 1.*

In Fig. 3, when the norm r is not too small, or equivalently when the purity $u = (1 + r^2)/2$ of the state to be estimated is not too small, a rising temperature T of the noise always increases the Fisher information F_q, and in this respect enhances the performance in estimation. On the contrary, at small r or small purity u, the Fisher information F_q starts to decrease when the temperature T rises from zero, up to some value of T (dependent on r) where F_q is at a minimum and the estimation performance at its worse. Then for still higher temperatures T, the increase of F_q with T tends to resume, at least for $r > 0$. At $r = 0$, the qubit is in the maximally mixed state $\mathbb{1}/2$. Thus, in the range of low r, decreasing r, i.e. decreasing the purity u of the state to be estimated, can lead to a higher Fisher information F_q, as visible in Fig. 3. This behavior is further illustrated in Fig. 4.

Fig. 4 shows at various noise temperatures T, the nonmonotonic evolution of the Fisher information F_q, when the norm r or purity $u = (1 + r^2)/2$ is varied, showing the possibility of higher F_q, or higher estimation performance, at lower purity (larger mixedness) of the estimated state. Since at increased mixedness a quantum state is often interpreted as a more noisy or random state, this behavior stemming from Eq. (3), is

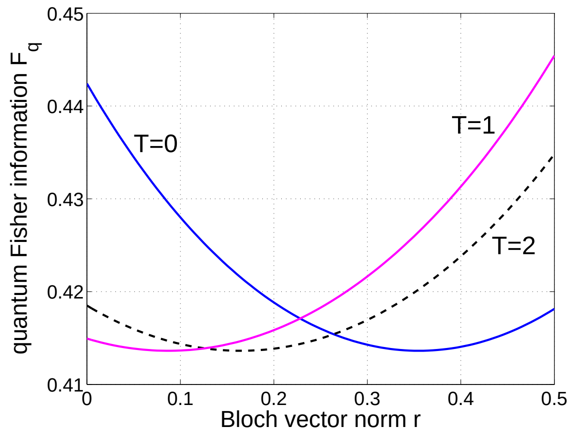

Fig. 4 *Quantum Fisher information F_q from Eq. (3) as a function of the norm r of Bloch vector $\vec{r}$, in conditions otherwise as in Fig. 3.*

another mark of some constructive action of noise which can occur in the Fisher information F_q in the presence of a thermal GAD noise.

Conclusion: Increasing the level of noise by increasing the temperature T of the thermal bath representing the decohering environment, can lead to enhancement of the estimation performance as assessed by the quantum Fisher information F_q of Eq. (3). This typically occurs when the qubit to be estimated is prepared with a nonoptimal orientation $\vec{r}^{\,\rm un}$. The beneficial action of the noise on F_q in Eq. (3) can be essentially attributed to the geometric action of the noise parameter $\vec{c}$ in the term $(r{\rm A}\vec{r}^{\,\rm un} + \vec{c})$ of Eq. (3) to increase F_q as T is raised. Especially, for the class of unital quantum noises [10] defined by $\vec{c} \equiv \vec{0}$, no enhancement of F_q is observed, and the effect requires a nonunital noise with $\vec{c} \not\equiv \vec{0}$, such as the thermal GAD noise. The Fisher information F_q of Eq. (3) improves through a monotonic increase as the temperature T is raised. This is in contrast with classical forms of enhancement by noise or stochastic resonance, where usually a finite optimal amount of noise maximizes the performance [6]. In practice here, it will be necessary to limit the temperature T before it becomes harmful to the quantum system under study.

Similar investigations of the effect of quantum thermal noise could be extended to other information tasks or other quantum systems, to further explore this important possibility demonstrated here that increased decoherence is not necessarily associated with poorer information performance.

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