Qubit state detection and enhancement by quantum thermal noise

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We consider the task fundamental to quantum communication and coding which consists in detecting between two possible states of a noisy qubit, with a performance assessed by the overall probability of detection error. The detection process operates in the presence of decoherence represented by a quantum thermal noise at an arbitrary temperature. With uneven prior probabilities of the two states, as the noise temperature is increased, nonmonotonic evolutions are reported for the performance, which does not uniformly degrades. Regimes are found where higher noise temperatures are more favorable to detection, with relation to stochastic resonance effects where noise reveals beneficial to information processing.

Qubit detection with noise: We address the common problem in quantum communication or binary coding consisting in detecting or discriminating between two noisy quantum states [1, 2]. We apply the approach to the qubit, which stands as a fundamental reference for quantum information, representing for instance situations with photons with their two states of polarization, or electrons with their two states of spin, individually serving as information carrier. A qubit with two-dimensional Hilbert space $\mathcal{H}_2$ is considered in a quantum state represented by the density operator ρ, which can be prepared as $\rho=\rho_0$ or $\rho=\rho_1$ respectively with prior probability P_0 or $P_1=1-P_0$, assuming $P_0\le P_1$ without loss of generality. The qubit states are parameterized in Bloch representation as [3]

$$\rho=\frac{1}{2}\big(\mathrm{I}_2+\vec{r}\cdot\vec{\sigma}\big)\,, \tag{1}$$

with the real 3-dimensional Bloch vector $\vec{r}\in\mathbb{R}^3$ of Euclidean norm $\|\vec{r}\,\|\le 1$, and $\vec{\sigma}$ a formal vector assembling the three 2×2 Pauli matrices $[\sigma_x,\sigma_y,\sigma_z]=\vec{\sigma}$, and I_2 the identity of $\mathcal{H}_2$, with $\vec{r}=\vec{r}_0$ or $\vec{r}_1$ respectively for ρ_0 or ρ_1. The qubit is then altered by decoherence as a quantum noise producing the noisy state $\rho\to\rho'=\mathcal{N}(\rho)$. The noise is generally represented by the completely positive trace-preserving superoperator $\mathcal{N}(\cdot)$ acting on the Bloch vectors according to the affine map [3, 2]

$$\vec{r}\longrightarrow\vec{r}\,'=A\vec{r}+\vec{c}\,, \tag{2}$$

where A is a 3×3 real matrix and $\vec{c}$ a real vector in $\mathbb{R}^3$. From a quantum measurement on the noisy state ρ', the detection problem is to decide whether the preparation was $\rho=\rho_0$ or $\rho=\rho_1$. A relevant metric of performance is the overall probability of detection error P_{er}, which is minimized by the following strategy [1, 2]. A two-outcome measurement is performed on the noisy qubit by means of a positive operator-valued measure with two elements $\{\mathsf{M}_0,\mathsf{M}_1\}$, standing as two positive operators decomposing the identity of $\mathcal{H}_2$, i.e. $\mathsf{M}_0+\mathsf{M}_1=\mathrm{I}_2$. The test (Hermitian) operator $\mathsf{T}=P_1\rho_1-P_0\rho_0$ is in Bloch representation

$$\mathsf{T}=\frac{1}{2}\big[(P_1-P_0)\mathrm{I}_2+\vec{\tau}\,'\cdot\vec{\sigma}\big]\,, \tag{3}$$

characterized by the test vector $\vec{\tau}=P_1\vec{r}_1-P_0\vec{r}_0=[\tau_x,\tau_y,\tau_z]^\top$ of $\mathbb{R}^3$ after alteration by the noise of Eq. (2) yielding

$$\vec{\tau}\,'=A\vec{\tau}+(P_1-P_0)\vec{c}=[\tau'_x,\tau'_y,\tau'_z]^\top\,. \tag{4}$$

Then the optimal measurement operator is $\mathsf{M}_1^{\mathrm{opt}}$ standing as the projector on the eigenspace of T associated with positive eigenvalues, while $\mathsf{M}_0^{\mathrm{opt}}=\mathrm{I}_2-\mathsf{M}_1^{\mathrm{opt}}$ is the complementary projector in $\mathcal{H}_2$. So from ρ', when $\mathsf{M}_1^{\mathrm{opt}}$ is measured this detects the preparation $\rho=\rho_1$, while $\mathsf{M}_0^{\mathrm{opt}}$ detects $\rho=\rho_0$. This is the optimal detection strategy where $P_{\mathrm{er}}=\Pr\{\mathsf{M}_1|\rho_0\}P_0+\Pr\{\mathrm{M}_0|\rho_1\}P_1$, with $\Pr\{\mathsf{M}_k|\rho_j\}=\mathrm{tr}(\mathsf{M}_k\rho_j)$ for $j,k=0,1$, reaches the minimum

$$P_{\mathrm{er}}=\frac{1}{2}\big(1-\|\vec{\tau}\,'\|\big),\quad\text{when } \|\vec{\tau}\,'\|\ge P_1-P_0, \tag{5}$$

$$P_{\mathrm{er}}=P_0,\quad\text{when } \|\vec{\tau}\,'\|<P_1-P_0. \tag{6}$$

We note that our general condition $P_0\le P_1$ implies $0\le P_1-P_0\le 1$, and that $\vec{\tau}=P_1\vec{r}_1-P_0\vec{r}_0$ is a vector of $\mathbb{R}^3$ which can be anywhere in the Bloch ball satisfying $\|\vec{\tau}\,\|\le 1$. We now want to study the impact of a specific quantum noise of great relevance to the qubit, which is the generalized amplitude damping noise or quantum thermal noise [3, 2]. It is defined in Eq. (2) by

$$A\vec{r}+\vec{c}=\begin{bmatrix}\sqrt{1-\gamma}&0&0\\0&\sqrt{1-\gamma}&0\\0&0&1-\gamma\end{bmatrix}\vec{r}+\begin{bmatrix}0\\0\\(2p-1)\gamma\end{bmatrix}. \tag{7}$$

This noise model describes the interaction of the qubit with an uncontrolled environment represented as a thermal bath at temperature T. The parameter $\gamma\in[0,1]$ is a damping factor which often can be expressed [3] as a function of the interaction time t of the qubit with the bath as $\gamma=1-e^{-t/T_1}$, where T_1 is a time constant for the interaction (such as the relaxation time T_1 of a spin in magnetic resonance). At long interaction time $t\to\infty$, then $\gamma\to 1$ and the qubit relaxes to the equilibrium mixed state $\rho_\infty=p\,|0\rangle\langle 0|+(1-p)\,|1\rangle\langle 1|$ of Bloch vector $\vec{r}_\infty=\vec{c}$. At equilibrium, the qubit has probabilities p of being measured in the ground state $|0\rangle$ and $1-p$ of being measured in the excited state $|1\rangle$. With the energies E_0 and $E_1>E_0$ respectively for the states $|0\rangle$ and $|1\rangle$, the equilibrium probabilities are governed by the Boltzmann distribution

$$p=\frac{1}{1+\exp[-(E_1-E_0)/(k_BT)]}\,. \tag{8}$$

In this way, in the quantum thermal noise of Eq. (7), the probability p is determined by the temperature T of the bath via Eq. (8). From Eq. (8), the probability p is a decreasing function of the temperature T. At $T=0$ the probability is $p=1$ for the ground state $|0\rangle$, while at $T\to\infty$ the ground state $|0\rangle$ and excited state $|1\rangle$ are equiprobable with $p=1/2$. Therefore, from Eq. (8), when the temperature T monotonically increases from 0 to ∞, the probability p monotonically decreases from 1 to $1/2$. The remarkable feature we will demonstrate in the sequel is that, as the noise temperature T rises from 0 to ∞, the performance P_{er} of the detection in Eqs. (5)–(6) does not necessarily degrade uniformly, but on the contrary can experience nonmonotonic evolution.

The evolution of P_{er} in Eqs. (5)–(6) with the temperature T is essentially controlled by the norm $\|\vec{\tau}\,'\|$ of the noisy test vector $\vec{\tau}\,'$ of Eq. (4), or equivalently its squared norm expressible as

$$\|\vec{\tau}\,'\|^2=(1-\gamma)(\tau_x^2+\tau_y^2)+\tau_z'^2 \tag{9}$$

for the thermal noise of Eq. (7), where the influence of the noise temperature T, via p, is conveyed only through the squared z-component

$$\tau_z'^2=\big[(1-\gamma)\tau_z+(P_1-P_0)(2p-1)\gamma\big]^2\,. \tag{10}$$

This term $\tau_z'^2$ is a $\cup$-shaped parabola in the variable p, however limited by the allowed range $p\in[1/2,1]$. The minimum of the parabola is zero and occurs when $(1-\gamma)\tau_z=-(P_1-P_0)(2p-1)\gamma$, corresponding for the variable p to the critical value

$$p_c=\frac{1}{2}-\frac{1}{2}\alpha_c\,, \tag{11}$$

with the scalar parameter

$$\alpha_c=\frac{1}{P_1-P_0}\frac{1-\gamma}{\gamma}\tau_z\,. \tag{12}$$

For uneven prior probabilities $P_0\ne P_1$, as the temperature T rises from 0 to ∞, inducing p to decrease from 1 to $1/2$, it results that three regimes of evolution of $\tau_z'^2$ in Eq. (10), and subsequently of P_{er} in Eqs. (5)–(6), are accessible, depending on the situation of p_c of Eq. (11) in relation to the allowed interval $[1/2,1]\ni p$. These evolutions will take place between the two extreme values, at $T=0$ (i.e. at $p=1$) determined in Eq. (10) by $\tau_z'^2(T=0)=[(1-\gamma)\tau_z+(P_1-P_0)\gamma]^2$ fixing $P_{\mathrm{er}}(T=0)$ in Eqs. (5)–(6), and at $T=\infty$ (i.e. at $p=1/2$) determined by $\tau_z'^2(T=\infty)=[(1-\gamma)\tau_z]^2$ fixing $P_{\mathrm{er}}(T=\infty)$. Especially, depending on the conditions, one can have $P_{\mathrm{er}}(T=0)<P_{\mathrm{er}}(T=\infty)$, which is the natural condition of an error in detection which worsens as the noise temperature increases. But the opposite $P_{\mathrm{er}}(T=0)>P_{\mathrm{er}}(T=\infty)$ can also be found, as we shall see, as a counterintuitive manifestation of a beneficial role of decoherence. Between these two extremes at $T=0$ and $T=\infty$, as indicated, three regimes of evolution are accessible for the performance P_{er}, which we now analyze.

Increasing P_{er}*:* In Eq. (12), the condition $\alpha_c \geq 0$ is obtained by $\tau_z \geq 0$, and produces in Eq. (11) a $p_c \leq 1/2$ occurring before the interval $[1/2, 1] \ni p$. In such configurations, the ∪-shaped parabola of $\tau_z'^2$ in Eq. (10) increases as p increases in $[1/2, 1]$. This is equivalent in Eqs. (5)–(6) to a probability of detection error P_{er} which increases as the temperature T rises from 0 to ∞. This is somehow the expected natural behavior: as the temperature T of the thermal noise increases, the performance in detection steadily degrades. Such a regime of increasing P_{er} is obtained in conditions with $\alpha_c \geq 0$ in Eq. (12), which is ensured by any $\tau_z \geq 0$. Some illustrative conditions of this type are presented in Fig. 1. The illustrations of Fig. 1, and also of Figs. 2–3, are obtained with $\vec{r}_1 = \vec{n}$ and $\vec{r}_0 = -\vec{n}$, with $\vec{n}$ an arbitrary unit vector of $\mathbb{R}^3$, yielding a test vector $\vec{\tau} = \vec{n}$ with $\tau_z \in [-1, 1]$ for any P_1. Two such antipodal unit Bloch vectors $\vec{r}_0$ and $\vec{r}_1$ in $\mathbb{R}^3$ represent two initial signaling states ρ_0 and ρ_1 which are two orthogonal pure states of $\mathcal{H}_2$, which, were not the degradation by the quantum noise, could be distinguished with no error. Also in the illustrations, the conditions (not critical for the analysis) take for Eq. (8) an energy difference $E_1 - E_0 = 1$ in units where $k_B = 1$.

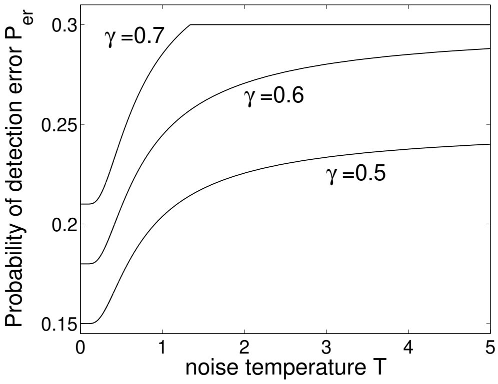

Fig. 1 *Increasing probability of detection error* P_{er} *of Eqs. (5)–(6), as a function of the noise temperature* T*, when* $\tau_z = 1$*, with prior probability* $P_0 = 0.3$*, at various damping* γ*.*

Resonant P_{er}*:* In Eq. (12) an $\alpha_c \in]-1, 0[$ leads in Eq. (11) to a p_c occurring inside the interval $[1/2, 1] \ni p$. In such configurations, as p increases in $[1/2, 1]$, the ∪-shaped parabola of $\tau_z'^2$ in Eq. (10) passes through its minimum of zero at $p = p_c$. This is equivalent in Eqs. (5)–(6) to a ∩-shaped resonant evolution of the probability of detection error P_{er}, as the temperature T rises from 0 to ∞. In particular, P_{er} culminates at a maximum corresponding to the zero $\tau_z'^2$, occurring at a critical temperature T_c related to p_c via Eq. (8). Such a regime of resonant P_{er} is obtained in conditions ensuring $\alpha_c \in]-1, 0[$ in Eq. (12), which requires $\tau_z < 0$ associated with appropriate tunings of the damping γ and difference $P_1 - P_0$. Some illustrative realizations of such conditions are shown in Fig. 2.

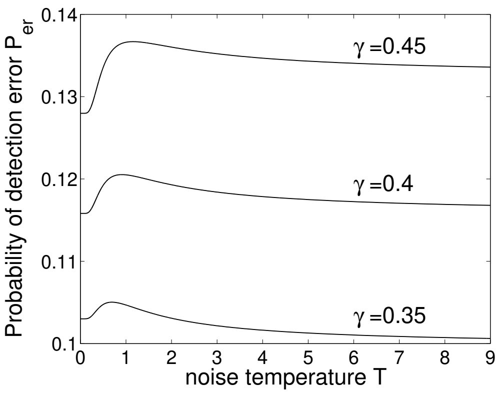

Fig. 2 *Resonant probability of detection error* P_{er} *of Eqs. (5)–(6), as a function of the noise temperature* T*, when* $\tau_z = -0.2$*, with prior probability* $P_0 = 0.2$*, at various damping* γ*.*

The resonant evolutions of Fig. 2 manifest a nontrivial action of the quantum thermal noise. They reveal that there exists a finite range of the temperature T around T_c where the quantum noise is specifically detrimental to the detection task, and that smaller but also larger noise temperatures can be more favorable for detection.

Decreasing P_{er}*:* In Eq. (12) an $\alpha_c \leq -1$ leads in Eq. (11) to a p_c occurring after the interval $[1/2, 1] \ni p$. In such configurations, the ∪-shaped parabola of $\tau_z'^2$ in Eq. (10) decreases as p increases in $[1/2, 1]$. This is equivalent in Eqs. (5)–(6) to a probability of detection error P_{er} which decreases as the temperature T rises from 0 to ∞. This is also an unusual behavior where raising the noise temperature is always beneficial to the detection efficacy. In practice, however, the temperature will have to be limited before it can cause damage to the quantum system. Such a regime of decreasing P_{er} is obtained in conditions with $\alpha_c \leq -1$, realizable with $\tau_z < 0$ and adapted damping γ and difference $P_1 - P_0$. Some illustrative conditions of this type are shown in Fig. 3.

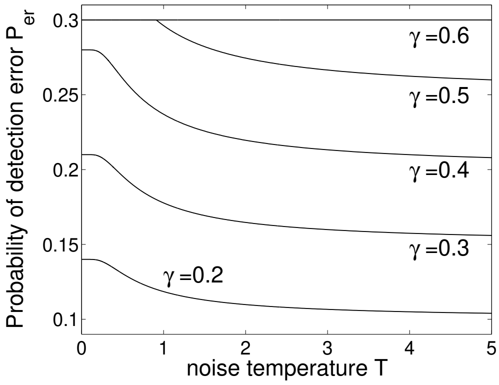

Fig. 3 *Decreasing probability of detection error* P_{er} *of Eqs. (5)–(6), as a function of the noise temperature* T*, when* $\tau_z = -1$*, with prior probability* $P_0 = 0.3$*, at various damping* γ*.*

Discussion: The nonmonotonic evolutions of the probability of detection error P_{er} with the temperature T of the thermal noise reveal the possibility of sophisticated behaviors of decoherence, which is not necessarily uniformly more detrimental as its amount increases. Configurations with higher noise temperatures can be more efficient for quantum detection from noisy qubit states. Such useful-noise effects are reminiscent of stochastic resonance phenomena where enhancement in the level of noise can reveal beneficial to various information processing tasks. Stochastic resonance effects have mainly been reported in the classical domain. For the quantum domain, stochastic resonance has been reported for informational tasks such as the transmission of information over noisy channels [4, 5], or for parametric estimation [6]. To our knowledge, it is reported here for the first time for a task of quantum detection from noisy qubit states affected by quantum thermal noise. Various extensions can be envisaged in different directions, to investigate the possibility of useful-noise effects, with other noise models or in other information processing tasks, for a better understanding and mastering of quantum decoherence and its nontrivial behaviors, which is fundamental to quantum information.

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