Fluctuation and Noise Letters
Vol. 18, No. 3 (2019) 1950015 (15 pages)

DOI: 10.1142/S0219477519500159

Stochastic Resonance with Unital Quantum Noise

Nicolas Gillard, Étienne Belin and François Chapeau-Blondeau

Laboratoire Angevin de Recherche
en Ingénierie des Systèmes (LARIS)
Université d'Angers, 62 avenue Notre Dame du Lac,
49000 Angers, France

Received 15 November 2018
Accepted 27 January 2019
Published 11 March 2019

Communicated by Nigel Stocks

The fundamental quantum information processing task of estimating the phase of a qubit is considered. Following quantum measurement, the estimation efficiency is evaluated by the classical Fisher information which determines the best performance limiting any estimator and achievable by the maximum likelihood estimator. The estimation process is analyzed in the presence of decoherence represented by essential quantum noises that can affect the qubit and belonging to the broad class of unital quantum noises. Such a class especially contains the bit-flip, the phase-flip, the depolarizing noises, or the whole family of Pauli noises. As the level of noise is increased, we report the possibility of non-standard behaviors where the estimation efficiency does not necessarily deteriorate uniformly, but can experience non-monotonic variations. Regimes are found where higher noise levels prove more favorable to estimation. Such behaviors are related to stochastic resonance effects in signal estimation, shown here feasible for the first time with unital quantum noises. The results provide enhanced appreciation of quantum noise or decoherence, manifesting that it is not always detrimental for quantum information processing.

Keywords: Quantum noise; stochastic resonance; quantum estimation; decoherence; improvement by noise.

1. Introduction

For information sciences, stochastic resonance identifies situations where the action of noise can prove beneficial to information processing [1–3]. Such stochastic resonance effects have been reported and analyzed in a broad variety of information processing tasks, including signal transmission [4–7], detection [8–10], estimation [11–13], or for image acquisition [14–17], and this mainly in the classical (non-quantum) domain. Quantum physics holds large potential for information processing, which is currently under intense investigation. Quantum noise or decoherence is a critical feature impacting the performance of quantum information processing, and better understanding and control of such noise are crucial to the progress of quantum information technologies [18–20]. A fundamental system of quantum information is the quantum bit or qubit. Various forms of stochastic resonance have recently been reported in informational processes involving the qubit in the presence of noise. Forms of stochastic resonance have been investigated for information communication over qubit channels [21–24]. For quantum sensors and metrology, stochastic resonance has recently been extended to situations of quantum state detection [25] or quantum state estimation [26, 27] from noisy qubits. These forms of stochastic resonance in [25–27] have been shown possible specifically with quantum thermal noise acting on the qubit, where the decohering environment is represented as a thermal bath at finite temperature. Regimes were reported in [25–27], where an increase in the noise temperature can prove favorable to the metrological performance from the noisy qubit, in quantum detection or estimation. The quantum thermal noise, as we shall explain below, owns the important property of being a non-unital quantum noise, exhibiting in some sense a lesser degree of symmetry, and this non-unital character is essential to the forms of stochastic resonance observed in [25–27]. There however exist other important quantum noises acting on the qubit that are unital noises, exhibiting higher degree of symmetry. The class of unital quantum noises contains many important noises for the qubit, such as the bit-flip, the phase-flip, the depolarizing noises, or the whole family of Pauli noises. The forms of stochastic resonance reported in [25–27] were not investigated in the presence of such common unital noises for the qubit. So far, no scenario has been reported in quantum metrology, detection or estimation, showing the possibility of stochastic resonance effects with such unital quantum noises, and it is not known whether this is possible or not. In the present paper, we show that this possibility holds, by reporting scenarios of phase estimation from a noisy qubit, where unital quantum noise can play a role for improving the efficiency in definite conditions.

2. Phase Estimation on a Noisy Qubit

For self-containedness of the present report, we begin this Section by briefly presenting the process of phase estimation on a noisy qubit which is in common with the recent study of [27]; then here phase estimation will be investigated specifically with unital quantum noise not addressed in [27] or elsewhere. A qubit with two-dimensional Hilbert space $\mathcal{H}_2$ is prepared in a quantum state represented by the density operator ρ_0 expressed in Bloch representation [28] as

$$\rho_0 = \frac{1}{2}(\mathrm{I}_2 + \mathbf{r}_0 \cdot \boldsymbol{\sigma}), \tag{1}$$

with I_2 the identity on $\mathcal{H}_2$, and $\boldsymbol{\sigma}$ a formal vector assembling the three 2×2 (traceless Hermitian unitary) Pauli matrices $[\sigma_x, \sigma_y, \sigma_z] = \boldsymbol{\sigma}$. The coordinates of ρ_0 in Eq. (1) are specified by the Bloch vector $\mathbf{r}_0 \in \mathbb{R}^3$, with norm $\|\mathbf{r}_0\| = 1$ for a pure state and $\|\mathbf{r}_0\| < 1$ for a mixed state. This qubit serves as a probe experiencing the transformation $\rho_0 \mapsto \mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger$ defined by the unitary operator

$$\mathsf{U}_\xi = \exp\left(-i\frac{\xi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right), \tag{2}$$

where $\mathbf{n} = [n_x, n_y, n_z]^\top$ is a unit vector of $\mathbb{R}^3$. In Bloch representation, the transformation U_ξ of Eq. (2) amounts to a rotation of the initial Bloch vector $\mathbf{r}_0$ by the angle ξ around the axis $\mathbf{n}$ in $\mathbb{R}^3$. This produces the rotated Bloch vector $\mathbf{r}_1(\xi)$ characterizing the transformed state $\rho_1(\xi) = \mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger$. With a given axis $\mathbf{n}$, we seek to estimate the unknown phase angle ξ, through measurement of the transformed qubit. Such scenario bears important practical relevance for different areas of metrology, for instance atomic clocks, interferometry, or magnetometry, when operated at their quantum limits [29].

For more realistic conditions, we consider that the rotated qubit state $\rho_1(\xi)$, before it becomes accessible to measurement for estimating ξ, is affected by quantum noise or decoherence. The action of a quantum noise is generally representable by a completely positive trace-preserving superoperator $\mathcal{N}(\cdot)$ producing the noisy quantum state $\rho_\xi = \mathcal{N}(\rho_1(\xi))$. This is equivalent to a Bloch vector $\mathbf{r}_\xi$ specifying ρ_ξ supplied by the affine transformation [28, 30]

$$\mathbf{r}_\xi = A\mathbf{r}_1(\xi) + \mathbf{c}, \tag{3}$$

with A a 3×3 real matrix and $\mathbf{c}$ a vector of $\mathbb{R}^3$ together characterizing the quantum noise.

A quantum measurement is then performed on the noisy qubit in state ρ_ξ in order to estimate the unknown value of the phase ξ. From the outcomes of the measurement, having the status of realizations of a classical random variable, an estimator $\widehat{\xi}$ is devised for the phase ξ. After classical estimation theory [31, 32], it is known that any estimator $\widehat{\xi}$ for ξ is endowed with a mean-squared error $\langle(\widehat{\xi}-\xi)^2\rangle$ which is lower bounded by the Cramér–Rao bound involving the reciprocal of the classical Fisher information $F_c(\xi)$. The larger the Fisher information $F_c(\xi)$, the more efficient the estimation can be. The maximum likelihood estimator [32] is known to achieve the best efficiency dictated by the Cramér–Rao bound and Fisher information $F_c(\xi)$, at least in the asymptotic regime of a large number of independent measurements. The classical Fisher information $F_c(\xi)$ stands in this way as a fundamental metric quantifying the best achievable efficiency in estimation. Such Fisher information has previously been applied to characterize stochastic resonance effects in classical signal estimation affected by classical noise [4, 33–36]. Here, we will apply it to investigate a task of quantum estimation affected by quantum noise. In view of its status as a metric expressing the best estimation efficiency, it is relevant to identify the conditions of optimality maximizing the Fisher information $F_c(\xi)$. In this respect, there exists a general upper bound [37, 38] formed by the quantum Fisher information $F_q(\xi)$ which limits the classical Fisher information $F_c(\xi)$ by imposing $F_c(\xi) \le F_q(\xi)$. For estimation of the phase ξ from a noisy qubit in a state ρ_ξ specified by the Bloch vector $\mathbf{r}_\xi$ of Eq. (3), the quantum Fisher information $F_q(\xi)$ is expressible [39] as

$$F_q(\xi) = \frac{[(A\mathbf{r}_1 + \mathbf{c})A(\mathbf{n} \times \mathbf{r}_1)]^2}{1 - (A\mathbf{r}_1 + \mathbf{c})^2} + [A(\mathbf{n} \times \mathbf{r}_1)]^2. \tag{4}$$

In particular, the quantum Fisher information $F_q(\xi)$ is maximized [40] with no noise at $F_q^{\max}(\xi) = 1$ by a probe $\mathbf{r}_0$ orthogonal to the rotation axis $\mathbf{n}$. Therefore, for phase estimation from a qubit [40], the Fisher informations, quantum $F_q(\xi)$ and classical $F_c(\xi)$, never exceed the maximum $F_q^{\max}(\xi) = 1$.

The quantum Fisher information $F_q(\xi)$ of Eq. (4) is intrinsic to the quantum state ρ_ξ and its relation to the phase parameter ξ, and does not refer to any specific measurement protocol on the state ρ_ξ. By contrast, the classical Fisher information $F_c(\xi)$ characterizes an explicit measurement protocol which is required for effective estimation. A general quantum measurement on a qubit is represented by a generalized measurement or positive operator valued measure [28] which is defined by K measurement operators $\mathsf{M}_k = b_k \mathrm{I}_2 + \mathbf{a}_k \cdot \boldsymbol{\sigma}$ which are positive operators on $\mathcal{H}_2$ with $(\mathbf{a}_k, b_k)$ real satisfying $\sum_{k=1}^K \mathbf{a}_k = \mathbf{0}$ and $\sum_{k=1}^K b_k = 1$, so as to realize $\sum_{k=1}^K \mathsf{M}_k = \mathrm{I}_2$. Especially, for all k, one has $\|\mathbf{a}_k\| \le b_k \le 1 - \|\mathbf{a}_k\|$ which is required to ensure $0 \le \mathsf{M}_k \le \mathrm{I}_2$, this implying $\|\mathbf{a}_k\| \le 1/2$. For estimating the phase ξ, when such a generalized measurement is applied to the qubit in the state ρ_ξ from Eq. (3), the probability of obtaining outcome k is $P_k = \mathrm{tr}(\rho_\xi \mathsf{M}_k) = b_k + \mathbf{r}_\xi \mathbf{a}_k$, and the classical Fisher information $F_c(\xi) = \sum_k (\partial_\xi P_k)^2 / P_k$ results [39] as

$$F_c(\xi) = \sum_{k=1}^{K} \frac{[\mathbf{a}_k A(\mathbf{n} \times \mathbf{r}_1)]^2}{b_k + \mathbf{a}_k(A\mathbf{r}_1 + \mathbf{c})}. \tag{5}$$

For a qubit, a most accessible measurement consists in measuring a spin observable $\Omega = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}$ with eigenvalues $\pm\|\boldsymbol{\omega}\| = \pm 1$. This is equivalent to implementing a von Neumann projective measurement defined by the $K = 2$ measurement operators $\mathsf{M}_\pm = (\mathrm{I}_2 \pm \boldsymbol{\omega} \cdot \boldsymbol{\sigma})/2$, with $\|\boldsymbol{\omega}\| = 1$, forming two projectors on two orthogonal directions in $\mathcal{H}_2$ defined by the Bloch vectors $\pm\boldsymbol{\omega}$ of $\mathbb{R}^3$. In this circumstance, the classical Fisher information $F_c(\xi)$ of Eq. (5) reduces to

$$F_c(\xi) = \frac{[\boldsymbol{\omega} A(\mathbf{n} \times \mathbf{r}_1)]^2}{1 - [\boldsymbol{\omega}(A\mathbf{r}_1 + \mathbf{c})]^2}. \tag{6}$$

The classical Fisher information $F_c(\xi)$ of Eq. (5) or (6) will be taken as a metric of efficiency for the quantum estimation task. We will show that, as the level of quantum noise increases, this estimation efficiency assessed by $F_c(\xi)$ is not bound to always deteriorate. On the contrary, we will demonstrate the existence of conditions, where higher levels of quantum noise can induce enhanced estimation efficiency, interpretable as a quantum form of stochastic resonance. Such possibility of noise-enhanced efficiency in quantum estimation has been shown recently in [26, 27] with quantum thermal noise, also designated as generalized amplitude damping noise.

The level of the quantum thermal noise, in the studies of [26, 27], is represented by the temperature of the decohering environment acting as a thermal bath affecting the qubit. This level is controlled by the variable $\mathbf{c}$ occurring in Eqs. (3)–(6) with $\mathbf{c}$ carrying into the process the influence of the temperature of the thermal noise. In particular, the presence of a non-vanishing $\mathbf{c} \not\equiv \mathbf{0}$ is essential for the noise effects reported in [26, 27], and by varying $\mathbf{c}$ one can vary the noise temperature and show the possibility, in definite conditions, of enhanced estimation efficiency at increasing temperature.

An alternative and important class of noises for the qubit consists in unital quantum noises, defined by the invariance property $\mathcal{N}(\mathrm{I}_2) = \mathrm{I}_2$, and associated with $\mathbf{c} \equiv \mathbf{0}$ in Eq. (3). Unital quantum noises have specific interesting properties [41–43], and many common qubit noises belong to this class, such as bit-flip, or phase-flip, or depolarizing noises, or the whole family of Pauli noises we will consider below. For quantum estimation processes, stochastic resonance or effects of enhancement by noise have never been reported with unital quantum noises. Here, in the sequel, we will show regimes of operation where these become possible.

3. Analysis with Unital Noises

For both unital noises and for the quantum thermal noise of [26, 27], it is known from [40] that maximization of the quantum Fisher information $F_q(\xi)$ of Eq. (4) requires a probe prepared in a pure initial state ρ_0 specified by a unit Bloch vector $\mathbf{r}_0$ orthogonal to the rotation axis $\mathbf{n}$. Only for more complicated noise models, such as the squeezed generalized amplitude damping noise considered in [40], does one need an optimal (pure) input probe $\mathbf{r}_0$ not orthogonal to the rotation axis $\mathbf{n}$.

For maximum estimation efficiency, we therefore prepare the input probe in a pure state ρ_0 with a unit Bloch vector $\mathbf{r}_0$ orthogonal to the rotation axis $\mathbf{n}$ and denoted $\mathbf{r}_0 = \mathbf{n}_\perp$. With $\mathbf{n} = [n_x, n_y, n_z]^\top$ given, we choose for definiteness $\mathbf{n}_\perp = [-n_y, n_x, 0]^\top / \sqrt{n_x^2 + n_y^2}$ in the plane (Ox, Oy) of $\mathbb{R}^3$, fixing a (non-critical) origin to count the rotation angle ξ. In the orthonormal basis $\{\mathbf{n}, \mathbf{n}_\perp, \mathbf{n}'_\perp = \mathbf{n} \times \mathbf{n}_\perp\}$ of $\mathbb{R}^3$ one then has for the rotated Bloch vector

$$\mathbf{r}_1(\xi) = \cos(\xi)\mathbf{n}_\perp + \sin(\xi)\mathbf{n}'_\perp, \tag{7}$$

and

$$\mathbf{n} \times \mathbf{r}_1(\xi) = -\sin(\xi)\mathbf{n}_\perp + \cos(\xi)\mathbf{n}'_\perp. \tag{8}$$

For unital noises with $\mathbf{c} \equiv \mathbf{0}$, the Fisher information of Eq. (5) for the generalized measurement, follows as

$$F_c(\xi) = \sum_{k=1}^{K} \frac{(\mathbf{a}_k[-\sin(\xi)A\mathbf{n}_\perp + \cos(\xi)A\mathbf{n}'_\perp])^2}{b_k + \mathbf{a}_k[\cos(\xi)A\mathbf{n}_\perp + \sin(\xi)A\mathbf{n}'_\perp]}. \tag{9}$$

And upon measuring the observable $\Omega = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}$, the Fisher information of Eq. (6) follows as

$$F_c(\xi) = \frac{(\boldsymbol{\omega}[-\sin(\xi)A\mathbf{n}_\perp + \cos(\xi)A\mathbf{n}'_\perp])^2}{1 - (\boldsymbol{\omega}[\cos(\xi)A\mathbf{n}_\perp + \sin(\xi)A\mathbf{n}'_\perp])^2}. \tag{10}$$

In general, for a given measurement protocol specified by the $(\mathbf{a}_k, b_k)$ in Eq. (9) or by the observable $\Omega = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}$ in Eq. (10), the resulting Fisher information $F_c(\xi)$ in Eq. (9) or (10) is dependent on the specific value of the phase angle ξ being estimated. This indicates that the estimation efficiency assessed by $F_c(\xi)$ is dependent in particular on the orientation in $\mathbb{R}^3$ of the rotated Bloch vector $\mathbf{r}_1(\xi)$ relative to the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$. However, it is not generally possible to adjust the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ to match the rotated Bloch vector $\mathbf{r}_1(\xi)$ so as to maximize the Fisher information $F_c(\xi)$, since the value of the angle ξ under estimation is unknown. Instead, one usually has to operate with fixed measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ and cope with a ξ-dependent estimation efficiency. In such circumstances, with fixed $\mathbf{a}_k$ or $\boldsymbol{\omega}$, we will show that there exist ranges or conditions on the angle ξ where the estimation efficiency $F_c(\xi)$ improves as the level of noise increases.

For the demonstration, we consider an important class of unital quantum noises relevant to the qubit and formed by Pauli noises [44]. A Pauli noise acts through random applications of the four Pauli operators $\{\sigma_0 \equiv \mathrm{I}_2, \sigma_x, \sigma_y, \sigma_z\}$ which form an orthogonal basis for operators on $\mathcal{H}_2$. In Kraus representation [28, 44], a Pauli noise implements the quantum operation

$$\rho \mapsto \mathcal{N}(\rho) = \sum_{\ell=0,x,y,z} p_\ell \sigma_\ell \rho \sigma_\ell^\dagger, \tag{11}$$

with the $\{p_\ell\}$ a probability distribution. This leads in Eq. (3) to a transformation of the Bloch vectors with $\mathbf{c} \equiv \mathbf{0}$ and the matrix

$$A = \begin{bmatrix} \alpha_x & 0 & 0 \\ 0 & \alpha_y & 0 \\ 0 & 0 & \alpha_z \end{bmatrix}, \tag{12}$$

with the real scalar coefficients

$$\alpha_x = p_0 + p_x - p_y - p_z, \tag{13}$$

$$\alpha_y = p_0 - p_x + p_y - p_z, \tag{14}$$

$$\alpha_z = p_0 - p_x - p_y + p_z, \tag{15}$$

referring to the frame (Ox, Oy, Oz) of $\mathbb{R}^3$. For the qubit, the class of Eqs. (11)–(12) in particular contains such important Pauli noises as the bit-flip, the phase-flip, the bit-phase-flip, the depolarizing noises [28, 44]. The three parameters α_ℓ in Eqs. (13)–(15) are compression factors satisfying $0 \le |\alpha_\ell| \le 1$ for all $\ell \in \{x, y, z\}$, to guarantee that the Bloch ball of valid Bloch vectors is mapped into itself. In this way, the transformation $\mathbf{r} \mapsto A\mathbf{r}$ by the noise compresses the Bloch vectors in $\mathbb{R}^3$. An increasing level of noise corresponds to a more pronounced compression. The maximum level of compression would occur as $\alpha_x = \alpha_y = \alpha_z = 0$, corresponding to a noisy state with a null Bloch vector $\mathbf{0}$ characterizing the maximally mixed qubit state $\mathrm{I}_2/2$ identifiable with the maximally noisy state. As the level of noise increases, the variation of the Fisher information $F_c(\xi)$ from Eq. (9) or (10) is then controlled by the geometric configuration of the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ in relation to the rotation axis $\mathbf{n}$ and the compression axes of the Pauli noise in $\mathbb{R}^3$. In the sequel we will show that, as the level of Pauli noise increases, the Fisher information quantifying the estimation efficiency, is not bound to monotonically deteriorate, but that on the contrary it can experience non-monotonic variations, where higher noise levels prove more favorable to estimation.

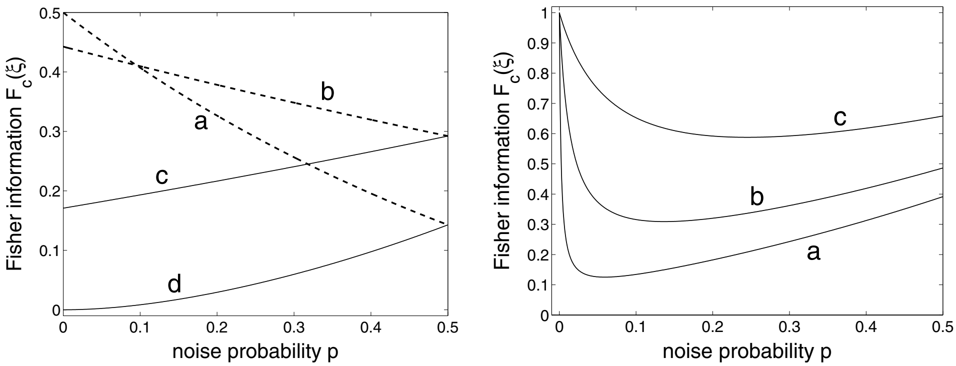
For the sequel, in the frame (Ox, Oy, Oz) of $\mathbb{R}^3$ the rotation axis $\mathbf{n} = [n_x, n_y, n_z]^\top$ can also be identified by the coelevation angle $\theta_n \in [0, \pi]$ relative to the axis Oz and the azimuth angle $\varphi_n \in [0, 2\pi[$ around Oz, to give $\mathbf{n} = [n_x = \sin(\theta_n)\cos(\varphi_n), n_y = \sin(\theta_n)\sin(\varphi_n), n_z = \cos(\theta_n)]^\top$. In the orthonormal basis $\{\mathbf{n}, \mathbf{n}_\perp, \mathbf{n}'_\perp\}$ of $\mathbb{R}^3$, it is convenient to identify the unit measurement vector $\boldsymbol{\omega}$ by the coelevation angle $\theta_\omega \in [0, \pi]$ relative to the axis $\mathbf{n}$ and the azimuth angle $\varphi_\omega \in [0, 2\pi[$ around $\mathbf{n}$.

4. Phase-Flip Noise

As a first example, we consider an important Pauli noise for the qubit which is the phase-flip noise. Phase-flip noise is able to represent any random phase shift experienced by a qubit as it scatters with no loss of energy, as with a traveling photon for instance [28]. Phase-flip noise is obtained in Eq. (11) with the four probabilities $p_0 = 1 - p$, also $p_x = p_y = 0$ and $p_z = p$. This leads in Eqs. (13)–(15) to the compression factors $\alpha_x = \alpha_y = 1 - 2p$ and $\alpha_z = 1$, indicating that the phase-flip noise compresses the Bloch vector of a qubit only along the two directions (Ox, Oy) of $\mathbb{R}^3$. At $p = 0$, the noise matrix A in Eq. (12) is the identity of $\mathbb{R}^3$ and it characterizes the noise-free situation. As the level of noise increases, the probability p rises above 0, and the compression along (Ox, Oy) gradually increases to get maximized at $\alpha_x = \alpha_y = 0$ when $p = 1/2$.

We now, examine the variation of the Fisher information $F_c(\xi)$ from Eq. (9) or (10) characterizing the efficiency of phase estimation from the noisy qubit, when the level of phase-flip noise increases from no noise at $p = 0$ to maximum noise at $p = 1/2$. The variation of the Fisher information $F_c(\xi)$ from Eq. (9) or (10) is here controlled by the geometric configuration of the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ in relation to the rotation axis $\mathbf{n}$ and the two compression axes (Ox, Oy) of the phase-flip noise.

Figure 1 illustrates the three regimes of variation that are found accessible to the Fisher information. Figure 1(A) in (a) and (b) shows the standard expected regime, where the Fisher information $F_c(\xi)$ decreases as the noise probability p increases, manifesting the common situation of an estimation efficiency which deteriorates as the level of noise increases. By contrast, Fig. 1(A) in (c) and (d) shows the possibility



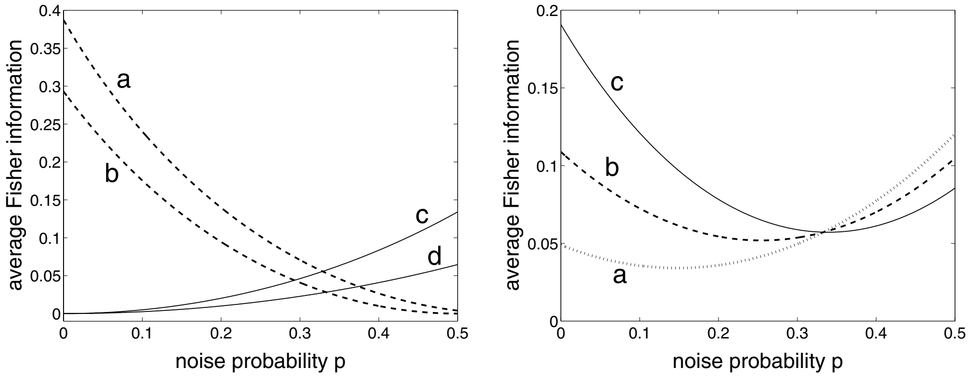
(A) (B)

Fig. 1. Fisher information $F_c(\xi)$ from Eq. (10) as a function of the probability p of the phase-flip noise, with a rotation axis $\mathbf{n} = [1, 0, 0]^\top$, for the angle $\xi = \pi/4$. The measurement vector $\boldsymbol{\omega}$ is in panel A with $\theta_\omega = \pi/4$ and $\varphi_\omega = 0.75\pi$ (a), 0.60π (b), 0.40π (c) and 0.25π (d); in panel B with $\theta_\omega = \pi/2$ and $\varphi_\omega = 0.27\pi$ (a), 0.30π (b) and 0.35π (c).

of a non-standard regime, where the Fisher information $F_c(\xi)$ increases as the noise level p increases. In such configurations, higher levels of noise, when accessible, are more favorable to the estimation efficiency. In addition, Fig. 1(B) shows another non-standard regime, with non-monotonic variations of the Fisher information $F_c(\xi)$. In Fig. 1(B), depending on the conditions, there exists a critical value of the noise probability $p \in [0, 1/2]$, where the Fisher information $F_c(\xi)$ gets minimized. This identifies critical noise levels that are specially detrimental to the estimation efficiency; and operating below, but also above, such critical noise levels is more favorable to estimation. More efficient information processing with increasing noise is identifiable with stochastic resonance effects. In this respect, the form of Fig. 1(B) with finite noise levels that are specifically detrimental, would rather characterize regimes of stochastic antiresonance, as also reported in [27].

Figure 1 presents various regimes of variation of the Fisher information $F_c(\xi)$ of Eq. (10) characterizing the estimation efficiency in the range around $\xi = \pi/4$. Such behaviors of $F_c(\xi)$ are generic and can be found analogously in other ranges of the parameter ξ. In addition, it is possible to obtain a ξ-independent characterization of the estimation efficiency, and show that it still has access to non-standard and non-monotonic regimes of variation with the level of noise. This can be accomplished by averaging the Fisher information $F_c(\xi)$ from Eq. (9) or (10) over the unknown angle ξ taken uniform in $[0, 2\pi[$. This uniform probability distribution for the angle ξ is however not critical, and stands as a reasonable assumption when strictly no prior knowledge is available for the range of ξ to be estimated. Such a ξ-averaged Fisher information obtained from $F_c(\xi)$ of Eq. (10) is presented in Fig. 2.

The ξ-averaged Fisher information in Fig. 2 illustrates that the three regimes of variation are preserved for the average estimation efficiency. Especially, in certain



(A) (B)

Fig. 2. Fisher information $F_c(\xi)$ from Eq. (10) averaged over the angle ξ, as a function of the probability p of the phase-flip noise. In panel A for (a) and (b) the rotation axis $\mathbf{n}$ is with $\theta_n = \pi/4$ and $\varphi_n = 0$, the measurement vector $\boldsymbol{\omega}$ is with $\varphi_\omega = 1.5\pi$, and $\theta_\omega = 0.29\pi$ (a), $\theta_\omega = 0.25\pi$ (b); for (c) and (d) $\theta_\omega = \varphi_\omega = 0$, $\varphi_n = 0$ and $\theta_n = \pi/4$ (c), $\theta_n = \pi/8$ (d). In panel B the rotation axis $\mathbf{n}$ is with $\theta_n = \pi/4$ and $\varphi_n = 0$, the measurement vector $\boldsymbol{\omega}$ is with $\varphi_\omega = 0$, and $\theta_\omega = 0.1\pi$ (a), $\theta_\omega = 0.15\pi$ (b) and $\theta_\omega = 0.2\pi$ (c).

conditions, estimation can be more efficient, on average, when operating at higher noise level.

The phase-flip noise tested here is an important unital noise for the qubit, which shows, as indicated, an invariant axis — the Oz axis — in the compression by the noise matrix A of Eq. (12). The bit-flip noise and the bit-phase-flip noise are two other important unital noises for the qubit, which analogously show an invariant axis in the compression — the Ox and the Oy axis, respectively. Therefore, similar non-standard and non-monotonic variations or stochastic resonance effects also hold equivalently for the estimation efficiency in the presence of bit-flip and bit-phase-flip noise. Moreover, an invariant axis of the noise is not a critical property required for such stochastic resonance to occur, as we shall see next.

5. Three Axes of Compression

We now use a Pauli noise according to Eqs. (11)–(15) that is able to compress the Bloch vectors in the three directions of $\mathbb{R}^3$. To have a control on the noise level or noise compression through a single scalar parameter p, we choose in Eq. (11) the four probabilities $p_0 = 1 - p$, $p_x = 0.5p$, $p_y = 0.3p$ and $p_z = 0.2p$. This leads in Eqs. (13)–(15) to the compression factors $\alpha_x = 1 - p$, $\alpha_y = 1 - 1.4p$ and $\alpha_z = 1 - 1.6p$ expressing how the Pauli noise compresses the Bloch vector of a qubit state along the three directions (Ox, Oy, Oz) of $\mathbb{R}^3$. At $p = 0$ with no compression is the noise-free situation. As the level of noise increases, the probability p rises above 0, and the most pronounced compression occurs along the Oz direction and gets maximized at $\alpha_z = 0$ when $p = 1/1.6 = 0.625$. With such Pauli noise with three effective directions of

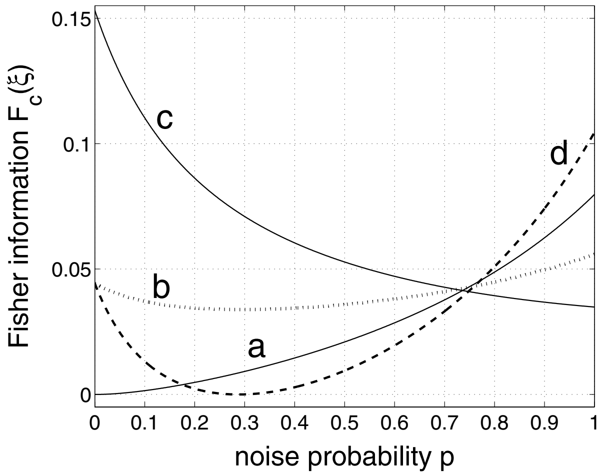


Fig. 3. Fisher information $F_c(\xi)$ from Eq. (10) as a function of the probability p of the Pauli noise, with a rotation axis $\mathbf{n}$ of $\theta_n = 0.5\pi$ and $\varphi_n = 0.6\pi$, for the angle $\xi = \pi/4$. The measurement vector $\boldsymbol{\omega}$ is with $\theta_\omega = 0.7\pi$, and $\varphi_\omega = 0.15\pi$ (a), $\varphi_\omega = 0.2\pi$ (b), $\varphi_\omega = 0.25\pi$ (c) and $\varphi_\omega = 0.3\pi$ (d).

compression in $\mathbb{R}^3$, we show that non-monotonic and non-standard variations of the efficiency in estimation are still possible as the level of noise increases. The variation of the Fisher information $F_c(\xi)$ from Eq. (9) or (10) will depend as before on the geometric configuration of the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ in relation to the rotation axis $\mathbf{n}$ and the three compression axes of the Pauli noise in $\mathbb{R}^3$.

Figure 3 shows various configurations of variation of the Fisher information $F_c(\xi)$ of Eq. (10), characterizing the estimation efficiency in the range around $\xi = \pi/4$ when measuring the observable $\Omega = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}$.

In Fig. 3, depending on the configuration of the measurement vector $\boldsymbol{\omega}$ in relation to the rotation axis $\mathbf{n}$, the same three regimes of variation of the Fisher information $F_c(\xi)$ as in Sec. 4 are observed, as the level p of the Pauli noise increases. For instance, Fig. 3(c) shows the standard situation where $F_c(\xi)$ monotonically decreases as p increases, identifying an efficiency in estimation which deteriorates as the level of noise increases. By contrast, Figs. 3(b) and 3(d) present non-monotonic variations of the efficiency $F_c(\xi)$ as the noise level p increases; this again reveals the existence of a critical noise level, around $p \approx 0.3$ in Fig. 3, where $F_c(\xi)$ gets minimized. Such a finite critical noise level is specially detrimental to the estimation efficiency, and smaller, but also larger noise levels, are more favorable to estimation, as illustrated in Fig. 3. Finally, Fig. 3(a) shows the possibility of a Fisher information $F_c(\xi)$ monotonically increasing as the noise level p grows, indicating here that raising the level of noise is always beneficial to the estimation efficiency.

In addition, Fig. 4 shows that the non-standard and non-monotonic regimes of variation are still accessible for the ξ-averaged Fisher information, when $F_c(\xi)$ from Eq. (10) is averaged over the angle ξ taken uniform in $[0, 2\pi[$.

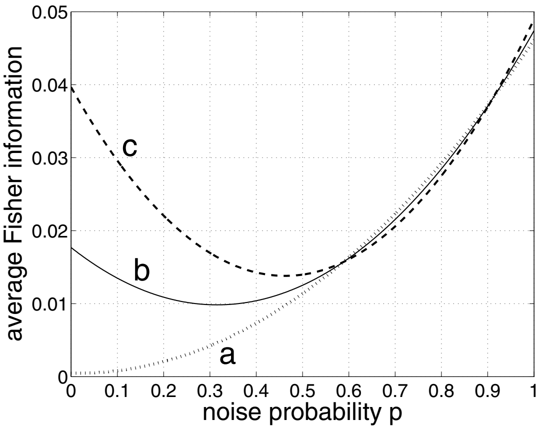


Fig. 4. Fisher information $F_c(\xi)$ from Eq. (10) averaged over the angle ξ, as a function of the probability p of the Pauli noise, with a rotation axis $\mathbf{n}$ of $\theta_n = \pi/4$ and $\varphi_n = 0$. The measurement vector $\boldsymbol{\omega}$ is with $\varphi_\omega = 0$, and $\theta_\omega = 0.01\pi$ (a), $\theta_\omega = 0.06\pi$ (b) and $\theta_\omega = 0.09\pi$ (c).

As the noise level p increases, Figs. 4(b) and 4(c) show non-monotonic anti-resonant variations, while Fig. 4(a) shows an increasing variation, for the ξ-averaged Fisher information. In such configurations, the non-standard variations of the ξ-averaged Fisher information indicate that benefit for estimation by increasing the level of noise is in this way globally accessible, and not restricted to specific values or ranges of the phase angle ξ being estimated.

One can especially note in Fig. 4 that the levels involved for the Fisher information are relatively small, compared to the overall maximum of 1 indicated in Sec. 2 for phase estimation from a qubit. This overall maximum Fisher information of 1 can only be reached at no noise, with an optimal input probe $\mathbf{r}_0$ and optimal measurement vectors $\boldsymbol{\omega}$ or $\mathbf{a}_k$ matched to the rotation axis $\mathbf{n}$. The conditions of Fig. 4 for instance are very far from these optimality conditions. Such situation may be imposed by the external context, that would force to operate at a low Fisher information. Nevertheless, the Fisher information can be enhanced by increasing the level of noise in definite configurations as we show; and although the resulting improvement itself may also be small, we find it is feasible in principle. Intrinsically low levels of the Fisher information can also be compensated by increasing the number of repetitions of the estimation process. The estimation process involving measurement of the qubit can be repeated on many independent qubits, independent photons or electrons for instance, that may be physically accessible in number. In such conditions the Fisher information is additive and can be raised to larger levels.

6. Discussion

We have considered the task of estimating the phase ξ of a qubit affected by quantum unital Pauli noise. The estimation efficiency is assessed by the classical Fisher information $F_c(\xi)$, a fundamental metric that quantifies the best efficiency limiting the performance of any conceivable estimators, and achievable by the maximum likelihood estimator in definite (usually asymptotic) conditions. When a large number N of estimation experiments are repeated on independent probing qubits, the mean-squared error of the maximum likelihood estimator reaches the Cramér–Rao lower bound and therefore the best efficiency dictated by the Fisher information $F_c(\xi)$ analyzed here. At small number N of repetitions, the Cramér–Rao bound may however not be tight, and other bounds are known with the potential ability to come closer to the mean-squared estimation error. This is for instance the case with Ziv-Zakai error bounds [45], which have recently been extended to the quantum domain [46], and could also be tested for stochastic resonance effects, quantum or classical.

Here, with the generic classical Fisher information metric, we have shown that, as the level of noise increases, the estimation efficiency does not necessarily deteriorate monotonically, but that on the contrary it can experience non-standard and non-monotonic regimes of variation where higher levels of noise can prove more favorable to estimation. These non-standard and non-monotonic regimes of variation of the Fisher information were reported here in Figs. 1–4 when measuring a spin observable $\Omega = \boldsymbol{\omega} \cdot \boldsymbol{\sigma}$. Yet, this condition is by no means critical, and similar regimes of stochastic resonance can also be obtained when using generalized measurements for estimation, and would follow from Eq. (9) instead of Eq. (10).

An essential aspect is that such stochastic resonance effects in quantum estimation are shown feasible here for the first time with the important class of Pauli unital noises. The variation of the Fisher information $F_c(\xi)$ from Eq. (9) or (10) is controlled by the geometric configuration of the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ in relation to the rotation axis $\mathbf{n}$ and the compression axes of the Pauli noise in $\mathbb{R}^3$. In practical quantum metrology, the geometry of these vectors may be constrained or imposed to some extent, and not fully adjustable at will. In given configurations of them, identifiable through the methodology developed here, we have shown that increasing the level of noise can be an option to contribute to enhancing the estimation efficiency.

Beyond the proof of feasibility in principle of regimes of improvement by noise, in practice to benefit from such possibility implies that the level of noise, quantified in Figs. 1–4 by the probability p, can be known to some extent, to appreciate whether its increase is profitable and have control on its tuning when implemented. For this purpose, estimation techniques can be employed [47–49], which to some extent rest on the same principles as those reported in Sec. 2. They can be implemented prior to the phase estimation task to evaluate the level of noise. Efficient states for a probing qubit can be devised to maximize the performance in estimating the level of noise, with here also the possibility in principle of reaching high accuracy by repeating the process with a large number of independent probing qubits.

For the qubit, the Pauli noises we tested, characterized by a diagonal matrix A in Eq. (12), can be considered as a generic form for unital noises. Any matrix A in Eq. (3) characterizing a generic unital noise, is a real matrix that maps the Bloch ball of $\mathbb{R}^3$ into itself. By the polar decomposition [28], it can always be expressed as $A = RS$, where S is a real symmetric matrix and R a real unitary matrix. The symmetric matrix S has a diagonal form similar to Eq. (12) that expresses the compression of the Bloch vectors by the unital noise occurring in $\mathbb{R}^3$ along the three directions set by the three orthogonal eigenvectors of S. By appropriately defining the computational orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_2$ for the qubit, it is always possible to refer the Bloch ball in $\mathbb{R}^3$ to the orthogonal frame formed by the three eigenvectors of S, so that S will be in diagonal form. Next, the real unitary matrix R implements a rotation in $\mathbb{R}^3$. Such a rotation conserves the norms and does not compress the Bloch vectors. It maps pure quantum states into pure quantum states, and as such it represents a coherent part (rather than a genuine noisy part) in the evolution of the quantum states by the unital noise, and it does not affect the possibility of stochastic resonance regimes. The same regimes of variation of the Fisher information $F_c(\xi)$ as reported in Secs. 3–5 for Pauli unital noises, can be expected to hold equivalently for arbitrary unital noises. It will just be a matter of interpreting the geometric configuration in $\mathbb{R}^3$ of the measurement vectors $\mathbf{a}_k$ or $\boldsymbol{\omega}$ in relation to the rotation axis $\mathbf{n}$ and the compression axes of the unital noise, in presence of the rotation by R. In this respect, with any generic unital noise, we expect to observe in equivalent geometric configurations, all the regimes of variation of $F_c(\xi)$ and the stochastic resonance effects that were found in Secs. 3–5 with the Pauli unital noises of Eq. (12).

The present results attest in novel and significant conditions and configurations that quantum noise or decoherence is not always a nuisance, but can sometimes prove beneficial to information processing. This provides better appreciation to cope with quantum noise or decoherence for quantum information processing.

References

[1] H. Chen, L. R. Varshney and P. K. Varshney, Noise-enhanced information systems, *Proc. IEEE* **102** (2014) 1607–1621.

[2] M. D. McDonnell, N. G. Stocks, C. E. M. Pearce and D. Abbott, *Stochastic Resonance: From Suprathreshold Stochastic Resonance to Stochastic Signal Quantization* (Cambridge University Press, Cambridge, 2008).

[3] F. Chapeau-Blondeau and D. Rousseau, Noise improvements in stochastic resonance: From signal amplification to optimal detection, *Fluctuation Noise Lett.* **2** (2002) L221–L233.

[4] M. D. McDonnell, N. G. Stocks, C. E. M. Pearce and D. Abbott, Optimal information transmission in nonlinear arrays through suprathreshold stochastic resonance, *Phys. Lett. A* **352** (2006) 183–189.

[5] T. Hoch, G. Wenning and K. Obermayer, Optimal noise-aided signal transmission through populations of neurons, *Phys. Rev. E* **68** (2003) 011911, 1–11.

[6] L. B. Kish, G. P. Harmer and D. Abbott, Information transfer rate of neurons: Stochastic resonance of Shannon's information channel capacity, *Fluctuation Noise Lett.* **1** (2001) L13–L19.

[7] F. Chapeau-Blondeau, Noise-assisted propagation over a nonlinear line of threshold elements, *Electron. Lett.* **35** (1999) 1055–1056.

[8] H. Chen and P. K. Varshney, Theory of stochastic resonance effect in signal detection: Part II — Variable detectors, *IEEE Trans. Signal Process.* **56** (2008) 5031–5041.
[9] S. Zozor and P. O. Amblard, On the use of stochastic resonance in sine detection, *Signal Process.* **82** (2002) 353–367.
[10] F. Chapeau-Blondeau, Stochastic resonance and optimal detection of pulse trains by threshold devices, *Digital Signal Process.* **9** (1999) 162–177.
[11] A. Patel and B. Kosko, Optimal mean-square noise benefits in quantizer-array linear estimation, *IEEE Signal Process. Lett.* **17** (2010) 1005–1009.
[12] H. Chen, P. K. Varshney and J. H. Michels, Noise enhanced parameter estimation, *IEEE Trans. Signal Process.* **56** (2008) 5074–5081.
[13] F. Chapeau-Blondeau, Noise-aided nonlinear Bayesian estimation, *Phys. Rev. E* **66** (2002) 032101.
[14] A. Delahaies, F. Chapeau-Blondeau, D. Rousseau and F. Franconi, Tuning the noise in magnetic resonance imaging to maximize nonlinear information transmission, *Fluctuation Noise Lett.* **12** (2013) 1350005.
[15] Y. Yang, Z. Jiang, B. Xu and D. W. Repperger, An investigation of two-dimensional parameter-induced stochastic resonance and applications in nonlinear image processing, *J. Phys. A* **42** (2009) 145207.
[16] S. Blanchard, D. Rousseau, D. Gindre and F. Chapeau-Blondeau, Constructive action of the speckle noise in a coherent imaging system, *Opt. Lett.* **32** (2007) 1983–1985.
[17] A. Histace and D. Rousseau, Constructive action of noise for impulsive noise removal in scalar images, *Electron. Lett.* **42** (2006) 393–395.
[18] B. Vacchini, Quantum noise from reduced dynamics, *Fluctuation Noise Lett.* **15** (2016) 1640003.
[19] B. Ye and L. Qiu, $1/f$ noise in Ising quantum computers, *Fluctuation Noise Lett.* **13** (2014) 1450006.
[20] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt and R. J. Schoelkopf, Introduction to quantum noise, measurement, and amplification, *Rev. Mod. Phys.* **82** (2010) 1155–1208.
[21] J. J. L. Ting, Stochastic resonance for quantum channels, *Phys. Rev. E* **59** (1999) 2801–2803.
[22] J. J. L. Ting, Noise effects for the depolarizing channel, *Phys. Lett. A* **259** (1999) 349–354.
[23] G. Bowen and S. Mancini, Stochastic resonance effects in quantum channels, *Phys. Lett. A* **352** (2006) 272–275.
[24] F. Chapeau-Blondeau, Quantum state discrimination and enhancement by noise, *Phys. Lett. A* **378** (2014) 2128–2136.
[25] N. Gillard, E. Belin and F. Chapeau-Blondeau, Qubit state detection and enhancement by quantum thermal noise, *Electron. Lett.* **54** (2018) 38–39.
[26] F. Chapeau-Blondeau, Qubit state estimation and enhancement by quantum thermal noise, *Electron. Lett.* **51** (2015) 1673–1675.
[27] N. Gillard, E. Belin and F. Chapeau-Blondeau, Stochastic antiresonance in qubit phase estimation with quantum thermal noise, *Phys. Lett. A* **381** (2017) 2621–2628.
[28] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[29] V. Giovannetti, S. Lloyd and L. Maccone, Advances in quantum metrology, *Nature Photonic.* **5** (2011) 222–229.
[30] F. Chapeau-Blondeau, Optimization of quantum states for signaling across an arbitrary qubit noise channel with minimum-error detection, *IEEE Trans. Inf. Theo.* **61** (2015) 4500–4510.

[31] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, New York, 1991).

[32] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory* (Prentice Hall, Englewood Cliffs, 1993).

[33] P. E. Greenwood, L. M. Ward and W. Wefelmeyer, Statistical analysis of stochastic resonance in a simple setting, *Phys. Rev. E* **60** (1999) 4687–4695.

[34] D. Rousseau, F. Duan and F. Chapeau-Blondeau, Suprathreshold stochastic resonance and noise-enhanced Fisher information in arrays of threshold devices, *Phys. Rev. E* **68** (2003) 031107.

[35] Y. Wang and L. Wu, Stochastic resonance and noise-enhanced Fisher information, *Fluctuation Noise Lett.* **5** (2005) L435–L442.

[36] F. Duan, F. Chapeau-Blondeau and D. Abbott, Exploring weak-periodic-signal stochastic resonance in locally optimal processors with a Fisher information metric, *Signal Process.* **92** (2012) 3049–3055.

[37] C. W. Helstrom, The minimum variance of estimates in quantum signal detection, *IEEE Trans. Inf. Theo.* **14** (1968) 234–242.

[38] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72** (1994) 3439–3443.

[39] F. Chapeau-Blondeau, Optimizing qubit phase estimation, *Phys. Rev. A* **94** (2016) 022334.

[40] F. Chapeau-Blondeau, Optimized probing states for qubit phase estimation with general quantum noise, *Phys. Rev. A* **91** (2015) 052310.

[41] C. King and M. B. Ruskai, Minimal entropy of states emerging from noisy quantum channels, *IEEE Trans. Inf. Theo.* **47** (2001) 192–209.

[42] C. King, Additivity for unital qubit channels, *J. Math. Phys.* **43** (2002) 4641–4653.

[43] A. Shaham, A. Halevy, L. Dovrat, E. Megidish and H. S. Eisenberg, Entanglement dynamics in the presence of controlled unital noise, *Sci. Rep.* **5** (2015) 10796.

[44] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, 2013).

[45] J. Ziv and M. Zakai, Some lower bounds on signal parameter estimation, *IEEE Trans. Inf. Theo.* **15** (1969) 386–391.

[46] M. Tsang, Ziv-Zakai error bounds for quantum parameter estimation, *Phys. Rev. Lett.* **108** (2012) 230401.

[47] A. Chiuri, V. Rosati, G. Vallone, S. Pádua, H. Imai, S. Giacomini, C. Macchiavello and P. Mataloni, Experimental realization of optimal noise estimation for a general Pauli channel, *Phys. Rev. Lett.* **107** (2011) 253602.

[48] L. Ruppert, D. Virosztek and K. Hangos, Optimal parameter estimation of Pauli channels, *J. Phys. A* **45** (2012) 265305.

[49] D. Collins, Mixed-state Pauli-channel parameter estimation, *Phys. Rev. A* **87** (2013) 032301.