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Optimized entanglement for quantum parameter estimation from noisy qubits

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For parameter estimation from an N-component composite quantum system, it is known that a separable preparation leads to a mean-squared estimation error scaling as $1/N$ while an entangled preparation can in some conditions afford a smaller error with $1/N^2$ scaling. This quantum superefficiency is however very fragile to noise or decoherence, and typically disappears with any small amount of random noise asymptotically at large N. To complement this asymptotic characterization, here we characterize how the estimation efficiency evolves as a function of the size N of the entangled system and its degree of entanglement. We address a generic situation of qubit phase estimation, also meaningful for frequency estimation. Decoherence is represented by the broad class of noises commuting with the phase rotation, which includes depolarizing, phase-flip and thermal quantum noises. In these general conditions, explicit expressions are derived for the quantum Fisher information quantifying the ultimate achievable efficiency for estimation. We confront at any size N the efficiency of the optimal separable preparation to that of an entangled preparation with arbitrary degree of entanglement. We exhibit the $1/N^2$ superefficiency with no noise, and prove its asymptotic disappearance at large N for any nonvanishing noise configuration. For maximizing the estimation efficiency, we characterize the existence of an optimum N_{opt} of the size of the entangled system along with an optimal degree of entanglement. For nonunital noises, maximum efficiency is usually obtained at partial entanglement. Grouping the N qubits into independent blocks formed of N_{opt} entangled qubits restores at large N a nonvanishing efficiency that can improve over that of N independent qubits optimally prepared. Also, one inactive qubit included in the entangled probe sometimes stands as the most efficient setting for estimation. The results further attest with new characterizations the subtlety of entanglement for quantum information in the presence of noise, showing that when entanglement is beneficial, maximum efficiency is not necessarily obtained by maximum entanglement but instead by a controlled degree and finite optimal amount of it.

Keywords: Quantum estimation; entanglement; quantum noise; quantum information.

1. Introduction

Quantum information processing can exploit resources nonexisting classically and offer unparalleled means for enhanced efficiency. This is the case with quantum entanglement, and in this paper we further investigate some important aspects of entanglement affording specific benefit for quantum estimation. Typically, in a task of quantum parameter estimation,[1–7] one has access to a quantum system with a state ρ_ξ carrying the dependence with the unknown parameter ξ to be estimated. After measuring the quantum system in state ρ_ξ, the measurement outcomes are processed by means of an estimator $\hat{\xi}$ to infer a value for the unknown parameter ξ. Following any measurement protocol, any conceivable estimator $\hat{\xi}$ for ξ is endowed with a mean-squared error $\langle(\hat{\xi} - \xi)^2\rangle$ which is lower bounded by the Cramér–Rao bound involving the reciprocal of the classical Fisher information $F_c(\xi)$.[8,9] Estimators are known, such as the maximum likelihood estimator, that can reach the Cramér–Rao bound in definite (usually asymptotic) conditions. Higher Fisher information $F_c(\xi)$ generally entails higher efficiency in estimation, and one has then the faculty to select the measurement protocol so as to maximize $F_c(\xi)$. In this respect, there is a fundamental upper bound[2,10] provided by the quantum Fisher information $F_q(\xi)$ which sets a limit to the classical Fisher information $F_c(\xi)$, i.e. fixing $F_c(\xi) \leq F_q(\xi)$. In turn, constructive methodologies (usually adaptive) exist[3,11–15] yielding a measurement protocol reaching $F_c(\xi) = F_q(\xi)$. The quantum Fisher information $F_q(\xi)$ has thus the status of a fundamental metric characterizing the ultimate best efficiency achievable in quantum estimation, and we shall use it as such in this report.

A definite measurement operating on a single copy of the quantum system in state ρ_ξ can be repeated N times on N independent copies of the quantum system. In this situation of independent state preparation and independent measurement, the quantum Fisher information is additive[16] and amounts to $NF_q(\xi)$. This is associated with a mean-squared error $\langle(\hat{\xi} - \xi)^2\rangle$ evolving as $1/N$, forming the standard or shot-noise limit of the error.[17] There however exists a possibility of improving over such $1/N$ efficiency, by exploiting the specifically quantum property of entanglement. With the N-component composite quantum system, exploiting entanglement for estimation can lead to a smaller mean-squared error scaling as $1/N^2$ in definite conditions, and forming the Heisenberg limit or superefficiency property.[17–19] Various schemes with separable preparation or measurement versus entangled preparation or measurement are reviewed in Refs. 17, 19–23, to report the $1/N^2$ superefficiency resulting from entanglement.

There are however different forms of entanglement, and different forms for the parametric dependence on ξ, and not all behave equally in relation to the possible improvement by the $1/N^2$ scaling. When the parameter ξ to be estimated characterizes a unitary transformation U_ξ such as a phase or frequency, it is known that the estimation can benefit from the $1/N^2$ superefficiency by exploiting entanglement.[16,17,20] It has however been found that this superefficiency is very fragile when

exposed to decoherence or quantum noise. Quantum estimation, initiated in noise-free conditions, has more recently been investigated in the presence of noise or decoherence.[20,22,24,25] It has been realized that for estimating a parameter ξ of a unitary U_ξ, any small amount of depolarizing noise is sufficient, for the efficiency, to ruin the $1/N^2$ scaling of an entangled process, in the asymptotic limit of large N, and return it to the $1/N$ scaling of the separable process.[16,20,26]

To complement this asymptotic characterization, in the present paper we characterize how the estimation efficiency evolves as a function of the size N of the entangled system and also of the degree of entanglement. As an important generic estimation task, we consider estimating the phase ξ of a unitary transformation U_ξ acting on a qubit, yet with a characterization which is meaningful as well for frequency estimation. Decoherence on the qubit is represented by the broad class of noises commuting with the phase rotation, which includes depolarizing, phase-flip, and thermal quantum noises. The efficiency of estimation from noisy qubits is assessed by the quantum Fisher information $F_q(\xi)$. For this generic estimation task, along the same line, previous studies examined the case of separable qubits[27] or the effect of an entangled qubit pair in the presence of depolarizing noise.[28] Now in the present report, estimation takes place on any number N of qubits in the presence of a broader class of quantum noises. The quantum Fisher information $F_q(\xi)$ is derived for any system size N and degree of entanglement; $F_q(\xi)$ can then be analyzed in the various noise configurations and, in particular, it allows us to identify the existence of optimal size and degree of entanglement maximizing the estimation efficiency.

2. The Estimation Task

A quantum system in a D-dimensional Hilbert space $\mathcal{H}_D$ has its state represented by the density operator ρ_ξ dependent upon an unknown parameter ξ. When measuring ρ_ξ for estimating ξ, the overall best efficiency is controlled by the quantum Fisher information $F_q(\xi)$ contained in the density operator ρ_ξ about the parameter ξ.[2,10] By referring to the eigendecomposition of ρ_ξ in its orthonormal eigenbasis $\rho_\xi = \sum_{j=1}^{D} \lambda_j |\lambda_j\rangle\langle\lambda_j|$, one has access to the expression[2,6,29]

$$F_q(\xi) = 2 \sum_{j,k} \frac{|\langle\lambda_j|\partial_\xi\rho_\xi|\lambda_k\rangle|^2}{\lambda_j + \lambda_k}, \tag{1}$$

where the sums in Eq. (1) include all terms corresponding to eigenvalues $\lambda_j + \lambda_k \neq 0$.

For the qubit with two-dimensional Hilbert space $\mathcal{H}_2$, which is a fundamental system of quantum information, we consider the essential task of estimating the phase ξ of a generic transformation defined by the unitary operator

$$U_\xi = \exp\left(-i\frac{\xi}{2}\mathbf{n}\cdot\boldsymbol{\sigma}\right), \tag{2}$$

where $\mathbf{n}$ is a real unit vector of $\mathbb{R}^3$ which is given, and $\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$ is a formal vector assembling the three (unitary Hermitian) Pauli operators $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$ and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. For the estimation, an initial qubit acting as a probe is prepared in the quantum state with density operator ρ_0 experiencing the transformation $\rho_0 \mapsto \mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger$. For more realistic conditions, we consider that the ξ-dependent transformed state $\mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger$ is not directly accessible to measurement for estimating ξ, but after the action of a quantum noise represented by the completely positive trace-preserving linear superoperator $\mathcal{N}(\cdot)$. The ξ-dependent noisy state $\rho_\xi = \mathcal{N}(\mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger)$, when used for estimating ξ, is then associated with the overall best efficiency controlled by the quantum Fisher information $F_q(\xi)$ of Eq. (1).

For the qubit, it is convenient to refer to the Bloch representation,[30,31] where the initial state ρ_0 of the probe is characterized by the Bloch vector $\mathbf{r}_0$ in $\mathbb{R}^3$. Then the transformation by U_ξ of Eq. (2) amounts to a rotation of $\mathbf{r}_0$ around the axis $\mathbf{n}$ by the angle ξ in $\mathbb{R}^3$. The axis $\mathbf{n}$ can be assigned the coelevation angle θ_n and azimuth φ_n characterizing the unit vector $\mathbf{n} = [\sin(\theta_n)\cos(\varphi_n), \sin(\theta_n)\sin(\varphi_n), \cos(\theta_n)]^\top$ of $\mathbb{R}^3$. We then define in $\mathcal{H}_2$ the pure quantum state $|\mathbf{n}\rangle = |0'\rangle = \cos(\theta_n/2)|0\rangle + e^{i\varphi_n}\sin(\theta_n/2)|1\rangle$ as the state having $\mathbf{n}$ as its Bloch vector. The pure state with Bloch vector $-\mathbf{n}$ is denoted $|-\mathbf{n}\rangle = |1'\rangle = \sin(\theta_n/2)|0\rangle - e^{i\varphi_n}\cos(\theta_n/2)|1\rangle$ and is orthogonal to $|0'\rangle$ in $\mathcal{H}_2$. Then in $\mathcal{H}_2$ the pure state $|+'\rangle = (|0'\rangle + |1'\rangle)/\sqrt{2}$ has a Bloch vector $\mathbf{n}_\perp$ in $\mathbb{R}^3$ orthogonal to the rotation axis $\mathbf{n}$. In Bloch representation, the action of the noise $\mathcal{N}(\cdot)$ on the qubit is equivalent to an affine transformation of its Bloch vector[30] as

$$\mathbf{r} \mapsto A\mathbf{r} + \mathbf{c}, \tag{3}$$

with A a 3×3 real matrix and $\mathbf{c}$ a real vector of $\mathbb{R}^3$, mapping the Bloch ball onto itself.

To specify the noise acting on the qubit, we choose a noise process which commutes with the rotation U_ξ around the axis $\mathbf{n}$. A similar assumption is performed in Ref. 32 for rotations around the Oz axis, "to separate cleanly the effect of the parameter from the effects of decoherence", when the process under estimation, itself, introduces a coherent rotation around this invariant axis. Especially, this provides the faculty to represent with one single noise operation, separate decohering actions which could take place before and after rotation of the probe by U_ξ of Eq. (2), and which can then be lumped into a single one. Commutation with the coherent rotation of the probe by U_ξ, is a necessary and sufficient condition for the noise to be lumpable in this way. We can now transpose the noise model of Ref. 32 invariant to rotations around the Oz axis, to a noise model with invariance relative to the arbitrary axis $\mathbf{n}$ here. Any noise model on the qubit can be completely defined by specifying the action of the superoperator $\mathcal{N}(\cdot)$ on a basis for operators on $\mathcal{H}_2$. We define the alternative Pauli operators as $\sigma_x' = |0'\rangle\langle 1'| + |1'\rangle\langle 0'|$, $\sigma_y' = i|1'\rangle\langle 0'| - i|0'\rangle\langle 1'|$ and $\sigma_z' = |0'\rangle\langle 0'| - |1'\rangle\langle 1'|$. Associated with I_2 the identity on $\mathcal{H}_2$, the set $\{\mathrm{I}_2, \sigma_x', \sigma_y', \sigma_z'\}$ as well as $\{\mathrm{I}_2, \sigma_x, \sigma_y, \sigma_z\}$ form two orthogonal bases for operators on $\mathcal{H}_2$. Transposing

Eq. (4.3) of Ref. 32, the quantum noise we consider is defined by

$$\mathcal{N}(\mathrm{I}_2) = \mathrm{I}_2 + \mu_0 \sigma'_z, \tag{4}$$

$$\mathcal{N}(\sigma'_z) = \mu_1 \sigma'_z, \tag{5}$$

$$\mathcal{N}(\sigma'_x \pm i\sigma'_y) = \mu_2 e^{\mp i\omega t}(\sigma'_x \pm i\sigma'_y). \tag{6}$$

Based on Ref. 32, the noise parameters here are $\mu_1 = e^{-t/T_1} \in [0, 1]$ and $\mu_2 = e^{-t/T_2} \in [0, 1]$, with a longitudinal relaxation time T_1 and a transverse dephasing time T_2 satisfying $T_2 \leq 2T_1$ for complete positivity, equivalent to $\mu_2^2 \leq \mu_1$; also $\mu_0 = \mu(1 - \mu_1)$ with $\mu \in [-1, 1]$. Following Ref. 32, the noise model of Eqs. (4)–(6) represents the most general noise model independently acting on single qubits, with here an arbitrary invariance axis $\mathbf{n}$, and an underlying temporal dynamics stationary and continuously differentiable. With this generality, the noise model incorporates in Eq. (6) an angle ωt representing the possibility of a coherent rotation around $\mathbf{n}$ contributed by the noise. Such a coherent rotation, if present, leads any procedure for estimating the phase shift ξ accompanied with a noise interaction over the duration t, to actually estimate $\omega t + \xi$. As such, this coherent rotation by ωt has no influence on the statistical fluctuation limiting the estimation accuracy; accordingly, as we shall see, it has no impact on the quantum Fisher information $F_q(\xi)$ quantifying the estimation efficiency.

When the Bloch vectors in $\mathbb{R}^3$ are referred to the orthonormal basis $\{\mathbf{n}_\perp, \mathbf{n}'_\perp = \mathbf{n} \times \mathbf{n}_\perp, \mathbf{n}\}$, the quantum noise of Eqs. (4)–(6) has in Eq. (3) the matrix

$$A = \begin{bmatrix} \mu_2 \cos(\omega t) & -\mu_2 \sin(\omega t) & 0 \\ \mu_2 \sin(\omega t) & \mu_2 \cos(\omega t) & 0 \\ 0 & 0 & \mu_1 \end{bmatrix}, \tag{7}$$

and the vector $\mathbf{c} = [0, 0, \mu_0]^\top$. From the form of A in Eq. (7), the action of the noise is to compress the Bloch vector by μ_1 in the direction $\mathbf{n}$ and by μ_2 in the plane $(\mathbf{n}_\perp, \mathbf{n}'_\perp)$ orthogonal to $\mathbf{n}$, with larger compression as a larger level of noise. At very long exposition time $t \to \infty$ to the noise, μ_1 and μ_2 go to zero effecting maximum compression, while the noisy qubit relaxes to the equilibrium state with Bloch vector $\mathbf{c} = \mu_0 \mathbf{n}$. From A and $\mathbf{c}$, one has access to an explicit expression for the quantum Fisher information $F_q(\xi)$ of Eq. (1), by means of Eq. (35) of Ref. 27, from which it follows here $F_q(\xi) = (\mu_2 r_{0\perp})^2$, with $r_{0\perp}$ the magnitude of the component orthogonal to the rotation axis $\mathbf{n}$ of the Bloch vector $\mathbf{r}_0$ characterizing the initial state ρ_0 of the probe. Consequently, with the noise model of Eqs. (4)–(6), maximization of the quantum Fisher information $F_q(\xi)$ is achieved by any probe initialized in a pure state with a unit Bloch vector $\mathbf{r}_0$ orthogonal to the rotation axis $\mathbf{n}$, i.e. verifying $(r_{0\perp})^2 = \|\mathbf{r}_0\|^2 = 1$, to reach the maximum $F_q^{\max}(\xi) = \mu_2^2$. The pure state $|+'\rangle = (|0'\rangle + |1'\rangle)/\sqrt{2}$ realizes such an optimal probe, having the density operator $\rho_0^{\mathrm{opt}} = |+'\rangle\langle+'|$ and Bloch vector $\mathbf{r}_0 = \mathbf{n}_\perp$. More complicated quantum noises exist, with an

optimal probe maximizing $F_q(\xi)$ which is not orthogonal to the rotation axis $\mathbf{n}$ in Bloch representation, as exemplified in Ref. 6. These are noises that do not share the commutation property with the rotations around $\mathbf{n}$ as it holds with the noise model of Eqs. (4)–(6).

The experiment where the probe qubit is prepared in the optimal state $\rho_0^{\mathrm{opt}} = |+'\rangle\langle+'|$ can be repeated N times on N independent and identical preparations of the probe qubit. This is equivalent to considering a composite probe of N qubits prepared in the separable state $(\rho_0^{\mathrm{opt}})^{\otimes N}$. In such circumstance the quantum Fisher information is additive,[16] and achieves the maximum $N\mu_2^2$ associated with $1/N$ scaling of the mean-squared estimation error. As announced in the Introduction, better estimation efficiency may be accessible by exploiting instead an N-qubit probe prepared in an entangled state, as we now examine.

3. With N Entangled Qubits

To select an efficient yet tractable family of N-qubit entangled states, we turn to the Hermitian operator $\mathbf{n} \cdot \boldsymbol{\sigma}$ appearing in Eq. (2), which represents a spin observable with eigenvalues ± 1 and eigenstates $|\pm\mathbf{n}\rangle$ in $\mathcal{H}_2$. As a result, the unitary U_ξ of Eq. (2) has the two eigenvalues $\exp(-i\xi/2)$ and $\exp(i\xi/2)$, respectively associated with the same two eigenstates $|\mathbf{n}\rangle = |0'\rangle$ and $|-\mathbf{n}\rangle = |1'\rangle$ in $\mathcal{H}_2$. We then choose for the N-qubit input probe in $\mathcal{H}_2^{\otimes N}$ the density operator $\rho_0 = |\psi_0\rangle\langle\psi_0|$ with the pure quantum state

$$|\psi_0\rangle = \sqrt{1-\kappa}|0'\rangle^{\otimes N} + \sqrt{\kappa}|1'\rangle^{\otimes N} = \sqrt{1-\kappa}|0'_N\rangle + \sqrt{\kappa}|1'_N\rangle, \tag{8}$$

where $\kappa \in [0,1]$ is a Schmidt coefficient allowing to control the degree of entanglement in $|\psi_0\rangle$, spanning from no entanglement at $\kappa \in \{0,1\}$, to maximal entanglement at $\kappa = 1/2$, to partial entanglement at $\kappa \notin \{0,1/2,1\}$. It is especially useful to test configurations of partial entanglement, because situations are known in quantum information where optimal or efficient processing occurs at partial entanglement, away from both separable and maximally entangled conditions[24,28,33–35]; and this will be observed here, in definite conditions. Since the states $|0'\rangle$ and $|1'\rangle$ are eigenstates of U_ξ in Eq. (2), the N-qubit transformed state is $|\psi_1\rangle = \mathsf{U}_\xi^{\otimes N}|\psi_0\rangle = \sqrt{1-\kappa}\,e^{-iN\xi/2}|0'_N\rangle + \sqrt{\kappa}\,e^{iN\xi/2}|1'_N\rangle$, with density operator $\rho_1 = |\psi_1\rangle\langle\psi_1|$ as

$$\rho_1(\xi) = (1-\kappa)|0'_N\rangle\langle0'_N| + \kappa|1'_N\rangle\langle1'_N| + \sqrt{(1-\kappa)\kappa}(e^{iN\xi}|1'_N\rangle|0'_N\rangle + e^{-iN\xi}|0'_N\rangle\langle1'_N|) \tag{9}$$

and for the quantum state $\rho_1(\xi)$ of Eq. (9), one obtains the derivative

$$\partial_\xi \rho_1(\xi) = iN\sqrt{(1-\kappa)\kappa}(e^{iN\xi}|1'_N\rangle\langle0'_N| - e^{-iN\xi}|0'_N\rangle\langle1'_N|). \tag{10}$$

For applying the noise model of Eqs. (4)–(6) to the N-qubit entangled state of Eq. (9), it is convenient to use the equivalent characterization by the

four transformations

$$\mathcal{N}(|0'\rangle\langle 0'|) = \frac{1}{2}(1 + \alpha_0)|0'\rangle\langle 0'| + \frac{1}{2}(1 - \alpha_0)|1'\rangle\langle 1'|, \tag{11}$$

$$\mathcal{N}(|1'\rangle\langle 1'|) = \frac{1}{2}(1 - \alpha_1)|0'\rangle\langle 0'| + \frac{1}{2}(1 + \alpha_1)|1'\rangle\langle 1'|, \tag{12}$$

$$\mathcal{N}(|0'\rangle\langle 1'|) = \alpha_2|0'\rangle\langle 1'|, \tag{13}$$

$$\mathcal{N}(|1'\rangle\langle 0'|) = \alpha_2^*|1'\rangle\langle 0'|, \tag{14}$$

with the parameters $\alpha_0 = \mu_1 + \mu_0$ and $\alpha_1 = \mu_1 - \mu_0$, and also $\alpha_2 = \mu_2 e^{-i\omega t}$.

From the one-qubit characterization of Eqs. (11)–(14), when the noise acts independently on each qubit, its action extends to N-qubit states by N-fold tensor product. For instance, for the N-qubit state $|0'_N\rangle\langle 1'_N|$ appearing in $\rho_1(\xi)$ of Eq. (9), one has $|0'_N\rangle\langle 1'_N| = (|0'\rangle\langle 1'|)^{\otimes N}$, and from Eq. (13) one can write $\mathcal{N}^{\otimes N}(|0'_N\rangle\langle 1'_N|) = \mathcal{N}^{\otimes N}((|0'\rangle\langle 1'|)^{\otimes N}) = (\mathcal{N}(|0'\rangle\langle 1'|))^{\otimes N} = \alpha_2^N(|0'\rangle\langle 1'|)^{\otimes N} = \alpha_2^N|0'_N\rangle\langle 1'_N|$.

In this way, the action of the quantum noise on the N-qubit state $\rho_1(\xi)$ of Eq. (9), produces the N-qubit noisy state $\rho_\xi = \mathcal{N}^{\otimes N}(\rho_1)$ which follows from Eqs. (11)–(14). By linearity of the superoperator $\mathcal{N}(\cdot)$, we have for the derivative $\partial_\xi \rho_\xi = \partial_\xi \mathcal{N}^{\otimes N}(\rho_1) = \mathcal{N}^{\otimes N}(\partial_\xi \rho_1)$ which applied to Eq. (10) gives

$$\partial_\xi \rho_\xi = iN\sqrt{(1-\kappa)\kappa}(\alpha_2^{*N}e^{iN\xi}|1'_N\rangle\langle 0'_N| - \alpha_2^N e^{-iN\xi}|0'_N\rangle\langle 1'_N|). \tag{15}$$

To characterize the quantum Fisher information $F_q(\xi)$ of Eq. (1), it is not enough to characterize the differentiated state $\partial_\xi \rho_\xi$ as accomplished in Eq. (15). One also needs to characterize the N-qubit noisy state ρ_ξ with its eigenvalues λ_j and eigenstates $|\lambda_j\rangle$, for $j = 1$ to 2^N.

From Eq. (9), as a first step, the noisy state $\rho_\xi = \mathcal{N}^{\otimes N}(\rho_1)$ can be expressed as

$$\rho_\xi = (1-\kappa)\mathcal{N}^{\otimes N}(|0'_N\rangle\langle 0'_N|) + \kappa\mathcal{N}^{\otimes N}(|1'_N\rangle\langle 1'_N|)$$
$$+ \sqrt{(1-\kappa)\kappa}\,(\alpha_2^{*N}e^{iN\xi}|1'_N\rangle\langle 0'_N| + \alpha_2^N e^{-iN\xi}|0'_N\rangle\langle 1'_N|). \tag{16}$$

The N-qubit noisy state $\mathcal{N}^{\otimes N}(|0'_N\rangle\langle 0'_N|)$ is obtained with an N-fold tensor product of Eq. (11) as $\mathcal{N}^{\otimes N}(|0'_N\rangle\langle 0'_N|) = (\mathcal{N}(|0'\rangle\langle 0'|))^{\otimes N}$ yielding

$$\mathcal{N}^{\otimes N}(|0'_N\rangle\langle 0'_N|) = \sum_{\mathbf{u}\in\{0',1'\}^N}\left[\frac{1}{2}(1+\alpha_0)\right]^{N-w(\mathbf{u})}\left[\frac{1}{2}(1-\alpha_0)\right]^{w(\mathbf{u})}|\mathbf{u}\rangle\langle\mathbf{u}|, \tag{17}$$

where the summation extends over all N-bit binary sequences $\mathbf{u}$ made from the bits $\{0', 1'\}$ among the 2^N possible such sequences, and $w(\mathbf{u})$ denotes the Hamming weight or number between 0 and N of bits $1'$ in the sequence $\mathbf{u}$. In a similar way, the N-qubit noisy state $\mathcal{N}^{\otimes N}(|1'_N\rangle\langle 1'_N|)$ is obtained with an N-fold tensor product of

Eq. (12) as $\mathcal{N}^{\otimes N}(|1'_N\rangle\langle 1'_N|) = (\mathcal{N}(|1'\rangle\langle 1'|))^{\otimes N}$ yielding

$$\mathcal{N}^{\otimes N}(|1'_N\rangle\langle 1'_N|) = \sum_{\mathbf{u}\in\{0',1'\}^N} \left[\frac{1}{2}(1-\alpha_1)\right]^{N-w(\mathbf{u})} \left[\frac{1}{2}(1+\alpha_1)\right]^{w(\mathbf{u})} |\mathbf{u}\rangle\langle\mathbf{u}|. \qquad (18)$$

It can be noted from Eqs. (17) and (18) that $\mathcal{N}^{\otimes N}(|0'_N\rangle\langle 0'_N|)$ and $\mathcal{N}^{\otimes N}(|1'_N\rangle\langle 1'_N|)$ contain only diagonal terms under the form $|\mathbf{u}\rangle\langle\mathbf{u}|$. In this way, in the N-qubit noisy state ρ_ξ of Eq. (16), only the two terms made with $|1'_N\rangle\langle 0'_N|$ and $|0'_N\rangle\langle 1'_N|$ are off-diagonal terms of ρ_ξ.

Moreover, by orthogonality of the 2^N normalized state vectors $|\mathbf{u}\rangle$, one obtains from Eq. (16) for any particular $\mathbf{u}$ differing from $0'_N$ and $1'_N$,

$$\rho_\xi|\mathbf{u}\rangle = \left((1-\kappa)\left[\frac{1}{2}(1+\alpha_0)\right]^{N-w(\mathbf{u})} \left[\frac{1}{2}(1-\alpha_0)\right]^{w(\mathbf{u})}\right.$$
$$\left. + \kappa\left[\frac{1}{2}(1-\alpha_1)\right]^{N-w(\mathbf{u})} \left[\frac{1}{2}(1+\alpha_1)\right]^{w(\mathbf{u})}\right)|\mathbf{u}\rangle, \qquad (19)$$

establishing any such $|\mathbf{u}\rangle$ differing from $|0'_N\rangle$ and $|1'_N\rangle$ as an eigenstate of ρ_ξ with the corresponding eigenvalue readable from Eq. (19). This determines $2^N - 2$ (mutually orthogonal) eigenstates for ρ_ξ, which together span the $(2^N - 2)$-dimensional sub-space orthogonal to the two-dimensional subspace spanned by $(|0'_N\rangle, |1'_N\rangle)$. Since ρ_ξ is Hermitian and has therefore a total of 2^N mutually orthogonal eigenstates, the two remaining eigenstates of ρ_ξ are to be found in this two-dimensional subspace spanned by $(|0'_N\rangle, |1'_N\rangle)$. We look for them under the form

$$|\lambda\rangle = a_0|0'_N\rangle + a_1|1'_N\rangle. \qquad (20)$$

The transformation by ρ_ξ of such a $|\lambda\rangle$ occurs in the plane $(|0'_N\rangle, |1'_N\rangle)$ and can be described by the matrix operation

$$\begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{01}^* & \rho_{11} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \rho_\xi^{(2)}|\lambda\rangle, \qquad (21)$$

with the 2×2 matrix $\rho_\xi^{(2)}$ defining the operation of the operator ρ_ξ restricted to the plane $(|0'_N\rangle, |1'_N\rangle)$. The form of Eq. (21) and the four matrix elements of $\rho_\xi^{(2)}$ are derived from Eq. (16) acting on state vectors belonging to the plane $(|0'_N\rangle, |1'_N\rangle)$ like $|\lambda\rangle$ of Eq. (20). In this way, one obtains

$$\rho_{00} = (1-\kappa)\left[\frac{1}{2}(1+\alpha_0)\right]^N + \kappa\left[\frac{1}{2}(1-\alpha_1)\right]^N, \qquad (22)$$

$$\rho_{11} = (1-\kappa)\left[\frac{1}{2}(1-\alpha_0)\right]^N + \kappa\left[\frac{1}{2}(1+\alpha_1)\right]^N, \qquad (23)$$

$$\rho_{01} = \sqrt{(1-\kappa)\kappa}\,\alpha_2^N e^{-iN\xi}. \qquad (24)$$

We are faced with an eigendecomposition restricted to the plane $(|0'_N\rangle, |1'_N\rangle)$, with a characteristic equation $\det(\rho^{(2)}_\xi - \lambda I_2) = 0$ equivalent to $\lambda^2 - (\rho_{00} + \rho_{11})\lambda + \rho_{00}\rho_{11} - \rho_{01}\rho^*_{01} = 0$. This second-degree equation in λ has the discriminant $\Delta = (\rho_{00} - \rho_{11})^2 + 4|\rho_{01}|^2$, and two roots providing the two (real) eigenvalues that we seek as $\lambda_\pm = (\rho_{00} + \rho_{11} \pm \sqrt{\Delta})/2$. Especially useful to us in the sequel is the sum $\lambda_+ + \lambda_- = \rho_{00} + \rho_{11}$, i.e.

$$\lambda_+ + \lambda_- = (1 - \kappa)\beta_0 + \kappa\beta_1, \tag{25}$$

with the two parameters

$$\beta_0 = \left[\frac{1}{2}(1 + \alpha_0)\right]^N + \left[\frac{1}{2}(1 - \alpha_0)\right]^N, \tag{26}$$

$$\beta_1 = \left[\frac{1}{2}(1 + \alpha_1)\right]^N + \left[\frac{1}{2}(1 - \alpha_1)\right]^N. \tag{27}$$

Now that the two eigenvalues $\lambda_\pm$ are expressed, the two corresponding eigenstates $|\lambda_\pm\rangle$ readily follow by solving the two linear systems $\rho^{(2)}_\xi|\lambda_\pm\rangle = \lambda_\pm|\lambda_\pm\rangle$. We do not write explicitly the resulting expressions, knowing that for the sequel we shall see that it is enough to use the characterization $|\lambda_+\rangle = a_0|0'_N\rangle + a_1 e^{iN(\omega t + \xi)}|1'_N\rangle$ and $|\lambda_-\rangle = a_1 e^{-iN(\omega t + \xi)}|0'_N\rangle - a_0|1'_N\rangle$, with a_0 and a_1 real, for two orthonormal eigenstates $|\lambda_\pm\rangle$ lying in the plane $(|0'_N\rangle, |1'_N\rangle)$.

Now for our objective of computing the quantum Fisher information $F_q(\xi)$ of Eq. (1), due to the form of $\partial_\xi\rho_\xi$ in Eq. (15) involving only the two basis operators $|0'_N\rangle\langle 1'_N|$ and $|1'_N\rangle\langle 0'_N|$, it is clear that only the two eigenstates $|\lambda_\pm\rangle$ lying in the plane $(|0'_N\rangle, |1'_N\rangle)$ can contribute a non-vanishing scalar $\langle\lambda_j|\partial_\xi\rho_\xi|\lambda_k\rangle$ in Eq. (1). Moreover, with the above form determined for $|\lambda_\pm\rangle$, it follows that each of the two rectangular terms $\langle\lambda_+|\partial_\xi\rho_\xi|\lambda_+\rangle$ and $\langle\lambda_-|\partial_\xi\rho_\xi|\lambda_-\rangle$ is identically zero. Only the two diagonal terms $\langle\lambda_+|\partial_\xi\rho_\xi|\lambda_-\rangle = \langle\lambda_-|\partial_\xi\rho_\xi|\lambda_+\rangle^*$ do not vanish, but yield

$$\langle\lambda_+|\partial_\xi\rho_\xi|\lambda_-\rangle = iN\sqrt{(1 - \kappa)\kappa}\,\alpha^N_2 e^{-iN\xi}. \tag{28}$$

Relying on the sum $\lambda_+ + \lambda_-$ of Eq. (25), the quantum Fisher information $F_q(\xi)$ of Eq. (1) finally evaluates to

$$F_q(\xi) = \frac{4(1 - \kappa)\kappa N^2\mu^{2N}_2}{(1 - \kappa)\beta_0 + \kappa\beta_1}. \tag{29}$$

It is observed that, by exploiting an N-qubit probe in the state of Eq. (8) matched to the rotation axis $\mathbf{n}$ through the two qubit states $|0'\rangle$ and $|1'\rangle$, respectively having Bloch vector $\mathbf{n}$ and $-\mathbf{n}$, then we obtain in Eq. (29) a quantum Fisher information $F_q(\xi)$ which is independent of the axis $\mathbf{n}$. If this were not done, the quantum Fisher information $F_q(\xi)$ would in general depend on the situation of the probe in relation to the rotation axis $\mathbf{n}$. This can be seen starting with the case of a one-qubit probe,

having $F_q(\xi) = (\mu_2 r_{0\perp})^2$ as explained in the paragraph of Eq. (7), which is a Fisher information that changes with the angle between $\mathbf{n}$ and the Bloch vector $\mathbf{r}_0$ characterizing the probe.

In addition, the quantum Fisher information $F_q(\xi)$ of Eq. (29) is also found independent of the unknown phase angle ξ we want to estimate for the qubit rotation around the axis $\mathbf{n}$. This parameter-independent performance is an interesting feature, not always present for parametric estimation, ensuring that the overall best estimation efficiency assessed by $F_q(\xi)$ is independent of the specific value of the phase ξ being estimated. This independence here is related to the qubit noise model having symmetry relative to the axis $\mathbf{n}$ according to Eqs. (4)–(6) so as to commute with the rotation U_ξ around $\mathbf{n}$. Other noises with no such symmetry would in general lead to a quantum Fisher information $F_q(\xi)$ showing dependence on the unknown phase angle ξ. This is for instance verified in Ref. 6, with qubit noises not having the above symmetry and with a one-qubit probe showing a quantum Fisher information $F_q(\xi)$ with dependence on the unknown phase ξ.

In this way, with the broad class of qubit noises of Eqs. (4)–(6), by selecting the probe according to Eq. (8), we are able to characterize with Eq. (29) a quantum Fisher information $F_q(\xi)$ which is independent of both $\mathbf{n}$ and ξ, and therefore identifies an overall best estimation efficiency uniform for any axis $\mathbf{n}$ and any phase ξ.

4. Analysis of the Quantum Fisher Information

The decoherence-free situation with the noise model of Eqs. (4)–(6) is characterized by $\mu_0 = 0$, $\mu_1 = \mu_2 = 1$ and $\omega = 0$; this translates for Eqs. (11)–(13) into $\alpha_0 = \alpha_1 = \alpha_2 = 1$, and $\beta_0 = \beta_1 = 1$ in Eqs. (26)–(27). The quantum Fisher information in Eq. (29) follows as $F_q(\xi) = 4(1-\kappa)\kappa N^2$. This corresponds to the $1/N^2$ scaling of the mean-squared estimation error accessible at no noise. This superefficiency is obtained thanks to entanglement of the N qubits of the probe. Any nondegenerate degree of entanglement, with $\kappa \notin \{0, 1\}$ in Eq. (8), is appropriate to give access to superefficiency with $1/N^2$ scaling of the mean-squared estimation error. Comparatively, the N independent qubits of the optimal separable probe $|+'\rangle^{\otimes N}$ would reach a quantum Fisher information $F_q(\xi) = N$ associated with the (less efficient) $1/N$ scaling of the mean-squared estimation error. In this respect, at no noise, quantum correlation among the qubits via entanglement is always beneficial to the estimation efficiency.

In addition, $F_q(\xi) = 4(1-\kappa)\kappa N^2$ is maximized for $\kappa = 1/2$ when the prefactor $4(1-\kappa)\kappa = 1$. This indicates that, at no noise, in the family of probe states of Eq. (8), it is the maximally entangled state with Schmidt coefficient $\kappa = 1/2$ that forms the most efficient probe for estimation. We will see that this is not always the case with noise, where maximum entanglement at $\kappa = 1/2$ in Eq. (8) is not necessarily the optimum maximizing $F_q(\xi)$ of Eq. (29).

4.1. *Large probe size*

In the presence of noise, Eq. (29) expresses for the quantum Fisher information $F_q(\xi)$ a sophisticated dependence with the size N of the entangled probe, that significantly departs from simple evolutions as N^2 or as N. The quantum Fisher information $F_q(\xi)$ of Eq. (29), in accordance with its general definition, is always nonnegative. A significant property conveyed by Eq. (29) is that, in the presence of any nonvanishing noise according to the model of Eqs. (4)–(6), the Fisher information $F_q(\xi)$ of Eq. (29) always returns to zero asymptotically at large size N of the entangled probe. This is established as follows, when $\mu_0 \geq 0$, given that when $\mu_0 \leq 0$ one arrives at the same conclusion by symmetry of the situation. When $\mu_0 \geq 0$, one has $0 \leq \alpha_1 \leq \alpha_0 < 1$, so that at large N, one obtains for $F_q(\xi)$ of Eq. (29) the asymptotic form

$$F_q(\xi) \to 4\kappa \frac{N^2 \mu_2^{2N}}{[(1+\alpha_0)/2]^N} = 4\kappa N^2 \left(\frac{2\mu_2^2}{1+\mu_1+\mu_0} \right)^N. \tag{30}$$

The condition $\mu_2^2 \leq \mu_1$ guaranteed by complete positivity of the noise, implies that $0 \leq 2\mu_2^2/(1+\mu_1+\mu_0) < 1$ for the exponentiated factor in Eq. (30). As a consequence, $F_q(\xi)$ controlled by Eq. (30) always decays exponentially to zero at large $N \to \infty$. This means that large sizes $N \to \infty$ of the entangled probe are always inefficient for estimation, for any configuration of a nonvanishing noise according to Eqs. (4)–(6). Other studies have come to a similar conclusion, with various types of noise and metrics of efficiency.[16,20,26] We have here an alternative proof based on the behavior of the quantum Fisher information and holding for the broad class of noises of Eqs. (4)–(6). This outcome materializes the fragility to noise evoked in the Introduction for the superefficient estimation. This asymptotic behavior of $F_q(\xi)$ indicates that, at large size $N \to \infty$, the block of N entangled qubits behaving in a correlated way gets dominated by the noise and becomes inoperative for estimation, and this for any degree of entanglement controlled by the Schmidt coefficient κ in Eq. (8). Comparatively, the optimal separable probe $|+'\rangle^{\otimes N}$ of N independent qubits would maintain a quantum Fisher information $F_q(\xi) = N\mu_2^2$ linearly increasing with N, remaining in this way operative for estimation at any size N. Here, with noise, quantum correlation among the qubits via entanglement becomes detrimental to the estimation efficiency for large probe size $N \to \infty$, at any nonzero level of noise.

This contrasted behavior without or with noise manifests the sophisticated and versatile role of quantum correlation by entanglement for information processing. Depending on the conditions, the correlated behavior among entangled qubits can reveal beneficial (with no noise) or asymptotically detrimental (with noise).

4.2. *Optimal probe size*

Another significant property conveyed by Eq. (29) is the existence of an optimal value of the size N of the entangled probe, able to maximize the Fisher information $F_q(\xi)$ of Eq. (29), this for any configuration of the noise of Eqs. (4)–(6) and any degree

of entanglement κ. An analytical prediction can even be made for an optimal size N_{opt} of the entangled probe, by looking for the maximum of the asymptotic form of $F_q(\xi)$ given by Eq. (30), which comes out analytically, for any fixed κ, as

$$N_{\mathrm{opt}} = \frac{2}{\ln\left(\frac{1+\mu_1+\mu_0}{2\mu_2^2}\right)}, \tag{31}$$

this expression being expected as accurate for noise configurations where the optimal probe size occurs in the range $N_{\mathrm{opt}} \gg 1$.

Alternatively, for arbitrary noise configurations, numerical study of $F_q(\xi)$ from Eq. (29), allows one to verify the existence and value of an optimal size N_{opt} of the entangled probe. This can be exemplified for instance with a noise configuration of great relevance to the qubit, under the form of the depolarizing noise, which is obtained in Eqs. (4)–(6) when $\mu_0 = 0$, also $\mu_1 = \mu_2 = \alpha \in [0, 1]$ and $\omega = 0$, associated in Eqs. (11)–(13) with $\alpha_0 = \alpha_1 = \alpha_2 = \alpha \in [0, 1]$. This describes the situation of a unital (i.e. verifying $\mathcal{N}(\mathrm{I}_2) = \mathrm{I}_2$) isotropic quantum noise, with the relaxation times $T_1 = T_2$, and thus no privileged axis.

For the depolarizing noise described by the single depolarizing factor α, the parameters β_0 and β_1 in Eqs. (26)–(27) coincide, so that the quantum Fisher information $F_q(\xi)$ of Eq. (29) reduces to

$$F_q(\xi) = 4(1 - \kappa)\kappa \frac{N^2 (2\alpha^2)^N}{(1+\alpha)^N + (1-\alpha)^N}. \tag{32}$$

For the depolarizing noise, Eq. (32) shows that $F_q(\xi)$ is always maximized for $\kappa = 1/2$ with the prefactor in Eq. (32) culminating at $4(1 - \kappa)\kappa = 1$. This indicates that, in the family of probe states of Eq. (8), it is always the maximally entangled state at Schmidt coefficient $\kappa = 1/2$ that forms the most efficient setting for estimation, for any factor α of the depolarizing noise and any probe size N.

For different values of the depolarizing noise factor α, the quantum Fisher information $F_q(\xi)$ of Eq. (32) at $\kappa = 1/2$ is depicted in Fig. 1 as a function of the number N of entangled qubits of the probe.
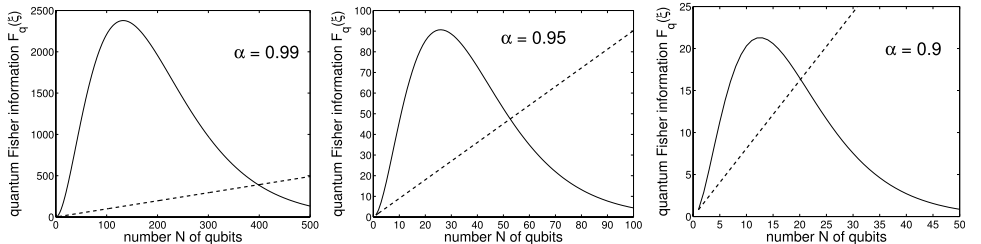

Fig. 1. As a function of the number N of qubits in the probe, quantum Fisher information $F_q(\xi)$ from Eq. (32) at $\kappa = 1/2$ for a maximally entangled probe in Eq. (8) (solid line), and $F_q(\xi) = N\alpha^2$ for the optimal separable probe $|+'\rangle^{\otimes N}$ (dashed line), at 3 values of the depolarizing noise factor α.

Figure 1 verifies the two generic properties announced for the quantum Fisher information $F_q(\xi)$ of Eq. (29), that $F_q(\xi)$ always returns to zero asymptotically at large size N of the entangled probe, and that $F_q(\xi)$ gets maximized at a finite (noise-dependent) optimal probe size N_{opt}. Figure 1 also represents the Fisher information $F_q(\xi) = N\alpha^2$ achieved by N separable qubits in state $|+'\rangle^{\otimes N}$, and clearly demonstrates broad ranges of the probe size N where the entangled probe is more efficient than the optimal separable probe for estimation, as manifested by a superior Fisher information. As also visible in Fig. 1, the value of the optimal size N_{opt} of the entangled probe usually depends on the specific noise configuration. Figure 2 represents the optimal size $N_{\text{opt}}(\alpha)$ of the entangled probe maximizing $F_q(\xi)$ of Eq. (32), as a function of the factor α of the depolarizing noise, along with the corresponding maximum Fisher information $F_q^{\max}(\xi)$ reached at $N_{\text{opt}}(\alpha)$ in Eq. (32) when $\kappa = 1/2$.
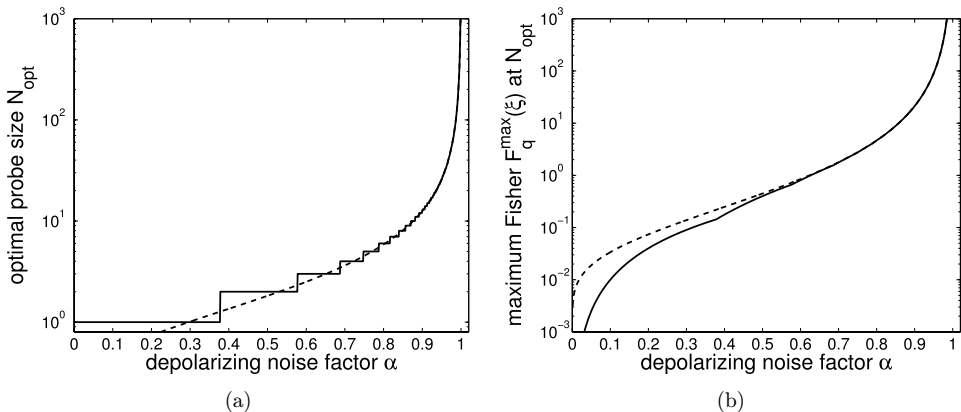

Fig. 2. As a function of the depolarizing noise factor α, (a) optimal size N_{opt} of the entangled probe maximizing the quantum Fisher information $F_q(\xi)$ of Eq. (32) (solid line) and its analytical expression of Eq. (34) at small noise when $\alpha \to 1$ (dashed line), (b) corresponding maximum Fisher information $F_q^{\max}(\xi)$ reached in Eq. (32) at each $N_{\text{opt}}(\alpha)$ when $\kappa = 1/2$ (solid line) and its analytical expression from Eq. (33) at small noise when $\alpha \to 1$ (dashed line).

Figure 2(a) shows an optimal size $N_{\text{opt}}(\alpha)$ of the entangled probe which increases as the noise factor α increases. At large noise when $\alpha \to 0$, the optimal size N_{opt} goes to 1, expressing in another way that entanglement or correlation among several qubits ceases to be efficient at large noise. Above $\alpha \approx 0.377$ in Fig. 2(a) the optimal size N_{opt} of the probe jumps above 1 and then N_{opt} gradually increases towards very large values as α approaches 1 when the noise faints. At small noise when $\alpha \to 1$, entanglement gets more and more efficient, as expressed by larger optimal size N_{opt} with increasing α. This is also reflected in Fig. 2(b) showing the maximum Fisher information $F_q^{\max}(\xi)$ at N_{opt}. This maximum $F_q^{\max}(\xi)$ also gets very large as $\alpha \to 1$, expressing larger efficiency of entanglement at small noise.

For large size N of the entangled probe with depolarizing noise, Eq. (32) has the asymptotic form

$$F_q(\xi) = 4(1-\kappa)\kappa N^2 \left(\frac{2\alpha^2}{1+\alpha}\right)^N.$$

(33)

In the regime of small noise $\alpha \to 1$, when the optimal probe size N_{opt} is large, the analytical prediction of Eq. (31) becomes

$$N_{\mathrm{opt}}(\alpha) = \frac{2}{\ln\left(\frac{1+\alpha}{2\alpha^2}\right)},$$

(34)

for maximizing $F_q(\xi)$ of Eq. (33), as shown in Fig. 2(a). When the value $N = N_{\mathrm{opt}}$ from Eq. (34) is placed in Eq. (33), an analytical prediction results for the maximum Fisher information $F_q^{\mathrm{max}}(\xi)$ in the small-noise regime $\alpha \to 1$, also shown in Fig. 2(b). As visible in Fig. 2, these two analytical predictions from Eqs. (34) and (33) for the small-noise regime $\alpha \to 1$, provide a very good match.

The maximum Fisher information $F_q^{\mathrm{max}}(\xi)$ at N_{opt} in Fig. 2 is achieved by the entangled probe of Eq. (8) at $\kappa = 1/2$, and it is meaningful to compare it with the Fisher information $N_{\mathrm{opt}}\alpha^2$ achieved by the optimal separable probe $|+'\rangle^{\otimes N}$ with same size $N = N_{\mathrm{opt}}$. This is accomplished in Fig. 3 showing the ratio $F_q^{\mathrm{max}}(\xi)/(N_{\mathrm{opt}}\alpha^2)$ as a function of the depolarizing noise factor α.
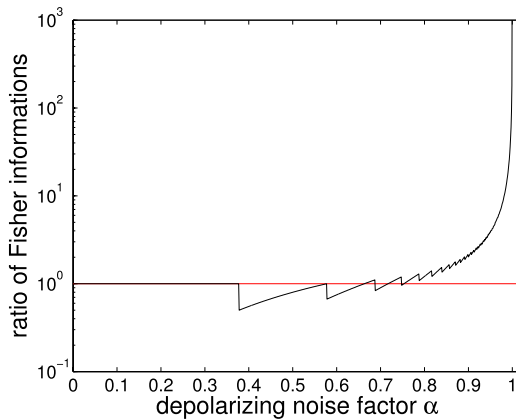

Fig. 3. (Color online) As a function of the depolarizing noise factor α, the ratio $F_q^{\mathrm{max}}(\xi)/(N_{\mathrm{opt}}\alpha^2)$ of the maximum Fisher information $F_q^{\mathrm{max}}(\xi)$ reached at each α by $N_{\mathrm{opt}}(\alpha)$ maximally entangled qubits as in Fig. 2, to the Fisher information $N_{\mathrm{opt}}\alpha^2$ reached by the optimal separable probe $|+'\rangle^{\otimes N_{\mathrm{opt}}}$ of N_{opt} independent qubits. The baseline at ordinate 1 is shown in red.

At large noise when $\alpha \to 0$ in Fig. 3 the ratio $F_q^{\mathrm{max}}(\xi)/(N_{\mathrm{opt}}\alpha^2)$ goes to 1, which indicates a similar efficiency for the maximally entangled probe and the separable probe, when they coincide at $N_{\mathrm{opt}} = 1$ in this range of noise. For intermediate α in Fig. 3, the ratio can get below 1, indicating some intermediate ranges of noise where

the separable probe is in this respect more efficient than the entangled probe, at sizes $N_{\mathrm{opt}} > 1$. However, above $\alpha \approx 0.754$ in Fig. 3, the ratio always exceeds 1 and can take very large values as the noise vanishes with $\alpha \to 1$ and the size N_{opt} grows. This expresses that at small noise the entangled probe is always more efficient than the separable probe, with a superiority which can become arbitrary large as $\alpha \to 1$ and increasing size N_{opt} of the entangled probe.

A complementary viewpoint is provided by the Fisher information per qubit presented in Fig. 4, which amounts to $F_q^{\mathrm{max}}(\xi)/N_{\mathrm{opt}}(\alpha)$ for the entangled probe and to α^2 for the separable probe.
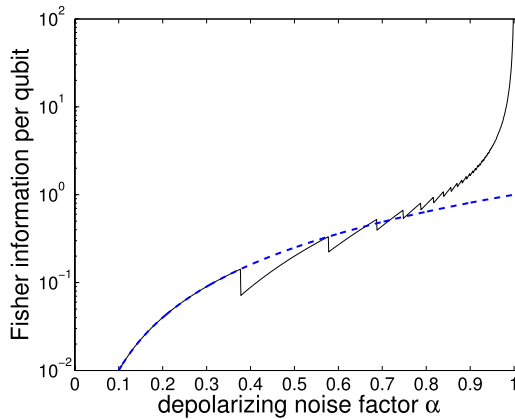

Fig. 4.   As a function of the depolarizing noise factor α, the maximum Fisher information per qubit F_q^{max} $(\xi)/N_{\mathrm{opt}}(\alpha)$ for the maximally entangled probe of N_{opt} qubits (solid line), and α^2 representing the Fisher information per qubit of the optimal separable probe $|+'\rangle^{\otimes N_{\mathrm{opt}}}$ of N_{opt} independent qubits (dashed line).

Consistently in Fig. 4, for a noise factor above $\alpha \approx 0.754$ the Fisher information per qubit is always superior for the entangled probe compared to the separable probe. For the separable probe, the Fisher information per qubit α^2 can never exceed 1. By contrast, for the entangled probe the Fisher information per qubit $F_q^{\mathrm{max}}(\xi)/N_{\mathrm{opt}}(\alpha)$ can take very large values as $\alpha \to 1$ when the noise vanishes. This is a striking manifestation of the superiority that entanglement can afford over separable states in definite circumstances. Here, entanglement can bring an arbitrarily large superiority compared to separable states at their optimum.

In addition, Fig. 5 represents the ratio $F_q^{\mathrm{max}}(\xi)/N_{\mathrm{opt}}^2(\alpha)$ and shows that this ratio tends to a constant ≈ 0.135 as the noise diminishes when $\alpha \to 1$. The analytical expressions provided by Eqs. (34) and (33) for the small-noise regime $\alpha \to 1$, lead here to an analytical ratio $F_q^{\mathrm{max}}(\xi)/N_{\mathrm{opt}}^2(\alpha) = e^{-2} \approx 0.135$, which precisely matches the limit observed in Fig. 5. This behavior points to the asymptotic trend $F_q^{\mathrm{max}}(\xi) \sim 0.135 N_{\mathrm{opt}}^2$, i.e. a Fisher information $F_q^{\mathrm{max}}(\xi)$ quadratic in the optimal probe size N_{opt} at small noise $\alpha \to 1$, reminiscent of the quadratic Fisher information $F_q(\xi) = 4(1 - \kappa)\kappa N^2$ in Eq. (32) at no noise when $\alpha = 1$.
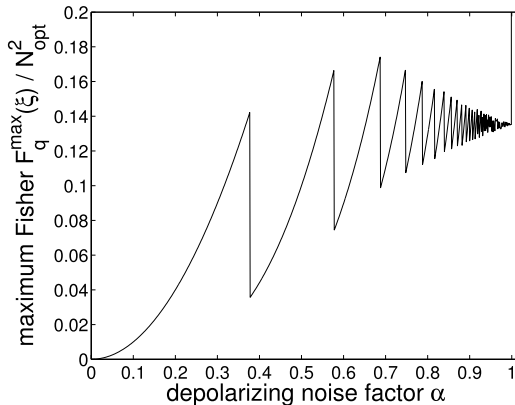
Fig. 5. As a function of the depolarizing noise factor α, the ratio $F_q^{\max}(\xi)/N_{\mathrm{opt}}^2(\alpha)$ for the maximum Fisher information $F_q^{\max}(\xi)$ of the maximally entangled probe with N_{opt} qubits, illustrating the asymptotic trend $F_q^{\max}(\xi) \sim 0.135 N_{\mathrm{opt}}^2$ as $\alpha \to 1$.

4.3. *Optimum at partial entanglement*

It is also relevant to examine the evolution of the quantum Fisher information $F_q(\xi)$ of Eq. (29), at any given size N of the entangled probe, as a function of the Schmidt coefficient $\kappa \in [0, 1]$ quantifying the degree of entanglement of the probe from Eq. (8). It turns out that $F_q(\xi)$ of Eq. (29) gets maximized at an optimal degree of entanglement κ_{opt} which is computable analytically as

$$\kappa_{\mathrm{opt}} = \frac{1}{1 + \sqrt{\beta_1/\beta_0}}. \tag{35}$$

For the class of unital noises $\mu_0 = 0$ in Eq. (4), so that $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$ in Eqs. (26)–(27) for any size N. For this class of noises, it then results from Eq. (35) that $\kappa_{\mathrm{opt}} = 1/2$, indicating that the quantum Fisher information $F_q(\xi)$ of Eq. (29) is always maximized by a maximally entangled probe in Eq. (8). By contrast, for nonunital noises with $\mu_0 \neq 0$ in Eq. (4), then in general $\alpha_0 \neq \alpha_1$ and $\beta_0 \neq \beta_1$ in Eqs. (26)–(27). Depending on the sign of $\mu_0 \in [-1, 1]$, the optimal degree of entanglement κ_{opt} of Eq. (35) can be found below or above $1/2$. In addition, through β_0 and β_1 in Eqs. (26)–(27), the optimal κ_{opt} of Eq. (35) is usually dependent on the size N of the probe. The expression of $\kappa_{\mathrm{opt}}(N)$ from Eq. (35) can be placed into Eq. (29) to yield an expression for the Fisher information as $F_q(\xi; \kappa_{\mathrm{opt}}(N))$. This expression can then be maximized according to the size N of the probe; although this can be difficult to perform analytically, based on Eqs. (29) and (35) this is quite feasible numerically. In this way one obtains the maximizer N_{opt} and therefrom $(\kappa_{\mathrm{opt}}, N_{\mathrm{opt}})$ defining the optimal configuration of the probe of Eq. (8) with size N_{opt} and degree of entanglement κ_{opt} maximizing the quantum Fisher information $F_q(\xi)$ in Eq. (29). Depending on the values of the parameters of the nonunital noise, one will usually obtain an

optimal probe with finite size N_{opt} and partial entanglement κ_{opt}, representing the most efficient setting of the probe of Eq. (8) for estimation.

For illustration we consider the case of the amplitude damping noise,[30,31] which describes a qubit asymptotically relaxed to the equilibrium state $|0'\rangle$, and which can be obtained in Eqs. (4)–(6) by $\mu_2 = \sqrt{\mu_1}$ and $\mu = 1$ so that $\mu_0 = 1 - \mu_1$. Figure 6 shows, as a function of the noise factor μ_1, the optimal entanglement setting $(\kappa_{\text{opt}}, N_{\text{opt}})$ maximizing the quantum Fisher information $F_q(\xi)$ of Eq. (29).
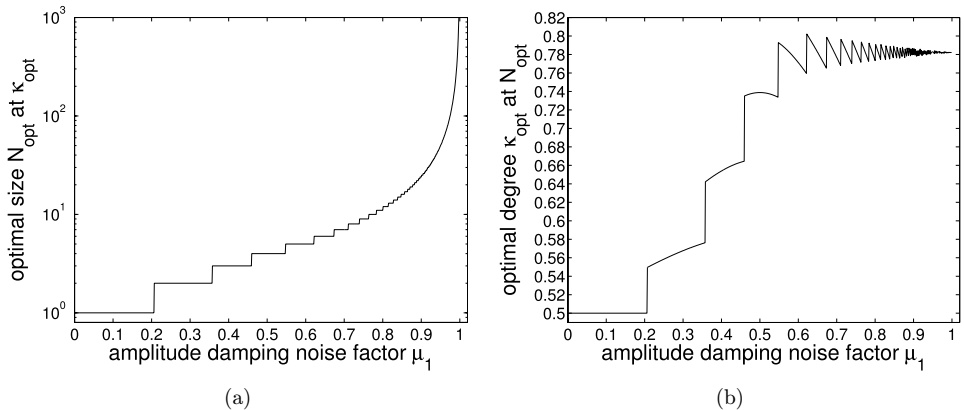

Fig. 6.   As a function of the amplitude damping noise factor μ_1, (a) optimal size N_{opt} of the entangled probe, associated with (b) optimal degree of entanglement κ_{opt}, together maximizing the quantum Fisher information $F_q(\xi)$ of Eq. (29).

The optimal entanglement setting $(\kappa_{\text{opt}}, N_{\text{opt}})$ of Fig. 6 realizes the maximum $F_q^{\max}(\xi)$ for the quantum Fisher information $F_q(\xi)$ of Eq. (29). This maximum $F_q^{\max}(\xi)$ achieved by N_{opt} optimally entangled qubits, is compared in Fig. 7 with the Fisher information $N_{\text{opt}}\mu_2^2$ achieved by N_{opt} independent qubits prepared in the optimal separable input state $|+'\rangle^{\otimes N_{\text{opt}}}$.

Figure 7 shows a ratio $F_q^{\max}(\xi)/(N_{\text{opt}}\mu_2^2)$ which, above the noise factor $\mu_1 \approx 0.486$, always exceeds 1, and which can take very large values as the noise vanishes with $\mu_1 \to 1$ and the size N_{opt} grows. This expresses that at intermediate or small noise, the optimally entangled probe is always more efficient than the separable probe, with a superiority which can become arbitrary large as $\mu_1 \to 1$ and increasing size N_{opt} of the entangled probe. Moreover, in this range of the noise factor μ_1, the optimal degree of entanglement κ_{opt} is clearly above $1/2$ and lies between 0.66 and 0.81 in Fig. 6(b), indicating that maximum estimation efficiency is achieved at partial entanglement. The inset of Fig. 7 depicts the ratio $F_q^{\max}(\xi; \kappa = \kappa_{\text{opt}})/F_q^{\max}(\xi; \kappa = 1/2)$ where $F_q^{\max}$ $(\xi; \kappa = 1/2)$ is obtained by maximizing $F_q(\xi)$ at $\kappa = 1/2$ instead of the more efficient maximization at $\kappa = \kappa_{\text{opt}}$. Consistently this ratio is above 1 and it represents the gain of efficiency by operating at the optimal partial entanglement κ_{opt} instead of operating at maximum entanglement $\kappa = 1/2$. The ratio saturates around 1.3 indicating,
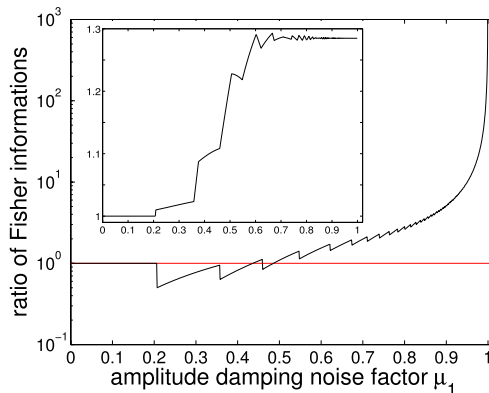
Fig. 7. (Color online) As a function of the amplitude damping noise factor μ_1, the ratio $F_q^{\max}(\xi)/(N_{\mathrm{opt}}\mu_2^2)$ of the maximum Fisher information $F_q^{\max}(\xi)$ reached at each μ_1 by $N_{\mathrm{opt}}(\mu_1)$ qubits with optimal degree of entanglement $\kappa_{\mathrm{opt}}(\mu_1)$ as in Fig. 6, to the Fisher information $N_{\mathrm{opt}}\mu_2^2$ reached by the optimal separable probe $|+'\rangle^{\otimes N_{\mathrm{opt}}}$ of N_{opt} independent qubits. The baseline at ordinate 1 is shown in red. The inset is the ratio $F_q^{\max}(\xi; \kappa = \kappa_{\mathrm{opt}})/F_q^{\max}(\xi; \kappa = 1/2)$.

in these noise conditions, a maximum gain of around 30% by optimizing the degree of entanglement. The maximum gain of around 30% is observed in Fig. 7 for any noise factor μ_1 above 0.6. At $\mu_1 = 0.6$, Fig. 6(a) indicates an optimal size $N_{\mathrm{opt}} = 5$ of the entangled probe.

Reference 24 tested another family of partially entangled states for frequency estimation with phase-flip noise, and it reports improvement of up to 10% compared to the maximally entangled probe, in the range of the small probe sizes handled in Ref. 24. Here, we can handle entangled probes of arbitrary size in Eq. (29), with maximum improvement of around 30% observed in Fig. 7. Large entangled states with arbitrary degree of entanglement are generally difficult to handle analytically (and even numerically), and it is useful as here to identify scenarios with entangled states that remain analytically tractable at arbitrary sizes, while bringing a meaningful contribution to the picture. In another context of estimation Refs. 20, 34 report other situations of optimality at partial entanglement of the probe, with a maximum improvement of $\sim 40\%$ above the separable probe, while here much larger gain above the separable probe are observed in Fig. 7. This indicates the possibility of significant variability of the quantitative behaviors, depending on the specific conditions, in the presence of noise and decoherence which represent sophisticated situations, where intuition can be defeated, and requiring separate analysis for clarification.

The case of the amplitude damping noise in Figs. 6–7 is merely illustrative, and this important property of optimality at partial entanglement is rather generic and occurs in many configurations of nonunital noises inserted in Eq. (29). Also we note that when the ratio $F_q^{\max}(\xi; N_{\mathrm{opt}})/(N_{\mathrm{opt}}\mu_2^2)$ of Fisher informations with entangled or separable probe, similar to the ratio shown in Fig. 7 or 3, is strictly above 1, then this is consistently associated for the entangled probe with an optimal size $N_{\mathrm{opt}} > 1$ as in

Fig. 6(a) or 2(a). In such circumstance, for the asymptotic regime where a large number N of qubits are available, an efficient strategy, as also considered for instance in Ref. 36, would be to group the N qubits into independent blocks of N_{opt} optimally entangled qubits. For such strategy with entanglement, the asymptotic regime of large N would be characterized, in terms of Fisher information, by an efficiency growing linearly with N, yet with a level strictly superior to the efficiency of the strategy using the optimal separable probe (which also grows linearly with N). Therefore, for any conditions with a ratio $F_q^{\mathrm{max}}(\xi; N_{\mathrm{opt}})/(N_{\mathrm{opt}}\mu_2^2)$ above 1, entangled blocks of optimal size N_{opt} always improve the asymptotic efficiency above that of optimally prepared independent qubits. And the improvement factor can be evaluated, for any noise according to Eqs. (4)–(6), through the analysis of Eq. (29) as exemplified in this section. This identifies a broad class of conditions with a strategy where entangled probes improve over the optimal separable probes for efficient estimation.

5. With a Fraction of Inactive Qubits

It is possible to gain additional flexibility in exploiting the N-qubit entangled probe of Eq. (8), by letting only a fraction N_1 of the N entangled qubits interact with the unitary U_ξ plus noise process, while the complementary fraction $N_2 = N - N_1$ of the entangled qubits remain completely untouched in the operation. In practice this could represent a situation where, after preparing the N-qubit entangled probe, N_2 qubits are kept sheltered in the laboratory while N_1 qubits are sent outside to interact with the noise and unitary process to be estimated. Comparable schemes with inactive entangled quantum systems or ancillas have also been considered for various scenarios of quantum metrology, with no noise,[17,19,37,38] or with noise,[5,20,22] often with two subsystems — active and inactive — of equal size. A prior motivation here is that the active qubits interact both with the process to be estimated (which is useful) and with the noise (which is detrimental), so a mixed strategy involving some entangled inactive qubits may globally lead to a more favorable trade-off (as we are going to verify in some configurations).

From the initial N-qubit probe $|\psi_0\rangle$ of Eq. (8), with N_1 active qubits we obtain the transformed state $|\psi_1\rangle = \mathsf{U}_\xi^{\otimes N_1} \otimes \mathrm{I}_2^{\otimes N_2}|\psi_0\rangle = \sqrt{1-\kappa}\, e^{-iN_1\xi/2}|0'_N\rangle + \sqrt{\kappa}\, e^{iN_1\xi/2}|1'_N\rangle$, with density operator $\rho_1 = |\psi_1\rangle\langle\psi_1|$ as

$$\begin{aligned}
\rho_1(\xi) = (1-\kappa)|0'_N\rangle\langle 0'_N| + \kappa|1'_N\rangle\langle 1'_N| + \sqrt{(1-\kappa)\kappa} \\
\times (e^{iN_1\xi}|1'_N\rangle\langle 0'_N| + e^{-iN_1\xi}|0'_N\rangle\langle 1'_N|)
\end{aligned} \tag{36}$$

instead of Eq. (9). The noisy state is $\rho_\xi = \mathcal{N}^{\otimes N_1} \otimes \mathcal{I}^{\otimes N_2}(\rho_1)$, with $\mathcal{I}(\cdot)$ the identity superoperator for the qubit, yielding

$$\begin{aligned}
\rho_\xi = (1-\kappa)\mathcal{N}^{\otimes N_1} \otimes \mathcal{I}^{\otimes N_2}(|0'_N\rangle\langle 0'_N|) + \kappa\mathcal{N}^{\otimes N_1} \otimes \mathcal{I}^{\otimes N_2}(|1'_N\rangle\langle 1'_N|) \\
+ \sqrt{(1-\kappa)\kappa}(\alpha_2^{*N_1}e^{iN_1\xi}|1'_N\rangle\langle 0'_N| + \alpha_2^{N_1}e^{-iN_1\xi}|0'_N\rangle\langle 1'_N|)
\end{aligned} \tag{37}$$

replacing Eq. (16). Next, we have the noisy basis state

$$\mathcal{N}^{\otimes N_1} \otimes \mathcal{I}^{\otimes N_2}(|0'_N\rangle\langle 0'_N|) = \left[\frac{1}{2}(1+\alpha_0)|0'\rangle\langle 0'| + \frac{1}{2}(1-\alpha_0)|1'\rangle\langle 1'|\right]^{\otimes N_1} \otimes (|0'\rangle\langle 0'|)^{\otimes N_2}$$

$$= \sum_{\mathbf{u}\in\{0',1'\}^{N_1}} \left[\frac{1}{2}(1+\alpha_0)\right]^{N_1-w(\mathbf{u})}$$

$$\left[\frac{1}{2}(1-\alpha_0)\right]^{w(\mathbf{u})} |\mathbf{u}\rangle\langle\mathbf{u}| \otimes (|0'\rangle\langle 0'|)^{\otimes N_2} \quad (38)$$

in place of Eq. (17), and a comparable expression for $\mathcal{N}^{\otimes N_1} \otimes \mathcal{I}^{\otimes N_2}(|1'_N\rangle\langle 1'_N|)$ replacing Eq. (18). By the same approach, we obtain as before an explicit characterization of both ρ_ξ and $\partial_\xi\rho_\xi$. For the eigendecomposition of ρ_ξ the structure of the eigenstates is conserved, with only two eigenstates belonging to the plane $(|0'_N\rangle, |1'_N\rangle)$ to impact the Fisher information $F_q(\xi)$, and determined via the matrix elements

$$\rho_{00} = (1-\kappa)\left[\frac{1}{2}(1+\alpha_0)\right]^{N_1} + \kappa\left[\frac{1}{2}(1-\alpha_1)\right]^{N_1}\delta_{NN_1}, \quad (39)$$

$$\rho_{11} = \kappa\left[\frac{1}{2}(1+\alpha_1)\right]^{N_1} + (1-\kappa)\left[\frac{1}{2}(1-\alpha_0)\right]^{N_1}\delta_{NN_1}, \quad (40)$$

$$\rho_{01} = \sqrt{(1-\kappa)\kappa}\,\alpha_2^{N_1}e^{-iN_1\xi}, \quad (41)$$

incorporating the Kronecker delta δ_{NN_1} as an essential evolution relative to Eqs. (22)–(24). The two associated eigenvalues sum according to $\lambda_+ + \lambda_- = \rho_{00} + \rho_{11}$ as in Eq. (25), with now the two parameters

$$\beta_0 = \left[\frac{1}{2}(1+\alpha_0)\right]^{N_1} + \left[\frac{1}{2}(1-\alpha_0)\right]^{N_1}\delta_{NN_1}, \quad (42)$$

$$\beta_1 = \left[\frac{1}{2}(1+\alpha_1)\right]^{N_1} + \left[\frac{1}{2}(1-\alpha_1)\right]^{N_1}\delta_{NN_1}. \quad (43)$$

Through the same path, the quantum Fisher information $F_q(\xi)$ of Eq. (1) finally evaluates to

$$F_q(\xi) = \frac{4(1-\kappa)\kappa N_1^2 \mu_2^{2N_1}}{(1-\kappa)\beta_0 + \kappa\beta_1}, \quad (44)$$

which coincides with Eq. (29) only when $N_1 = N$.

The Fisher information of Eq. (44) indicates that the efficiency obtained with a given fraction N_1 of active qubits, is independent of the total size N and is a function of N_1 only, provided $N \geq N_1 + 1$. In this respect, as soon as the size $N = N_1 + 1$ can the efficiency of Eq. (44) be obtained, i.e. with only $N_2 = 1$ inactive qubits. For N_1 given, there is no need to entangle more than $N = N_1 + 1$ qubits to obtain the efficiency of Eq. (44). The presence of $N_2 = 1$ inactive qubit does affect the efficiency,

but $N_2 = 2$ or more inactive qubits do not add anything except the burden of having to process a larger size N overloaded with superfluous inactive qubits.

In given noise configurations, due to the similarity of form with Eq. (29), the quantum Fisher information $F_q(\xi)$ in Eq. (44) also returns exponentially to zero at large $N_1 \to \infty$, and is maximized at an optimum of the size N_1 and at an optimal degree of entanglement κ_{opt}. Especially, this κ_{opt} is still given by Eq. (35) and is generally different from $1/2$ marking maximal entanglement.

To illustrate the benefit of handling a reduced fraction $N_1 < N$ of active qubits, instead of the totality of N active qubits, we again turn to the case of the amplitude damping noise as in Figs. 6–7. Figure 8 compares the quantum Fisher information $F_q(\xi)$ achieved by an N-qubit entangled probe with the totality of N active qubits, or with a reduced fraction $N_1 = N - 1$ of active qubits. In each scenario, the N-qubit entangled probe is prepared at the optimal degree of entanglement κ_{opt} of Eq. (35) as it follows either from Eqs. (26)–(27) or from Eqs. (42)–(43).
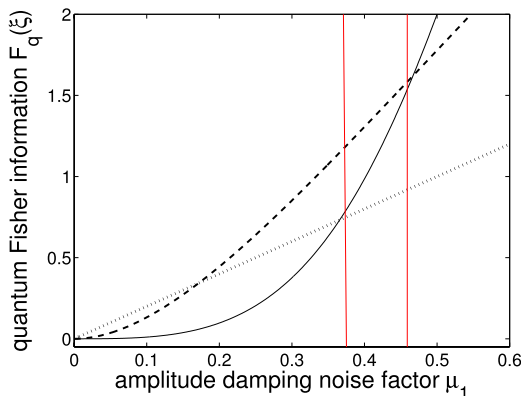

Fig. 8. (Color online) As a function of the amplitude damping noise factor μ_1, for a total probe size $N = 3$, the quantum Fisher information $F_q(\xi)$ of Eq. (29) with all N entangled qubits active (solid line), $F_q(\xi)$ of Eq. (44) with $N_1 = N - 1$ active qubits among the N entangled qubits (dashed line), and $F_q(\xi) = N_1 \mu_2^2$ for N_1 independent qubits optimally prepared in the separable state $|+'\rangle^{\otimes N_1}$ (dotted line). Between the two vertical red lines at $\mu_1 = 0.357$ and $\mu_1 = 0.459$ is the region of μ_1 where the size $N = 3 = N_{\text{opt}}$ is the optimum for the probe with all N active qubits as indicated in Fig. 6(a).

Figure 8 shows a situation where a probe with a reduced fraction $N_1 = N - 1$ of active qubits exhibits a Fisher information $F_q(\xi)$ surpassing that of the probe with the totality N of active qubits, this occurring for sufficiently large noise levels when $\mu_1 < 0.467$. A reduced fraction $N_1 = N - 1$ of active qubits can also surpass the Fisher information of N_1 independent active qubits optimally prepared in the separable state $|+'\rangle^{\otimes N_1}$. Figure 8 shows that, for a noise factor $\mu_1 \in [0.172, 0.467]$, the setting with a reduced fraction $N_1 = N - 1$ of active qubits surpasses the two other settings of the probe. In addition, the size $N = 3$ addressed in Fig. 8 is the optimal size N_{opt} for the probe with all N active qubits when the noise factor $\mu_1 \in [0.357, 0.459]$, as indicated in Fig. 6(a). Over this range of μ_1, where the probe

with the reduced fraction N_1 is the best probe, it surpasses the probe with all N active qubits when it is at its best. This represents a proof of principle that, in definite noise configurations, accommodating an inactive qubit in the N-qubit entangled probe, can stand as the most efficient setting for estimation. We emphasize that this inactive qubit must be coherently prepared as part of the entangled probe in Eq. (8), and coherently measured with it. Instead of being measured, if this inactive qubit were discarded, the effect would be described by tracing out this qubit in the state ρ_ξ of Eq. (37). The consequence would be to cancel in ρ_ξ the term carrying the dependence with ξ, and to produce in this way a reduced state independent of ξ, as a result of the randomization provoked by discarding the inactive qubit. And such a reduced state independent of ξ would of course be of no use for estimating ξ and would entail a zero Fisher information $F_q(\xi)$. To obtain the performance of Fig. 8, it is therefore essential to measure the inactive qubit coherently with the entangled probe, and as we have seen, although this qubit never interacts with the process U_ξ being estimated, its presence can improve the estimation efficiency. This is a manifestation of counter-intuitive properties that can result from quantum entanglement, and here bearing useful significance for estimation.

Here also the case of the amplitude damping noise presented in Fig. 8 is merely illustrative, and analysis of Eq. (44) shows that this property of maximum efficiency with an inactive qubit in the probe is generic, and exists in other noise configurations. For instance it exists with the depolarizing noise of Figs. 1–5, over some range of the noise factor α.

In general, the most efficient strategy for estimating the qubit phase ξ, will depend on the specific conditions. In particular, it will depend on whether one has access to a fixed budget N of entangled qubits, corresponding to the maximum size one is able to manage for the probe, or if on the contrary one is able to adjust the size N at the optimal level N_{opt} as in Fig. 2(a) or 6(a) irrespective of its (high) value. It will also depend on the type and level of the noise, knowing that, qualitatively, as also reflected in other reports,[20,24] separable states are usually preferable at large noise while entangled states become superior at intermediate or at small noise levels. In any case, the optimal strategy, both in terms of probe size and degree of entanglement, can be worked out in given conditions by maximizing the quantum Fisher information $F_q(\xi)$ which is explicitly known via Eq. (29) or (44), and confronting to μ_2^2 per independent qubit each prepared in state $|+'\rangle$; and such a strategy is applicable to any axis $\mathbf{n}$ associated with the qubit rotation by the unknown phase angle ξ.

6. Discussion

The present study considered a generic estimation task consisting in estimating the phase ξ of a unitary transformation U_ξ on a qubit, equivalent in Bloch representation to a rotation by ξ around an arbitrary axis $\mathbf{n}$. The effect of quantum noise or decoherence was included under the form of the general noise model of Eqs. (4)–(6),

which by commuting with U_ξ has the ability to represent as a single lumped operation decohering actions which could take place anywhere over the whole process starting with preparation of the probe up to its measurement. The noise model of Eqs. (4)–(6) includes as special cases important decohering processes such as depolarizing noise, phase-flip noise, amplitude and generalized amplitude damping noise describing the interaction of the qubit with an environment represented by a thermal bath at arbitrary temperature.[30,39] For estimation of the phase ξ in the presence of noise, the quantum Fisher information $F_q(\xi)$ was used to assess the overall best achievable efficiency especially governing the mean-squared estimation error.

An optimal separable input probe was characterized as the pure qubit state $|+'\rangle$ matched to the arbitrary rotation axis $\mathbf{n}$, along with the maximum Fisher information μ_2^2 per qubit it achieves for the general noise model of Eqs. (4)–(6). The possibility of enhanced efficiency was then investigated by employing an N-qubit entangled probe prepared in the family of states $|\psi_0\rangle$ of Eq. (8) accommodating an arbitrary degree of entanglement κ. In such general conditions, an analytical expression was derived with Eq. (29) for the quantum Fisher information $F_q(\xi)$.

This expression of Eq. (29) shows that in the absence of noise, the N-qubit entangled probe of Eq. (8) always displays the superefficient scaling as N^2 of the Fisher information $F_q(\xi)$, for any non-degenerate degree of entanglement κ, and is in this respect superior to the Fisher information N achieved by the N-quibt optimal separable probe. This is the superefficiency property afforded by entanglement in noise-free conditions, also previously reported in other situations, and found here to hold across a whole family of partially entangled probes. Moreover, in the presence of noise or decoherence according to the model of Eqs. (4)–(6), the expression of Eq. (29) shows that, for all (non-vanishing) noise configurations, the Fisher information $F_q(\xi)$ always returns to zero asymptotically at large size N of the entangled probe. In addition, there always exists a finite size N_{opt} of the entangled probe, along with an optimal degree of entanglement κ_{opt}, together maximizing the Fisher information $F_q(\xi)$ at a level $F_q^{\mathrm{max}}(\xi)$ which is completely characterized, depending on the specific noise configuration. We established that for all unital noises (with $\mu_0 = 0$) in Eqs. (4)–(6), the optimal degree of entanglement is $\kappa_{\mathrm{opt}} = 1/2$, i.e. at maximum entanglement of the probe in Eq. (8). By contrast, for nonunital noises in Eqs. (4)–(6), optimality is often obtained at partial entanglement, for $\kappa_{\mathrm{opt}} \notin \{0, 1/2, 1\}$. The maximum $F_q^{\mathrm{max}}(\xi)$ achieved by $(N_{\mathrm{opt}}, \kappa_{\mathrm{opt}})$ in Eq. (29), can be confronted with the maximum Fisher information $N_{\mathrm{opt}}\mu_2^2$ achieved by the N_{opt}-quibt optimal separable probe, and is usually found superior across a significant range of noise conditions, essentially at intermediate or small noise levels. In such conditions, grouping the N qubits into independent blocks formed of N_{opt} entangled qubits, restores in the asymptotic regime of large N, a nonvanishing efficiency improving over that of N independent qubits optimally prepared.

The effect of letting only a reduced fraction N_1 among N entangled qubits in the probe interact with the process U_ξ plus noise was also investigated, by means of Eq. (44) which provides an explicit evaluation of the quantum Fisher information $F_q(\xi)$ in these conditions. It revealed that arranging for one inactive qubit (but more than one is unnecessary) in the entangled probe, sometimes stands as the most efficient setting for estimation. Such inactive qubit, although it never interacts with the process U_ξ being estimated, can nevertheless improve the estimation efficiency.

Owing to the importance of entanglement and decoherence for quantum estimation, several other important studies such as Refs. 5, 14, 16, 24, 32, 40 have dealt with the subject. However, our derivations of Eqs. (29) and (44) with their general conditions of applicability provide new elements here, especially with their analysis pointing to optimal probe size and degree of entanglement, according to the configurations over a broad class of quantum noises. Other studies often focused on frequency rather than phase estimation, especially to optimize the interaction time of the probe with the process being estimated, and with no optimal probe size N_{opt} and degree of entanglement κ_{opt} reported. Here, instead, we consider phase estimation with a fixed interaction time of the probe with the process being estimated, such as for instance the fixed traveling time of photons across an interferometer, so as to produce the phase shift ξ. We exhibit optimal conditions for the probe size N_{opt} and degree of entanglement κ_{opt}. As long as we can relate the phase ξ as $\xi = \nu T$ to an angular frequency ν acting over a fixed interaction time T, the associated Fisher informations can be related as $F_q(\nu) = T^2 F_q(\xi)$. Therefore, with a fixed time T, the evolutions and optimizations according to N and κ concerning $F_q(\xi)$ of Eq. (29) or (44) as reported here, apply equally to $F_q(\nu) = T^2 F_q(\xi)$, and consequently the present characterizations hold for phase as well as for frequency estimation. In these respects our results stand as useful complements, for a broader appreciation of the role of entanglement for quantum parameter estimation and quantum metrology, and for fuller understanding and mastering of entanglement for the benefit of information processing.

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