Physica A 414 (2014) 204–215

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Tsallis entropy for assessing quantum correlation with Bell-type inequalities in EPR experiment

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HIGHLIGHTS

- A new Bell-type inequality for nonlocal correlation in quantum systems is derived.
- The Tsallis entropy is used as a generalized metric of statistical dependence.
- It is applied to classical outcomes of quantum measurements, as in the EPR setting.
- Superiority and complementarity of the generalized Bell inequality is demonstrated.
- It is able to detect nonlocal quantum correlation from a larger set of observables.

ARTICLE INFO

Article history:
Received 14 April 2014
Received in revised form 13 July 2014
Available online 23 July 2014

Keywords:
Tsallis entropy
Quantum correlation
Bell inequalities
EPR experiment
Quantum information

ABSTRACT

A new Bell-type inequality is derived through the use of the Tsallis entropy to quantify the dependence between the classical outcomes of measurements performed on a bipartite quantum system, as typical of an EPR experiment. This new inequality is confronted with standard correlation-based Bell inequalities, and with other known Bell-type inequalities based on the Shannon entropy for which it constitutes a generalization. For an optimal range of the Tsallis order, the new inequality is able to detect nonlocal quantum correlation with measurements from a larger set of quantum observables. In this respect it is more powerful and also complementary compared to the previously known Bell-type inequalities.

1. Introduction

Nonlocal quantum correlation and entanglement are now considered as important resources for quantum information processing and communication, although they are far from being completely understood and mastered [1,2]. Distant parts of a quantum system can exhibit correlations or dependence that cannot be understood by classical theory. Such quantum correlations, challenging physical realism and locality, were first envisaged in the Einstein–Podolsky–Rosen (EPR) thought experiment of Ref. [3]. Quantitative tests practically implementable to detect such nonlocal quantum correlations were proposed with the Bell inequalities [4], later generalized with the CHSH inequalities from Ref. [5]. A Bell inequality [4], or a Bell-type inequality like the CHSH inequality of Ref. [5], is an inequality relating statistics on pairs of random variables constituted by the classical outcomes from measurements performed on the two parts of a bipartite quantum system. Such Bell inequalities concern classical variables accessible as measurement outcomes, and are derived based on the assumption of a joint probability distribution existing for these classical variables, as implied by local realism. However, measurements

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http://dx.doi.org/10.1016/j.physa.2014.07.053

performed on entangled quantum states can lead to violation of Bell inequalities [6]. Such violation was experimentally demonstrated in Refs. [7,8], as a concrete manifestation of nonlocal quantum correlations associated with entanglement. Entangled quantum states and nonlocal correlations are specially important as they are now recognized as specific and useful resources practically applicable to information processing [9,2], for instance for superdense coding [10], teleportation [11], quantum error correction [12], quantum cryptography [13], or quantum games and strategies [14–19].

In order to identify nonlocal quantum correlations (dependences) between measurements, common statistics considered first for Bell and CHSH inequalities were realized by simple linear cross-correlation [4,5]. Later, other metrics of statistical dependence were also examined. This is accomplished with metrics based on the Shannon entropy in Refs. [20,21]. Bell-type inequalities are derived in Refs. [20,21] with the Shannon entropy to assess statistical dependence between measurement outcomes from a bipartite quantum system. Such Bell-type inequalities can also be violated by measurements from an entangled quantum state, as reported in Refs. [20,21]. However, the (quantum projective) measurements leading to violation of correlation-based (as in Refs. [4,5]) or Shannon-entropy-based (as in Refs. [20,21]) Bell-type inequalities, do not coincide, as shown in Ref. [21]. For a same entangled state, a given measurement protocol may lead to violation of the correlation-based inequality and not of the Shannon-entropy-based inequality; and conversely for another measurement protocol. In this respect, the two classes of Bell-type inequalities are complementary for a broader ability to detect quantum correlation inherent to an entangled state, from some given quantum projective measurements.

In the present study, we will investigate the possibility of another class of Bell-type inequalities. We will turn to the Tsallis entropy as a basis to assess statistical dependence between the measurement outcomes performed on the bipartite quantum system involved in a Bell inequality. The Tsallis entropy is a generalization to the Shannon entropy. Especially, the Tsallis entropy is a nonadditive generalization to the additive Shannon entropy [22]. As a consequence, the Tsallis entropy does not share all the properties of the Shannon entropy. This is especially relevant for the behavior of the joint and conditional entropies, which play an essential role in the derivation of Bell inequalities, and which differ for the Tsallis and Shannon cases. We will however demonstrate the possibility of Bell inequalities based on the Tsallis entropy, which generalize those based on the Shannon entropy, and which in some sense can be considered as more powerful.

In the present paper, we first briefly review in Section 2 the derivation of Bell inequalities based on the Shannon entropy as in Refs. [20,21]. This will especially serve as a useful guideline for our extension to the Tsallis entropy. Section 3 briefly reviews some basic properties of the Tsallis entropy, with a special focus on joint and conditional Tsallis entropies relevant to the derivation of Bell-type inequalities for a bipartite quantum system. Next, in Section 4, we explicitly derive an original Bell-type inequality based on the Tsallis entropy, and generalizing the previously known Bell inequalities based on the Shannon entropy as in Refs. [20,21]. Then, Sections 5–7 examine the behavior of the new Tsallis–Bell inequality to quantify correlations in an EPR experiment. Violations of the new Tsallis–Bell inequality are exhibited, showing superior capabilities compared to previously known Bell-type inequalities, for detecting quantum nonclassical correlations from measurements on distant parts of a quantum system. Finally, Section 8 also discusses the relation of the present results with some other applications of the Tsallis entropy previously reported for quantum information.

2. Bell inequalities with Shannon entropy

For a generic random variable A assuming different states a with respective probabilities $\Pr\{A = a\} = P(a)$, the Shannon entropy $H(A)$ is the nonnegative quantity defined as Ref. [23]

$$H(A) = -\sum_a P(a)\log[P(a)]; \tag{1}$$

while for two random variables A and B the conditional Shannon entropy $H(A|B)$ is an averaged entropy defined as Ref. [23]

$$H(A|B) = \sum_b P(b)H(A|b) = -\sum_{a,b} P(a,b)\log[P(a|b)], \tag{2}$$

with $P(a|b)$ and $P(a, b)$ respectively the conditional and the joint probability distributions for A and B. For A and B independent, $P(a|b) = P(a)$ leads to $H(A|B) = H(A)$.

The joint Shannon entropy $H(A, B)$ is the entropy according to Eq. (1) based on the joint probability distribution $P(a, b)$; from $P(a, b) = P(a)P(b|a)$, it verifies a chain rule [23]

$$H(A, B) = H(A) + H(B|A), \tag{3}$$

especially giving $H(A, B) = H(A) + H(B)$ at independent A and B. The concavity ($\cap$) of the function $-x\log(x)$ yields the conditioning inequality

$$H(A|B) \le H(A), \tag{4}$$

and with a third random variable C,

$$H(A|B, C) \le H(A|B). \tag{5}$$

Now for four random variables A_1, A_2, B_1 and B_2, it follows that

$$H(A_1, B_2) \leq H(A_1, B_1, A_2, B_2) = H(A_1|B_1, A_2, B_2) + H(B_1|A_2, B_2) + H(A_2|B_2) + H(B_2). \tag{6}$$

The inequality in Eq. (6) results from the nonnegativity of the Shannon entropy and the chain rule of Eq. (3) when (A_1, B_2) is identified to A and (A_2, B_1) to B. The equality in Eq. (6) results from repeated applications of the chain rule of Eq. (3). Since from the conditioning inequality of Eq. (5) one has $H(A_1|B_1, A_2, B_2) \leq H(A_1|B_1)$ and $H(B_1|A_2, B_2) \leq H(B_1|A_2)$, it follows that

$$H(A_1, B_2) \leq H(A_1|B_1) + H(B_1|A_2) + H(A_2|B_2) + H(B_2), \tag{7}$$

which, by subtracting $H(B_2)$ on both sides and rearranging, yields

$$0 \leq H(A_1|B_1) + H(B_1|A_2) + H(A_2|B_2) - H(A_1|B_2). \tag{8}$$

Eq. (8), relating conditional Shannon entropies on pairs among four random variables, represents an entropic Bell inequality. Such entropic Bell inequalities have been introduced in Ref. [20]. They have been studied on three variables in Ref. [21].

Now our purpose is to examine the possibility of deriving a Bell inequality comparable to Eq. (8), yet in a generalized form involving the Tsallis entropy. As we shall see, the derivation of Section 2 cannot be directly repeated with the Tsallis entropy, because the Tsallis entropy, as a nonadditive generalization to the Shannon entropy, does not satisfy the chain rule of Eq. (3) which is an essential ingredient of the derivation. We will show however that adaptation of the chain rule and associated conditioning inequality comparable to Eq. (5) can be performed so as to derive, with the Tsallis entropy, a Bell inequality generalizing Eq. (8). We proceed with a review of the properties of the Tsallis entropy that will be relevant to our purpose of deriving a Bell inequality.

3. Tsallis entropy

For a random variable A with probability distribution $P(a)$, the Tsallis entropy $H_q(A)$ of order q is the nonnegative quantity defined as Refs. [24,22]

$$H_q(A) = \frac{1}{q-1}\left(1 - \sum_a P^q(a)\right) = \sum_a h_q[P(a)], \tag{9}$$

with

$$h_q(x) = \frac{x - x^q}{q-1} \tag{10}$$

a nonnegative concave ($\cap$) function of $x \in [0, 1]$ for any $q \geq 0$. At the limit $q = 1$, one obtains $H_1(A) = -\sum_a P(a)\ln[P(a)]$ matching the Shannon entropy of Eq. (1). In this way, the Tsallis entropy of Eq. (9) represents a generalization of the Shannon entropy of Eq. (1). The Tsallis entropy, however, does not share all the properties of the Shannon entropy. In particular, for two independent random variables, the Tsallis entropy is nonadditive, and for this reason it has been postulated to form the ground of a nonextensive generalization to statistical mechanics [24,22]. We want here to investigate the possibility of Bell-type inequalities based on the Tsallis entropy. For this purpose, it is very important to review and specify the behaviors of the Tsallis entropy on joint random variables, so as to identify properties available for the derivation of a Bell inequality along the line of Section 2.

With another random variable B assuming states b with probabilities $P(b)$, and related to A through conditional probabilities $P(a|b)$, the Tsallis entropy

$$H_q(A|b) = \sum_a h_q[P(a|b)], \tag{11}$$

conforms to Eq. (9) since $P(a|b)$ defines over a a probability distribution for any b. Through an average of $H_q(A|b)$ over $P(b)$, a conditional Tsallis entropy can be defined as

$$H_q(A|B) = \sum_b P(b)H_q(A|b) = \sum_{a,b} P(b)h_q[P(a|b)]. \tag{12}$$

When A and B are independent, then $P(a|b) = P(a)$ and $H_q(A|B) = H_q(A)$. On the contrary, when A is fully (deterministically) determined by B, then the conditional probabilities $P(a|b)$ can only assume values 1 or 0, and since $h_q(0) = h_q(1) = 0$, it follows that $H_q(A|b) = 0$ for any b, and its average $H_q(A|B) = 0$.

For any a one has

$$\sum_b P(b)h_q[P(a|b)] \leq h_q\Big[\sum_b P(b)P(a|b)\Big] = h_q[P(a)], \tag{13}$$

where the inequality in Eq. (13) follows from the concavity ($\cap$) of $h_q(\cdot)$. Then by summing Eq. (13) over a one obtains the inequality

$$H_q(A|B) \leq H_q(A). \tag{14}$$

In a similar way, based on the concavity ($\cap$) of $h_q(\cdot)$, with a third random variable C one has

$$H_q(A|B,C) \le H_q(A|B). \tag{15}$$

Eqs. (14)–(15) represent conditioning inequalities which are shared in common by the Tsallis entropy of any order $q \ge 0$ and by the Shannon entropy ($q = 1$) as expressed by Eqs. (4)–(5).

It is to note that another notion of conditional Tsallis entropy exists, which in place of the average of $H_q(A|b)$ over $P(b)$ in Eq. (12), averages over the so-called escort probability distribution $Q_q(b) = P^q(b)/\sum_b P^q(b)$ to define the alternative conditional entropy [25]

$$H_q^{\mathrm{esc}}(A|B) = \sum_b Q_q(b) H_q(A|b). \tag{16}$$

At independence of A and B, one has simultaneously $H_q(A|B) = H_q^{\mathrm{esc}}(A|B) = H_q(A)$. The two notions of conditional Tsallis entropy $H_q(A|B)$ of Eq. (12) and $H_q^{\mathrm{esc}}(A|B)$ of Eq. (16) also coincide at $q = 1$ to recap the common notion of conditional Shannon entropy in Eq. (2). At arbitrary q, the escort-based conditional entropy $H_q^{\mathrm{esc}}(A|B)$ of Eq. (16) displays interesting behavior for joint random variables (see Eq. (22)); however, since the inequality analog to Eq. (13) is no longer satisfied, $H_q^{\mathrm{esc}}(A|B)$ does not verify the conditioning inequalities of Eqs. (14)–(15) which played a part in the derivation of the entropic Bell inequality of Section 2.

For the derivation of entropic Bell inequalities, in addition to conditioning inequalities, a chain rule for the entropy is also essential, that we now examine for the Tsallis entropy. For the joint Tsallis entropy $H_q(A,B)$ based on the joint probabilities $P(a,b) = P(a|b)P(b)$, the difference with the marginal Tsallis entropy $H_q(A)$ can be expressed from the definition of Eq. (9), as

$$H_q(A,B) - H_q(A) = \frac{1}{q-1}\Big[-\sum_a P^q(a)\sum_b P^q(b|a) + \sum_a P^q(a)\Big], \tag{17}$$

which is also

$$H_q(A,B) - H_q(A) = \frac{1}{q-1}\sum_a P^q(a)\Big[1 - \sum_b P^q(b|a)\Big] \tag{18}$$

$$= \sum_a P^q(a) H_q(B|a). \tag{19}$$

A chain rule is then obtained for the Tsallis entropy as

$$H_q(A,B) = H_q(A) + \sum_a P^q(a) H_q(B|a). \tag{20}$$

The chain rule of Eq. (20) can be further evolved to a special form if one resorts to the escort distribution $Q_q(a) = P^q(a)/\sum_a P^q(a)$ previously mentioned. Inversion of Eq. (9) yields the sum $\sum_a P^q(a) = 1 - (q-1)H_q(A)$, which can be used for dividing Eq. (19) to give

$$\frac{H_q(A,B) - H_q(A)}{1 - (q-1)H_q(A)} = \sum_a Q_q(a) H_q(B|a) = H_q^{\mathrm{esc}}(B|A), \tag{21}$$

involving the escort-based conditional entropy $H_q^{\mathrm{esc}}(B|A)$ from Eq. (16), and leading to an alternative chain rule as

$$H_q(A,B) = H_q(A) + H_q^{\mathrm{esc}}(B|A) + (1-q)H_q(A)H_q^{\mathrm{esc}}(B|A). \tag{22}$$

The chain rules of Eq. (20) or Eq. (22) express a nonadditive character attached to the Tsallis entropy, as exploited in the nonextensive generalization of statistical mechanics [24,22]. Both chain rules of Eqs. (20) and (22) coincide at $q = 1$, to provide the standard chain rule verified by the Shannon entropy in Eq. (3). It is however the chain rule of Eq. (20), associated with the conditional Tsallis entropy of Eq. (12), which will be useful to us for the derivation of a Bell inequality; (and not the chain rule of Eq. (22) based on the conditional Tsallis entropy of Eq. (16)).

4. A Bell-type inequality with Tsallis entropy

The derivation of Eqs. (6)–(8) of the Bell inequality in Eq. (8) was based on the properties of the Shannon entropy. We will now demonstrate how it is possible to reproduce a comparable derivation with the Tsallis entropy. For four random variables A_1, A_2, B_1 and B_2, we start with a Tsallis analog of Eq. (6), as

$$H_q(A_1,B_2) \le H_q(A_1,B_1,A_2,B_2) = H_q(B_1,A_2,B_2) + \sum_{b_1,a_2,b_2} P^q(b_1,a_2,b_2) H_q(A_1|b_1,a_2,b_2). \tag{23}$$

The inequality in Eq. (23) results from the nonnegativity of the Tsallis entropy and the chain rule of Eq. (20) when (A_1, B_2) is identified to A and (A_2, B_1) to B. The equality in Eq. (23) results from the chain rule of Eq. (20).

Next in Eq. (23) for $H_q(B_1, A_2, B_2)$ by the chain rule of Eq. (20) one has

$$H_q(B_1, A_2, B_2) = H_q(A_2, B_2) + \sum_{a_2,b_2} P^q(a_2, b_2) H_q(B_1|a_2, b_2). \tag{24}$$

Through a similar step, in Eq. (24) for $H_q(A_2, B_2)$ by the chain rule of Eq. (20) one has

$$H_q(A_2, B_2) = H_q(B_2) + \sum_{b_2} P^q(b_2) H_q(A_2|b_2). \tag{25}$$

Similarly from Eq. (20), the first term in Eq. (23) is

$$H_q(A_1, B_2) = H_q(B_2) + \sum_{b_2} P^q(b_2) H_q(A_1|b_2). \tag{26}$$

By using Eqs. (24), (25), (26) in Eq. (23) and rearranging, one obtains a Tsallis analog of Eq. (8) as

$$\begin{aligned} 0 \le & \sum_{b_1,a_2,b_2} P^q(b_1, a_2, b_2) H_q(A_1|b_1, a_2, b_2) + \sum_{a_2,b_2} P^q(a_2, b_2) H_q(B_1|a_2, b_2) \\ & + \sum_{b_2} P^q(b_2) H_q(A_2|b_2) - \sum_{b_2} P^q(b_2) H_q(A_1|b_2). \end{aligned} \tag{27}$$

Eq. (27) cannot yet play the role of a Bell inequality because it does not involve only statistics on pairs of random variables alone. A step further can be taken based on the conditioning inequalities

$$\sum_{a_2,b_2} P^q(a_2, b_2) H_q(B_1|a_2, b_2) \le \sum_{a_2} P^q(a_2) H_q(B_1|a_2), \tag{28}$$

and

$$\sum_{b_1,a_2,b_2} P^q(b_1, a_2, b_2) H_q(A_1|b_1, a_2, b_2) \le \sum_{b_1} P^q(b_1) H_q(A_1|b_1), \tag{29}$$

which are however valid only for $q \ge 1$. Eq. (29) follows from Eq. (28); and Eq. (28) follows from the inequality $\sum_a P^q(a) + \sum_b P^q(b) \le 1 + \sum_{ab} P^q(a, b)$ applying for $q \ge 1$ for any joint probability distribution $P(a, b)$ when the power function x^q is superadditive, i.e. $(x+y)^q \ge x^q + y^q$, and gives $x^q \le x$ for any $x, y \in [0, 1]$. Then Eqs. (28)–(29) associated with Eq. (27) lead to

$$0 \le \sum_{b_1} P^q(b_1) H_q(A_1|b_1) + \sum_{a_2} P^q(a_2) H_q(B_1|a_2) + \sum_{b_2} P^q(b_2) H_q(A_2|b_2) - \sum_{b_2} P^q(b_2) H_q(A_1|b_2). \tag{30}$$

Eq. (30) involving only statistics on pairs of random variables, has the form of a Bell inequality expressed with the Tsallis entropy, and comparable to Eq. (8) expressed with the Shannon entropy. Especially, at $q = 1$, Eqs. (30) and (8) coincide. At any order $q \ge 1$ where Eq. (30) is expected to apply, one has access to a broad family of Bell-type inequalities. Eq. (30), in common with any standard Bell-type inequality, is established here as a necessary condition, that follows from the existence of a joint probability distribution $P(a_1, a_2, b_1, b_2)$ on four compatible random variables (A_1, A_2, B_1, B_2), and that is necessarily satisfied by the two-variable probability distributions derived from an underlying four-variable probability distribution $P(a_1, a_2, b_1, b_2)$. Violation of Eq. (30) would imply the impossibility of a four-variable probability distribution $P(a_1, a_2, b_1, b_2)$ connecting the two-variable probability distributions $P(a_1, b_1)$, $P(b_1, a_2)$, $P(a_2, b_2)$, $P(a_1, b_2)$ involved in Eq. (30). This is the common rationale of any standard Bell-type inequality, which is preserved here with the generalized Tsallis–Bell inequality of Eq. (30). We plan now to investigate the newly established Tsallis–Bell inequality of Eq. (30), for assessment of correlation or statistical dependence between the outcomes of measurements performed on two distant parts of a nonseparable quantum system in an EPR experiment.

5. EPR experiment

We consider the standard setting of an EPR experiment [6,26]. Two protagonists, conventionally called Alice and Bob, share a pair of entangled qubits prepared in a so-called Bell or EPR quantum state

$$|\psi^{\mathrm{AB}}\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big). \tag{31}$$

Alice has access to the first qubit of the pair, Bob to the second qubit, and each of them can separately perform a measurement on her/his qubit. Alice and Bob can measure observables of the form [26]

$$\mathrm{O}(\theta) = \sin(\theta)\mathrm{X} + \cos(\theta)\mathrm{Z}, \tag{32}$$

where X and Z are the standard Pauli operators [6] for a spin-1/2 measurement. In the computational basis $\{|0\rangle, |1\rangle\}$, the observable $\mathrm{O}(\theta)$ of Eq. (32) has the two eigenstates $|V_+(\theta)\rangle = [\cos(\theta/2), \sin(\theta/2)]^\top$ and $|V_-(\theta)\rangle = [\sin(\theta/2), -\cos(\theta/2)]^\top$ associated with the two eigenvalues ± 1.

Alice can choose a direction $\theta = \alpha$ and measure on her qubit the observable $\mathrm{A} \equiv \mathrm{O}(\theta = \alpha)$; Bob independently can choose a direction $\theta = \beta$ and measure on his qubit the observable $\mathrm{B} \equiv \mathrm{O}(\theta = \beta)$. Due to their quantum character, the outcome of such measurements occur at random, for Alice as the random variable $A = \pm 1$, and for Bob as $B = \pm 1$. Since these measurements take place on a qubit pair in an entangled state $|\psi^{\mathrm{AB}}\rangle$, one is interested in quantifying the correlation or statistical dependence between their outcomes.

For this purpose of quantifying statistical dependence, one has the joint probability

$$P(A = +1, B = +1) = \left|\langle \psi^{\mathrm{AB}} | V_+(\alpha) \otimes V_+(\beta)\rangle\right|^2 . \tag{33}$$

In the bipartite basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ one has

$$\begin{aligned} |V_+(\alpha) \otimes V_+(\beta)\rangle = {} & \cos(\alpha/2)\cos(\beta/2)|00\rangle + \cos(\alpha/2)\sin(\beta/2)|01\rangle \\ & + \sin(\alpha/2)\cos(\beta/2)|10\rangle + \sin(\alpha/2)\sin(\beta/2)|11\rangle, \end{aligned} \tag{34}$$

from where it follows that

$$P(A = +1, B = +1) = \frac{1}{4}[1 - \cos(\alpha - \beta)]. \tag{35}$$

In a similar way, one obtains

$$\begin{aligned} P(A = +1, B = -1) &= \left|\langle \psi^{\mathrm{AB}} | V_+(\alpha) \otimes V_-(\beta)\rangle\right|^2 \\ &= \frac{1}{4}[1 + \cos(\alpha - \beta)], \end{aligned} \tag{36}$$

and

$$P(A = -1, B = +1) = \frac{1}{4}[1 + \cos(\alpha - \beta)], \tag{37}$$

$$P(A = -1, B = -1) = \frac{1}{4}[1 - \cos(\alpha - \beta)]. \tag{38}$$

From the joint probabilities of Eqs. (35)–(38), by summation, the marginal probabilities follow as

$$P(A = +1) = P(A = -1) = P(B = +1) = P(B = -1) = \frac{1}{2}. \tag{39}$$

And from the Bayes rule, the conditional probabilities are

$$\begin{aligned} P(A = +1 | B = +1) &= \frac{P(A = +1, B = +1)}{P(B = +1)} \\ &= \frac{1}{2}[1 - \cos(\alpha - \beta)] = P(A = -1 | B = -1), \end{aligned} \tag{40}$$

and

$$P(A = +1 | B = -1) = P(A = -1 | B = +1) = \frac{1}{2}[1 + \cos(\alpha - \beta)]. \tag{41}$$

One has then access to the correlation between Alice's and Bob's measurements, as used in standard Bell inequalities, as the average

$$\langle AB \rangle = -\cos(\alpha - \beta). \tag{42}$$

For entropic Bell inequalities, one has access for instance for the conditional Tsallis entropy according to Eq. (11), to

$$H_q(A|B = -1) = H_q(A|B = +1) = h_q\left(\frac{1}{2}[1 - \cos(\alpha - \beta)]\right) + h_q\left(\frac{1}{2}[1 + \cos(\alpha - \beta)]\right). \tag{43}$$

We note that in general the measurement of quantum observables A and B performed by Alice and Bob on their respective qubit of the pair are compatible and can be made separately since they occur on two distinct qubits. On the contrary, measurement of two distinct observables $\mathrm{A}_1 \equiv \mathrm{O}(\theta = \alpha_1)$ and $\mathrm{A}_2 \equiv \mathrm{O}(\theta = \alpha_2)$ at two angles $\alpha_1 \neq \alpha_2$ on the same qubit, cannot be performed simultaneously; the commutator $[\mathrm{A}_1, \mathrm{A}_2] = \sin(\alpha_2 - \alpha_1)[\mathrm{Z}, \mathrm{X}] = \sin(\alpha_2 - \alpha_1)2i\mathrm{Y}$ does not vanish in general precluding simultaneous measurement of the incompatible observables A_1 and A_2 on the same qubit.

To quantify the correlation or statistical dependence between the outcomes of measurements performed by Alice and Bob, we will now investigate the capabilities of the new Tsallis–Bell inequality of Eq. (30), and its confrontation with other previously established Bell inequalities.

6. Measurements from three observables

For an effective analysis of the behavior of the new Tsallis–Bell inequality of Eq. (30), we examine the situation where Alice and Bob can choose to measure three distinct observables. This is the original Bell scenario considered in Ref. [4]. Also a comparable approach is taken in Refs. [20,21] for Bell inequalities based on the Shannon entropy, with a general formulation in four variables as with our Eq. (30), followed by a concrete analysis on reduced conditions allowing to observe nontrivial violation of the Bell inequalities.

Alice and Bob, each time they perform a measurement, can separately decide through independent local random choices, to measure one among two observables. For her two observables Alice settles $\mathrm{A}_1 \equiv \mathrm{O}(\alpha_1 = 0) = \mathrm{Z}$ and $\mathrm{A}_2 \equiv \mathrm{O}(\alpha_2)$ at an arbitrary angle α_2. In a comparable way, Bob settles his two observables as $\mathrm{B}_1 \equiv \mathrm{O}(\beta_1 = 0) = \mathrm{Z}$ and $\mathrm{B}_2 \equiv \mathrm{O}(\beta_2)$ at an arbitrary angle β_2. When Alice and Bob measure the observable Z on their respective qubit of the pair in the entangled state $|\psi^{\mathrm{AB}}\rangle$, their measurements A_1 and B_1 are perfectly correlated as $A_1 = -B_1$, yielding $\langle A_1 B_1\rangle = -1$ and $H(A_1|B_1) = 0$. Then the entropic Shannon–Bell inequality of Eq. (8) becomes

$$0 \leq H(B_1|A_2) + H(A_2|B_2) - H(A_1|B_2), \tag{44}$$

also expressible on three variables (A_2, B_1, B_2) alone since $H(A_1|B_2) = H(B_1|B_2)$. Eq. (44) is comparable with the entropic Bell inequalities on three variables studied in Ref. [21]. The same reason $A_1 = -B_1$ leads similarly to $H_q(A_1|b_1) = 0$ for any b_1, and $H_q(A_1|b_2) = H_q(B_1|b_2)$ for any b_2, so the new Tsallis–Bell inequality of Eq. (30) becomes

$$0 \leq \sum_{a_2} P^q(a_2) H_q(B_1|a_2) + \sum_{b_2} P^q(b_2) H_q(A_2|b_2) - \sum_{b_2} P^q(b_2) H_q(A_1|b_2). \tag{45}$$

And since, from Eq. (39) for a qubit pair in the maximally entangled state $|\psi^{\mathrm{AB}}\rangle$ of Eq. (31), the marginal probabilities of the measurements are all equal, one obtains for the Tsallis–Bell inequality,

$$0 \leq H_q(B_1|A_2) + H_q(A_2|B_2) - H_q(A_1|B_2), \tag{46}$$

applying for any $q \geq 1$, and which is also expressible on three variables (A_2, B_1, B_2) since $H_q(A_1|B_2) = H_q(B_1|B_2)$ for any q. The Tsallis–Bell inequality of Eq. (46) at $q = 1$ coincides with the Shannon–Bell inequality of Eq. (44), and constitutes a generalization at $q \geq 1$.

By permutations of the variables in Eq. (46), two other nonequivalent inequalities can be obtained as

$$0 \leq H_q(A_1|B_2) + H_q(B_2|A_2) - H_q(B_1|A_2), \tag{47}$$

$$0 \leq H_q(A_2|B_1) + H_q(A_1|B_2) - H_q(A_2|B_2). \tag{48}$$

Again, since $A_1 = -B_1$ and $H_q(A_1|B_2) = H_q(B_1|B_2)$, the Tsallis–Bell inequalities of Eqs. (46)–(48) represent inequalities on the three nonequivalent variables (A_2, B_1, B_2). Yet, under the forms of Eqs. (46)–(48), the three Tsallis–Bell inequalities are computable from statistics involving only pairs of random variables occurring as outcomes of compatible quantum measurements that can be simultaneously performed on each one of the two parts of the bipartite system. This is a requirement for a Bell inequality, so as to be experimentally evaluable to serve as an experimental test for quantum correlation.

For comparison, it is also interesting to establish a standard correlation-based Bell inequality on the three variables (A_2, B_1, B_2). For measurements confined to ± 1, the quantity $B_1 A_2 + (A_2 - B_1) B_2$ is 1 when $B_1 = A_2$, and it is $-1 \pm 2 \leq 1$ when $B_1 = -A_2$. One therefore has the average

$$\langle B_1 A_2\rangle + \langle A_2 B_2\rangle - \langle B_1 B_2\rangle \leq 1, \tag{49}$$

and by permutations,

$$\langle B_1 B_2\rangle + \langle B_2 A_2\rangle - \langle B_1 A_2\rangle \leq 1, \tag{50}$$

$$\langle A_2 B_1\rangle + \langle B_1 B_2\rangle - \langle A_2 B_2\rangle \leq 1 \tag{51}$$

which are similar to the three correlation-based Bell inequalities on three variables considered in Ref. [21]. For experimental evaluation from pairs of compatible quantum measurements on each part of the bipartite system, one again uses $A_1 = -B_1$ to obtain the equivalent inequalities

$$\langle B_1 A_2\rangle + \langle A_2 B_2\rangle + \langle A_1 B_2\rangle \leq 1, \tag{52}$$

$$-\langle A_1 B_2\rangle + \langle B_2 A_2\rangle - \langle B_1 A_2\rangle \leq 1, \tag{53}$$

$$\langle A_2 B_1\rangle - \langle A_1 B_2\rangle - \langle A_2 B_2\rangle \leq 1. \tag{54}$$

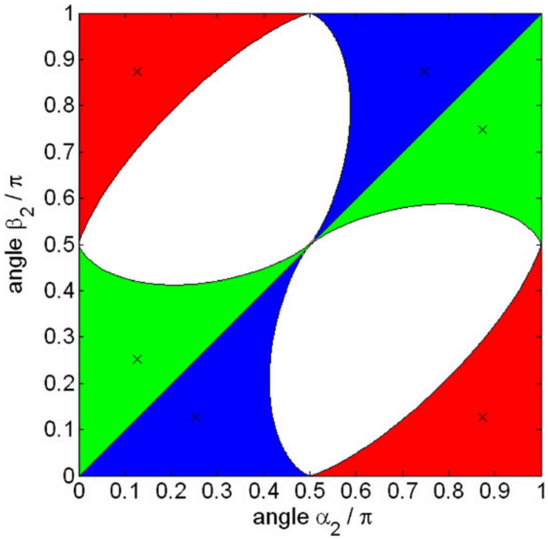

Fig. 1. For the measurement angles α_2 and β_2 determining the observables $\mathrm{A}_2 \equiv \mathrm{O}(\alpha_2)$ and $\mathrm{B}_2 \equiv \mathrm{O}(\beta_2)$ from Eq. (32) measured respectively by Alice and Bob on their qubit of a shared entangled pair in state $|\psi^{\mathrm{AB}}\rangle$ of Eq. (31), in the domain $(\alpha_2, \beta_2) \in [0, \pi]^2$, the colored regions indicate the measurement configurations (α_2, β_2) violating the Tsallis–Bell inequalities of Eqs. (46)–(48) at order $q = 1$, which coincide with the Shannon–Bell inequalities similar to Eq. (44). Six crosses ($\times$) locate the maximal violations when the right-hand side of inequalities Eqs. (46)–(48) uniformly reaches -0.134 Sh $= -0.093$ nat. The green region indicates violation of Eq. (46), with maximal violation at $(0.126\pi, 0.252\pi)$ and $(0.874\pi, 0.748\pi)$; the blue region is for Eq. (47), with maximal violation at $(0.252\pi, 0.126\pi)$ and $(0.748\pi, 0.874\pi)$; and the red for Eq. (48), with maximal violation at $(0.126\pi, 0.874\pi)$ and $(0.874\pi, 0.126\pi)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Violation of the Bell inequalities

The derivation of Bell inequalities connecting random variables is based on the existence of a joint probability distribution between the variables. This existence assumes local realism and implies compatibility of measurement jointly for all the variables. This classical assumption underlies the derivation of any Bell inequality, in a standard correlation form as in Eqs. (52)–(54), or based on the Shannon entropy as in Section 2, and this is also true with the Tsallis entropy for the derivation in Section 4. Bell inequalities can be violated by measurements performed on entangled quantum states, which break the classical assumption of local realism. Entanglement is a necessary condition for violation of Bell inequalities; therefore violation of Bell inequalities identifies nonclassical, i.e. quantum, nonlocal correlation or dependence which can exist between distant parts of a physical system. We investigate here the conditions of violation of the new Tsallis–Bell inequalities in their form of Eqs. (46)–(48), so as to test their ability to register nonclassical quantum correlation, especially as a function of the order $q \geq 1$.

Alice and Bob configure their measurements through the choice of the angles, respectively α_2 and β_2, determining their observables $\mathrm{A}_2 \equiv \mathrm{O}(\alpha_2)$ or $\mathrm{B}_2 \equiv \mathrm{O}(\beta_2)$. Given the symmetries of the quantum process, it is enough to consider the angles α_2 and β_2 in the interval $[0, \pi]$ to cover the relevant measurement configurations. We have tested the configurations $(\alpha_2, \beta_2) \in [0, \pi]^2$ that lead to violation of the Tsallis–Bell inequalities of Eqs. (46)–(48) at various $q \geq 1$.

Fig. 1 presents in the domain $(\alpha_2, \beta_2) \in [0, \pi]^2$ of the plane, all the measurement configurations (α_2, β_2) that lead to violation of the Tsallis–Bell inequalities of Eqs. (46)–(48) at order $q = 1$, which coincide with the corresponding Shannon–Bell inequalities similar to Eq. (44). The violations of each of the three inequalities of Eqs. (46)–(48) are shown separately in Fig. 1 with three different colors over the domain $(\alpha_2, \beta_2) \in [0, \pi]^2$. Also shown in Fig. 1 is the location of the configurations (α_2, β_2) achieving maximal violation of each of the three inequalities (46)–(48).

Fig. 1 shows that the violations of each of the three inequalities of Eqs. (46)–(48) occur separately in different regions of the domain $(\alpha_2, \beta_2) \in [0, \pi]^2$. This means that different choices of observables A_2 and B_2 may lead to violation of one Bell inequality and no violation of the two others among Eqs. (46)–(48) at $q = 1$. The three inequalities are therefore nonredundant, and their combination allows for the largest set of observables A_2 and B_2 capable of detecting nonclassical quantum correlation between distant measurements by Alice and Bob. An interesting possibility with the Tsallis–Bell inequalities of Eqs. (46)–(48), is that this set of violating observables can be enlarged by increasing the order $q > 1$. For illustration, Fig. 2 addresses the violation of the three Tsallis–Bell inequalities of Eqs. (46)–(48) at a higher order $q = 2.46$ superior to the Shannon order $q = 1$ of Fig. 1.

The results of Fig. 2 demonstrate enlarged regions in the domain $(\alpha_2, \beta_2) \in [0, \pi]^2$ leading to violation of the Tsallis–Bell inequalities of Eqs. (46)–(48) at $q = 2.46$, compared to the same inequalities at $q = 1$ in Fig. 1. As the order q increases between Figs. 1 and 2, one observes enlargement of each one of the three regions corresponding to violation of each one of the three Tsallis–Bell inequalities (46)–(48). Also, the location of the configurations (α_2, β_2) of maximal violation are

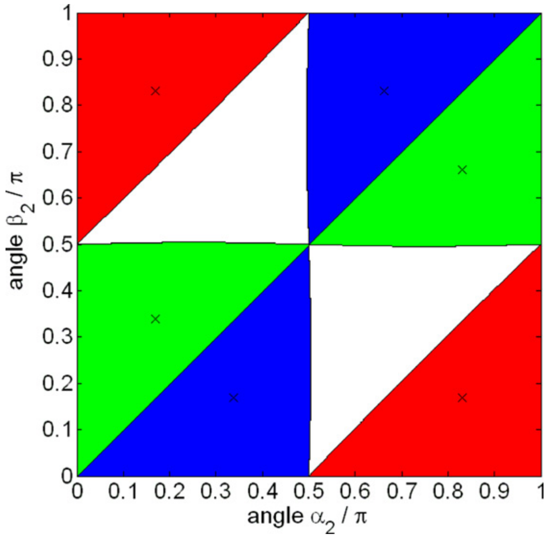

Fig. 2. Same as Fig. 1 but at the order $q = 2.46$, when the Tsallis–Bell inequalities of Eqs. (46)–(48) no longer coincide with the Shannon–Bell inequalities similar to Eq. (44). Six crosses (×) locate the maximal violations when the right-hand side of inequalities (46)–(48) uniformly reaches −0.161 Sh = −0.112 nat. The green region indicates violation of Eq. (46), with maximal violation at $(0.169\pi, 0.338\pi)$ and $(0.831\pi, 0.662\pi)$; the blue region is for Eq. (47), with maximal violation at $(0.338\pi, 0.169\pi)$ and $(0.662\pi, 0.831\pi)$; and the red for Eq. (48), with maximal violation at $(0.169\pi, 0.831\pi)$ and $(0.831\pi, 0.169\pi)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

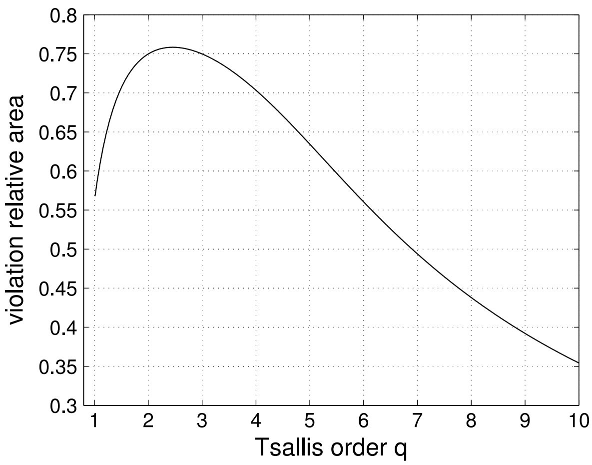

Fig. 3. Relative area in the domain $[0, \pi]^2$ of the total region of all measurement configurations (α_2, β_2) violating a Tsallis–Bell inequality among Eqs. (46)–(48), as a function of the order q.

slightly displaced between Figs. 1 and 2. Also, as indicated with Figs. 1 and 2, the magnitude of the maximal violation itself is enhanced as the order q increases. In this way, increasing the order q above 1 allows higher sensitivity for detecting nonclassical quantum correlation. This is materialized by a larger set of observables A_2 and B_2 capable of detecting nonclassical quantum correlation between the measurements by Alice and Bob. This is a benefit of the generalized entropy used for the new Tsallis–Bell inequalities of Eqs. (46)–(48), to allow for a larger set of observables for detecting quantum correlation.

We have quantified the total area in the domain $[0, \pi]^2$ associated with measurement configurations (α_2, β_2) violating a Tsallis–Bell inequality among Eqs. (46)–(48). This corresponds to the cumulated area of all colored regions in Fig. 1 or Fig. 2. Especially, we have observed that the three areas associated with the three Tsallis–Bell inequalities (46)–(48) at a given q are found equal, even though the region associated with inequality (48) (the red region in Figs. 1–2) differs in shape; and this equality is observed for any order q, manifesting a form of equivalence for the three inequalities (46)–(48). This total violation area has been evaluated as a function of the Tsallis order q, and the resulting evolution is depicted in Fig. 3.

In Fig. 3, a maximum relative area of 0.758 is obtained at the optimal order $q = q_{\rm opt} = 2.46$. This is at this optimal value of q that we chose to represent the violating regions (α_2, β_2) in Fig. 2. Therefore, from Fig. 3, an optimal order $q_{\rm opt} = 2.46$ exists that makes the Tsallis–Bell inequalities of Eqs. (46)–(48) maximally sensitive for the detection of quantum correlation.

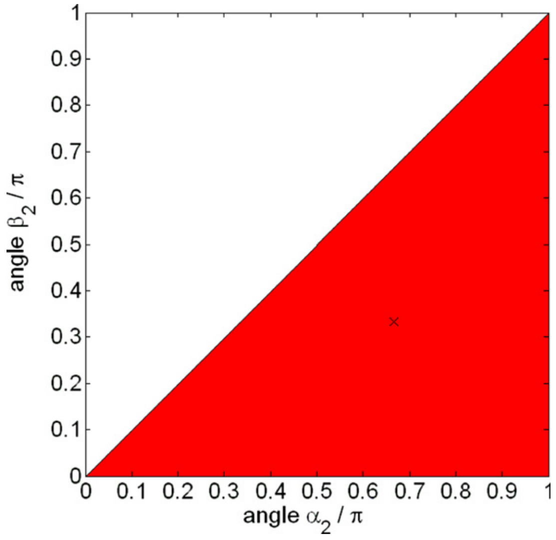

Fig. 4. Same as Figs. 1 and 2 but for the standard correlation-based Bell inequalities of Eqs. (52)–(54). No violation occurs for Eqs. (52) and (53). The red region indicates violation of Eq. (54), with the maximal violation at $(2\pi/3, \pi/3)$ indicated by the cross ($\times$), when the left-hand side of inequality (54) reaches 1.5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Especially, this optimal Tsallis order $q_{\text{opt}} = 2.46$ is markedly distinct from the Shannon order $q = 1$. Tsallis–Bell inequalities (46)–(48) around the optimal order $q_{\text{opt}} = 2.46$ are in this respect more sensitive than Shannon–Bell inequalities similar to Eq. (44), for the detection of quantum correlation. It is interesting to observe that in this way, the condition where the Tsallis–Bell inequality is maximally sensitive in Fig. 3, singles out a nontrivial optimal value $q_{\text{opt}} = 2.46$ of the Tsallis order. Maximum sensitivity does not occur at the trivial order $q = 1$ associated with the standard Shannon entropy. The sensitivity either does not monotonically evolve as q increases. At very large $q \to \infty$, one has for Eq. (10) the behavior $h_q(x) \sim x/q$, except at $x = 0$ or 1 where $h_q(x) = 0$; then from Eq. (11) one obtains $H_q(A|b) \sim \sum_a P(a|b)/q = 1/q$, for any b (configurations with $h_q(x = 0) = h_q(x = 1) = 0$ possibly occurring in the sum have no effect on $H_q(A|b)$). One therefore obtains an asymptotic inequality at $q \to \infty$ for Eq. (30) as $0 \le \sum_{b_1} P^q(b_1) + \sum_{a_2} P^q(a_2)$, which is obviously satisfied by any probability distributions, hence no measurement configuration exists that could lead to a violation. This explains the relative violation area in Fig. 3 going to zero at $q \to \infty$. There is then clearly in Fig. 3 a nonmonotonic action of the Tsallis order q on the sensitivity, with maximum efficacy at a nontrivial order $q_{\text{opt}} = 2.46$. Relatively few phenomena reveal a nonmonotonic response selecting a finite nontrivial optimal value $q_{\text{opt}} \neq 1$ for the Tsallis order [27–29], and the present Tsallis–Bell inequality can also be recognized for this reason.

As another basis for comparison, Fig. 4 shows the measurement configurations $(\alpha_2, \beta_2) \in [0, \pi]^2$ leading to violation of the standard correlation-based Bell inequalities of Eqs. (52)–(54).

It is visible from Fig. 4 that the measurement configurations (α_2, β_2) violating the correlation-based Bell inequalities of Eqs. (52)–(54), do not coincide everywhere with the configurations (α_2, β_2) violating the Tsallis–Bell inequalities of Eqs. (46)–(48) as shown in Figs. 1–2 at various orders q. This confirms that the metric used for statistical dependence, be it a standard cross-correlation or a conditional Shannon entropy or a generalized conditional Tsallis entropy, has a direct impact on the resulting Bell inequalities and their violation conditions, which do not occur for the same set of observables A_2 and B_2. This property was previously observed in Ref. [21] with the Shannon entropy, and is extended here in generalized conditions with the Tsallis entropy. In Fig. 4, a relative area of 0.5 is obtained for the region of all measurement configurations (α_2, β_2) in $[0, \pi]^2$ violating the standard correlation-based Bell inequalities of Eqs. (52)–(54). A slightly superior violation area of 0.567 is obtained with the Shannon–Bell inequalities similar to Eq. (44), as visible from Fig. 3 at order $q = 1$. At higher order $q > 1$, the Tsallis–Bell inequalities of Eqs. (46)–(48) lead to larger violation areas. As visible in Fig. 3, for any order $q \in (1, 5.89]$, the violation area is superior with the Tsallis–Bell inequalities compared to the Shannon-based Bell inequalities at $q = 1$. At $q = 5.89$, the violation area for the Tsallis–Bell inequalities based on $H_q(\cdot)$ returns to its value for the Shannon–Bell inequalities based on $H_1(\cdot)$; and moreover at $q = 5.89$, the violating regions are returned to the same shapes as in Fig. 1. For any order $q \in [1, 6.90]$ in Fig. 3, the violation area is superior with the Tsallis–Bell inequalities compared to the correlation-based Bell inequalities. In this respect, for the detection of nonclassical quantum correlation, the new Tsallis–Bell inequalities at orders q in the vicinity of $q_{\text{opt}} = 2.46$, exhibit enhanced sensitivity compared to the other previously known Bell-type inequalities, since they allow a larger set of observables A_2 and B_2 producing violating measurements. For the correlation-based Bell inequalities of Eqs. (52)–(54), the maximal violation of 1.5 given in Fig. 4 has only a relative meaning, since it is tied to the arbitrary values ± 1 ascribed to the two outcomes of a spin measurement. Meanwhile, the informational values of the maximal violations given in Figs. 1–2 are more intrinsically meaningful for statistical dependence.

8. Discussion

From the classical outcomes of measurements performed on a bipartite quantum system, nonlocal correlations can be detected by the violation of a Bell inequality. We have derived a new Bell-type inequality, Eq. (30). This new inequality is based on the Tsallis entropy as a metric to quantify the dependence between the classical random variables formed by the outcomes of quantum measurements performed on a bipartite quantum system. The new Bell-type inequality based on the Tsallis entropy generalizes previously known inequalities based on the Shannon entropy. We have studied the conditions of violation of this Tsallis–Bell inequality, in an EPR experiment when the measurements are performed on a bipartite quantum system in a maximally entangled state. We have shown that, for an appropriate range of the Tsallis order q, violation of the Tsallis–Bell inequality occurs with measurements from a larger set of quantum observables, compared to previously known Bell inequalities based on the Shannon entropy or on cross-correlation. In this respect, the new Tsallis–Bell inequality can be considered as more sensitive or more powerful than these previously known Bell inequalities. They are also complementary, because their violation does not always occur for the same measurement configurations, and putting together the new Tsallis–Bell inequality and the standard correlation-based Bell inequality offers the largest set of measurements capable of detecting nonlocal quantum correlation.

Other applications have been reported of the Tsallis entropy in quantum information, which however differ from the present results. A whole line of studies, inaugurated in Refs. [30–32], explored the application of the Tsallis entropy to provide a test for separability or nonseparability of a quantum state. A bibliographic review of these studies can be found in Ref. [22], and some very recent extensions in Ref. [33]. In these studies, the Tsallis entropy is used in its quantum form as $S_q(A) = [1 - \mathrm{Tr}(\rho_A^q)]/(q-1)$ operating on a density operator ρ_A for a quantum system A. The associated conditional quantum entropy can go negative, contrary to its classical counterpart associated with Eq. (9) or Eq. (1) which is always nonnegative [34]. Negativity of the conditional quantum entropy occurs only for entangled quantum states. In this way, nonnegativity of the conditional quantum entropy $S_q(A|B) \geq 0$ plays the role of an entropic Bell inequality satisfied when a bipartite density operator ρ_{AB} is separable. Violation of this condition with a negative conditional quantum entropy $S_q(A|B) < 0$ is used as a test of nonseparability or entanglement of a bipartite density operator ρ_{AB}, and controlling the order q of the quantum Tsallis entropy increases the power of the test assessed through the size of the set of detected entangled states [30,35]. An extension for characterizing entanglement in a multipartite quantum state is done in Ref. [36]. The problem we address here with the Tsallis entropy is different. The Tsallis entropy is not used here under its quantum form $S_q(A) = [1 - \mathrm{Tr}(\rho_A^q)]/(q-1)$ on density operators, so as to characterize a quantum state which in itself is not directly observable. Instead, the Tsallis entropy is used here under its classical form of Eq. (9) on the classical random variables resulting from measurements on a quantum state. In this way, the Tsallis entropy is used for an experimental test on measurable quantities, in the spirit of the original Bell and CHSH inequalities related to an EPR experiment [4,5,3]. As a result, the Tsallis–Bell inequality of Eq. (30) is new here and not present in previous studies.

Other applications of the Tsallis entropy in quantum information, and related to the line of studies of Refs. [30–32] to characterize density operators, were reported for the inference of quantum states with minimum fake entanglement [37–39], for metrics of disorder or mixedness [40–43] or purification [44] of quantum states, and for new metrics of quantumness [45–47].

The Tsallis–Bell inequality of Eq. (30) has been tested here with measurements performed on a qubit pair in the maximally entangled singlet state $|\psi^{\mathrm{AB}}\rangle$ of Eq. (31), which represent fundamental reference conditions for an EPR experiment. Beyond, the inequality can be tested on other quantum states, either pure or mixed states [48–50], and also with quantum systems of larger dimensionality, higher than the dimension two of the qubit [51,20,52–54]. All these conditions are accessible to exploration with the inequality in its general form given by Eq. (30). Other quantum experiments involving a statistical characterization of entanglement can also be approached with the present generalized methodology. In this direction, EPR experiment in the presence of noise could be examined [55–57] especially to test if the optimal Tsallis order q_{opt} is affected.

As in the present analysis, these broader conditions could reveal extended capabilities for the characterization of quantum states and measurements also stemming from the use of a generalized entropy. Also, this could bring further insight on the nonclassical correlation or dependence associated with entanglement in composite quantum systems. This would contribute to better understanding of the important resources for quantum information processing and communication formed by nonlocal quantum correlation and entanglement [58,2].

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