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Enhancing qubit information with quantum thermal noise

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HIGHLIGHTS

- Several generic informational quantities characterizing the qubit are analyzed.
- Qubit decoherence is represented by a quantum thermal noise at arbitrary temperature.
- Nontrivial regimes of variation are reported for the informational quantities.
- They do not always degrade but can show nonmonotonic variation at increasing temperature.
- Higher noise temperatures or increased decoherence may prove beneficial informationally.

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ABSTRACT

Informational quantities characterizing the qubit are analyzed in the presence of quantum thermal noise modeling the decoherence process due to interaction with the environment represented as a heat bath at arbitrary temperature. Nontrivial regimes of variation are reported for the informational quantities, which do not necessarily degrade monotonically as the temperature of the thermal noise increases, but on the contrary can experience nonmonotonic variations where higher noise temperatures can prove more favorable. Such effects show that increased quantum decoherence does not necessarily entail poorer informational performance, and they are related to stochastic resonance or noise-enhanced efficiency in information processing.

1. Introduction

For information processing it is known that, in some specific situations, noise is not necessarily a nuisance but can sometimes prove beneficial. Such possibility has been largely explored in the context of classical (non-quantum) information processing, especially in relation to the phenomenon of stochastic resonance under its many forms [1–7]. Investigation of stochastic resonance or noise-enhanced efficiency in information processing, has been extended to the quantum domain. Early studies on quantum stochastic resonance concentrated on noise-enhanced transmission of a periodic driving [8–13]. In different information processing contexts, stochastic resonance has been shown in networks of coupled spins [14–16], or in other high-dimensional quantum systems [17–19]. For tasks of quantum state detection or discrimination stochastic resonance has been reported in [20,21], and for quantum state estimation in [22,23].

For information transmission over noisy qubit channels – which is the main theme of the present report – stochastic resonance has been addressed by several studies. Refs. [24–26] considered Pauli qubit channels, forming a special class of unital noise channels, and found that enhancement by noise is dependent on the measure of performance and does not

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always exist for common informational measures. Ref. [27] showed stochastic resonance in a convex combination of a phase-flip channel and an amplitude damping channel. A related effect is investigated in [28,29] under the name of superactivation, when two noisy quantum channels with zero information capacity can be used together to provide a positive capacity, with illustration with the depolarizing channel in [29].

Here in this report, we concentrate on quantum thermal noise, which is a (nonunital) noise model of great significance where the decohering environment affecting the qubits is represented as a heat bath at an arbitrary temperature. This quantum noise model has been recently analyzed for noise-enhanced detection [21] or estimation [22,23] tasks. In the present report, to further assess the possibility of noise-enhanced performance, we analyze informational quantities relevant to the qubit and based on the von Neumann entropy. Especially, among the informational quantities we shall examine are the entropy exchange, the coherent information, the quantum mutual information, the information loss, the information noise, and the Holevo information. Each of these quantities comes with a significance in relation to specific informational processes, and can serve as a measure of performance related to such informational processes. Based on the geometric representation of qubit states with Bloch vectors and on the Kraus representation of the quantum thermal noise, analytical expressions will be derived for each informational quantity. Especially, these analytical expressions will allow us to analyze the impact of the temperature of the thermal noise on the informational quantities. Nontrivial regimes of variation will be reported here for the first time for such informational measures, demonstrating that they do not necessarily degrade monotonically as the temperature of the thermal noise increases, but that on the contrary they can experience nonmonotonic variations where higher noise temperatures can prove more favorable to information transmission.

2. Qubit entropy

A qubit with two-dimensional Hilbert space $\mathcal{H}_2$ is prepared in a quantum state represented by the density operator ρ parameterized in Bloch representation as [30]

$$\rho = \frac{1}{2}\left(\mathrm{I}_2 + \vec{r}\cdot\vec{\sigma}\right) , \tag{1}$$

with the real 3-dimensional Bloch vector $\vec{r} \in \mathbb{R}^3$ of Euclidean norm $\|\vec{r}\,\| \le 1$, and $\vec{\sigma}$ a formal vector assembling the three 2×2 Pauli matrices $[\sigma_x, \sigma_y, \sigma_z] = \vec{\sigma}$, and I_2 the identity of $\mathcal{H}_2$.

A qubit with Bloch vector $\vec{r}$ has a density operator ρ in Eq. (1) with two eigenvalues $\lambda_\pm = (1 \pm \|\vec{r}\,\|)/2$, so that its von Neumann entropy $S(\rho) = -\operatorname{tr}[\rho \log(\rho)]$ results as

$$S(\rho) = h\left(\frac{1+\|\vec{r}\,\|}{2}\right) + h\left(\frac{1-\|\vec{r}\,\|}{2}\right) , \tag{2}$$

with the auxiliary function $h(u) = -u\log_2(u)$. The von Neumann entropy $S(\rho)$ of Eq. (2) is a nonnegative and monotonically decreasing function of the Bloch vector norm $\|\vec{r}\,\|$. A qubit in a pure state ρ has $\|\vec{r}\,\| = 1$ and entropy $S(\rho) = 0$. A mixed state ρ has $\|\vec{r}\,\| < 1$ and entropy $S(\rho) > 0$, which reaches the maximum $S(\rho) = 1$ when $\|\vec{r}\,\| = 0$ for the maximally mixed state $\rho = \mathrm{I}_2/2$. It results that the entropy $S(\rho)$ is interpretable as a measure of disorder or unpredictability of the quantum state ρ, with $S(\rho)$ monotonically increasing as ρ passes from a pure quantum state to the maximally mixed state $\rho = \mathrm{I}_2/2$.

We consider that the qubit in state ρ is transmitted by a noisy communication channel generally representable by a completely positive trace-preserving superoperator $\mathcal{N}(\cdot)$ implementing the input–output transformation [30,31]

$$\rho \longrightarrow \rho' = \mathcal{N}(\rho) = \sum_{k=1}^{K} \Lambda_k \rho \Lambda_k^\dagger , \tag{3}$$

characterized by the K Kraus operators Λ_k satisfying $\sum_{k=1}^{K} \Lambda_k^\dagger \Lambda_k = \mathrm{I}_2$. This is equivalent to transforming the Bloch vectors by the affine map [30,31]

$$\vec{r} \longrightarrow \vec{r}\,' = A\vec{r} + \vec{c} , \tag{4}$$

with A a 3×3 real matrix and $\vec{c}$ a real vector in $\mathbb{R}^3$. We are specifically interested in studying the impact of a quantum noise channel $\mathcal{N}(\cdot)$ very important for the qubit, which is the generalized amplitude damping noise or quantum thermal noise [30,31]. Such thermal noise, unlike other less sophisticated noise models for the qubit, lends itself to nontrivial noise effects manifested by the entropy and other useful informational measures exhibiting unusual variations, as we shall see. The quantum thermal noise [30,31] is characterized in Eq. (3) by the $K = 4$ Kraus operators

$$\Lambda_1 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \tag{5}$$

$$\Lambda_2 = \sqrt{p}\begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \tag{6}$$

$$\Lambda_3 = \sqrt{1-p}\begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \tag{7}$$

$$\Lambda_4 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}, \tag{8}$$

with the associated affine map in Eq. (4) following as

$$\vec{r}\,' = A\vec{r} + \vec{c} = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix}\vec{r} + \begin{bmatrix} 0 \\ 0 \\ (2p-1)\gamma \end{bmatrix}. \tag{9}$$

This noise model describes the interaction of the qubit with an uncontrolled environment represented as a thermal bath at temperature T. The parameter $\gamma \in [0, 1]$ is a damping factor which often can be expressed [30] as a function of the interaction time t of the qubit with the bath as $\gamma = 1 - e^{-t/T_1}$, where T_1 is a time constant for the interaction (such as the spin-lattice relaxation time T_1 in magnetic resonance). At long interaction time $t \to \infty$, then $\gamma \to 1$ and the qubit relaxes to the equilibrium mixed state $\rho_\infty = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ of Bloch vector $\vec{r}_\infty = \vec{c}$. At equilibrium, the qubit has probabilities p of being measured in the ground state $|0\rangle$ and $1-p$ of being measured in the excited state $|1\rangle$. With the energies E_0 and $E_1 > E_0$ respectively for the states $|0\rangle$ and $|1\rangle$, the equilibrium probabilities are governed by the Boltzmann distribution

$$p = \frac{\exp[-E_0/(k_BT)]}{\exp[-E_0/(k_BT)] + \exp[-E_1/(k_BT)]} = \frac{1}{1+\exp[-(E_1-E_0)/(k_BT)]}. \tag{10}$$

In this way, in the quantum thermal noise of Eq. (9), the probability p is determined by the temperature T of the bath via Eq. (10). From Eq. (10), the probability p is a decreasing function of the temperature T. At $T = 0$ the probability is $p = 1$ for the ground state $|0\rangle$, while at $T \to \infty$ the ground state $|0\rangle$ and excited state $|1\rangle$ are equiprobable with $p = 1/2$. Therefore, from Eq. (10), when the temperature T monotonically increases from 0 to ∞, the probability p monotonically decreases from 1 to 1/2. In turn, the output noisy state ρ' determined by the Bloch vector $\vec{r}\,'$ of Eq. (9), is influenced by the noise temperature T only through the probability p. The remarkable feature we will demonstrate in the sequel is that, as the noise temperature T rises from 0 to ∞, the von Neumann entropy and other quantum informational measures associated to the output noisy state ρ', do not necessarily evolve monotonically, but on the contrary can experience nonmonotonic variations.

The input Bloch vector $\vec{r} = [r_x, r_y, r_z]^\top$ transformed by Eq. (9) yields the output Bloch vector $\vec{r}\,' = A\vec{r} + \vec{c}$ having the squared norm

$$\|\vec{r}\,'\|^2 = (1-\gamma)(r_x^2 + r_y^2) + r_z'^2, \tag{11}$$

with the squared z-component

$$r_z'^2 = \big[(1-\gamma)r_z + (2p-1)\gamma\big]^2 \tag{12}$$

carrying via p the dependence of $\|\vec{r}\,'\|^2$ with the noise temperature T. The von Neumann entropy $S(\rho')$ of the output state ρ' with Bloch vector $\vec{r}\,'$ is controlled by the norm $\|\vec{r}\,'\|$ inserted in Eq. (2). To analyze the influence of the noise temperature T on the entropy $S(\rho')$ of the noisy output state, we recall that $S(\rho')$ is a monotonically decreasing function of the norm $\|\vec{r}\,'\|$. In turn $\|\vec{r}\,'\|$, or $\|\vec{r}\,'\|^2$ in Eq. (11), is influenced by T only through the probability p in $r_z'^2$ of Eq. (12). The squared z-component $r_z'^2$ of Eq. (12) is a ∪-shaped parabola in the variable p, however limited by the allowed range $p \in [1/2, 1]$. The minimum of this parabola is zero and occurs when $(1-\gamma)r_z = -(2p-1)\gamma$, corresponding for the variable p to the critical value

$$p_c = \frac{1}{2} - \frac{1-\gamma}{2\gamma}r_z. \tag{13}$$

As the temperature T rises from 0 to ∞, inducing p to decrease from 1 to 1/2, it results that three regimes of variation of $\|\vec{r}\,'\|^2$ in Eq. (11), and subsequently of the output entropy $S(\rho')$ from Eq. (2), are accessible, depending on the situation of p_c of Eq. (13) in relation to the allowed interval $[1/2, 1] \ni p$. These variations will take place between the two extreme values, at $T = 0$ (i.e. at $p = 1$) determined in Eq. (12) by $r_z'^2(T=0) = [(1-\gamma)r_z + \gamma]^2$ setting the entropy $S(\rho'; T=0)$ via Eq. (2), and at $T = \infty$ (i.e. at $p = 1/2$) determined by $r_z'^2(T=\infty) = [(1-\gamma)r_z]^2$ setting $S(\rho'; T=\infty)$. Especially, depending on the conditions, one can have $S(\rho'; T=0) < S(\rho'; T=\infty)$, which is the natural expectation of a larger entropy of the noisy output at a larger noise temperature. But the opposite $S(\rho'; T=0) > S(\rho'; T=\infty)$ can also be found, as we shall see, manifesting the counterintuitive property of a smaller entropy at a larger noise temperature. Between these two extremes at $T = 0$ and $T = \infty$, as indicated three regimes of variation are accessible for the entropy $S(\rho')$ of the noisy output state, which we now analyze.

2.1. Increasing entropy

When the input state ρ is such that $r_z \geq 0$, then $p_c \leq 1/2$ in Eq. (13), so that the minimum of the ∪-shaped parabola $r_z'^2$ of Eq. (12) located at p_c always occurs before the allowed interval $[1/2, 1] \ni p$. As a consequence, $r_z'^2$ of Eq. (12) increases as the probability p increases in $[1/2, 1]$; the same is true for $\|\vec{r}\,'\|^2$ in Eq. (11), and this translates into a decreasing output entropy $S(\rho')$ in Eq. (2) when p increases in $[1/2, 1]$ or equivalently when the temperature T decreases from ∞ to 0. Therefore, as the

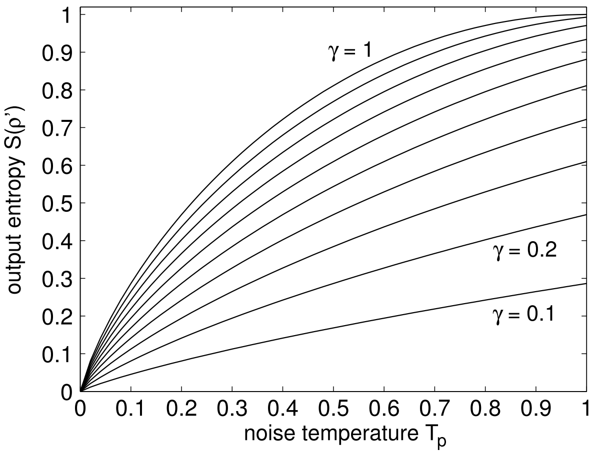

Fig. 1. Increasing output entropy $S(\rho')$ as a function of the noise temperature T_p, for the damping factor γ increasing from $\gamma = 0.1$ to $\gamma = 1$ by step 0.1, for an input state $\rho = |0\rangle\langle 0|$ of $\vec{r} = [0, 0, 1]^\top$.

temperature T of the thermal noise increases from 0 to ∞, the probability p resulting from Eq. (10) decreases from 1 to 1/2, and the output entropy $S(\rho')$ from Eq. (2) increases from $S(\rho'; T = 0)$ to $S(\rho'; T = \infty) > S(\rho'; T = 0)$. This is the natural expected regime where the entropy $S(\rho')$ of the output state ρ' of the noisy channel, increases as the noise temperature T increases. Typical illustrations for this regime of increasing entropy are presented in Fig. 1.

In the illustrations of Fig. 1 and following figures, the influence of the noise temperature is quantitatively displayed as a function of the auxiliary variable $T_p = 2(1 - p)$ interpreted as a reduced or equivalent noise temperature. In this way, from Eq. (10), this T_p is a monotonically increasing function of the temperature T, for any value of the energy difference $E_1 - E_0 > 0$. When T is 0 then p is 1 and T_p is 0, while when T is ∞ then p is 1/2 and T_p is 1. So a temperature T increasing from 0 to ∞ is monotonically mapped into a reduced temperature T_p increasing from 0 to 1. This offers the convenience of a finite range in terms of $T_p \in [0, 1]$ to display the influence of the noise temperature $T \in [0, \infty[$, and also disencumbers the quantitative analysis of the unimportant specific value of the energy difference $E_1 - E_0 > 0$.

2.2. Resonant entropy

When the input state ρ is such that $-\gamma < (1 - \gamma)r_z < 0$ or equivalently $-\gamma/(1 - \gamma) < r_z < 0$, then $p_c \in]1/2, 1[$ in Eq. (13), so that the minimum of the $\cup$-shaped parabola $r_z'^2$ of Eq. (12) located at p_c always occurs inside the allowed interval $[1/2, 1] \ni p$. As a consequence, both $r_z'^2$ of Eq. (12) and $\|\vec{r}'\|^2$ of Eq. (11) experience a $\cup$-shaped variation as p increases in $[1/2, 1]$ and pass through their minimum at $p = p_c$. This translates into an output entropy $S(\rho')$ in Eq. (2) which experiences a $\cap$-shaped resonant variation as p increases in $[1/2, 1]$ with $S(\rho')$ culminating at a maximum in $p = p_c$. This is equivalent to an output entropy $S(\rho')$ also experiencing a $\cap$-shaped resonant variation as the noise temperature T increases from 0 to ∞, with $S(\rho')$ culminating at a maximum for a critical temperature T_c related to p_c via Eq. (10). Typical illustrations for this regime of resonant entropy are presented in Fig. 2.

The resonant variations of Fig. 2 especially show that when the entropy $S(\rho')$ is interpreted as a measure of disorder or unpredictability of the quantum state ρ', then a range of finite temperatures exists where such measure of unpredictability of ρ' is maximized, depending on the conditions. This identifies ranges of finite temperatures that are specially detrimental to the purity or immunity of the quantum state, and that lower, but also higher, temperatures, will be less harmful to the quantum state in this respect.

2.3. Decreasing entropy

When the input state ρ is such that $(1 - \gamma)r_z \le -\gamma$ or equivalently $r_z \le -\gamma/(1 - \gamma)$, then $p_c \ge 1$ in Eq. (13), so that the minimum of the $\cup$-shaped parabola $r_z'^2$ of Eq. (12) located at p_c always occurs after the allowed interval $[1/2, 1] \ni p$. As a consequence, both $r_z'^2$ of Eq. (12) and $\|\vec{r}'\|^2$ of Eq. (11) decrease when p increases in $[1/2, 1]$. This translates into an increasing output entropy $S(\rho')$ in Eq. (2) when p increases in $[1/2, 1]$ or equivalently when the temperature T decreases from ∞ to 0. Therefore, as the temperature T of the thermal noise increases from 0 to ∞, now the output entropy $S(\rho')$ from Eq. (2) decreases from $S(\rho'; T = 0)$ to $S(\rho'; T = \infty) < S(\rho'; T = 0)$. Typical illustrations for this regime of decreasing entropy are presented in Fig. 3.

An input state ρ has a Bloch vector $\vec{r}$ with a z-component necessarily limited as $r_z \in [-1, 1]$, and the thermal noise has a damping factor $\gamma \in [0, 1]$. Across these feasible conditions, Fig. 4 represents the three domains in the plane (γ, r_z)

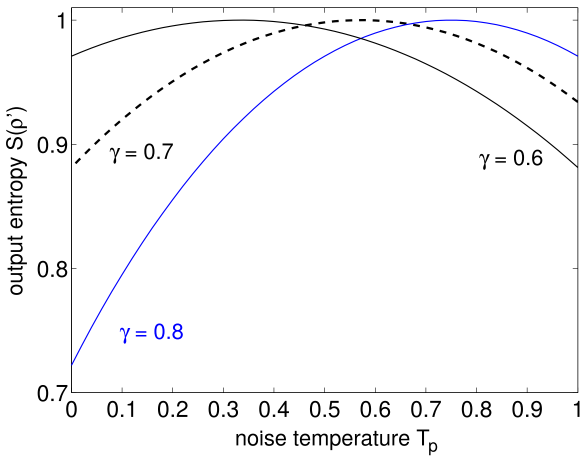

Fig. 2. Resonant output entropy $S(\rho')$ as a function of the noise temperature T_p, for the damping factor $\gamma = 0.6$, 0.7 and 0.8, for an input state $\rho = |1\rangle\langle 1|$ of $\vec{r} = [0, 0, -1]^\top$.

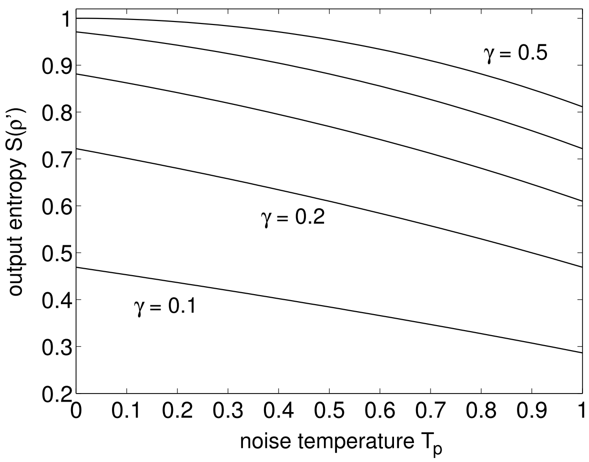

Fig. 3. Decreasing output entropy $S(\rho')$ as a function of the noise temperature T_p, for the damping factor γ increasing from $\gamma = 0.1$ to $\gamma = 0.5$ by step 0.1, for an input state $\rho = |1\rangle\langle 1|$ of $\vec{r} = [0, 0, -1]^\top$.

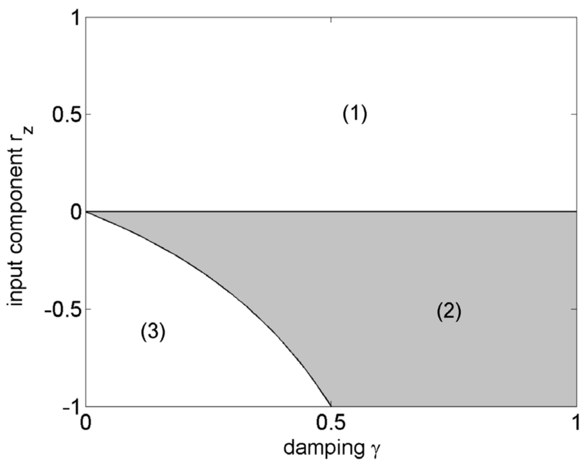

Fig. 4. For $(\gamma, r_z) \in [0, 1] \times [-1, 1]$, the three domains of variation, with the noise temperature T, of the entropy $S(\rho')$ of the output state ρ' from Eq. (2) for the channel with quantum thermal noise. Domain (1) is an increasing $S(\rho')$ when $r_z \geq 0$; domain (2) in gray is a resonant $S(\rho')$ when $-\gamma/(1-\gamma) < r_z < 0$; domain (3) is a decreasing $S(\rho')$ when $r_z \leq -\gamma/(1-\gamma)$. The curve separating domains (2) and (3) has equation $r_z = -\gamma/(1-\gamma)$.

corresponding to the three regimes of variation feasible for the output entropy $S(\rho')$ as a function of the noise temperature T, as controlled by the position of p_c in Eq. (13) in relation to the interval $[1/2, 1]$.

The present analysis especially shows that there does not exist a fourth regime of variation, where a nonmonotonic entropy $S(\rho')$ could antiresonate at a minimum for some critical temperature T, identifying some beneficial nonzero temperature where the unpredictability of the state ρ' would be minimized. Regarding nonmonotonic variations of the entropy $S(\rho')$, there only exist configurations as in Fig. 2, where $S(\rho')$ resonates at a maximum around a critical temperature. We will now examine other useful measures related to the fundamental measure of von Neumann entropy, and their variation with the noise temperature, and characterizing the flow of entropy into the environment and the performance for information communication over the noisy channel.

3. Entropy exchange

Another useful informational quantity is the entropy exchange $S(\rho, \mathcal{N})$, which represents [30,32] the amount of entropy generated in the environment by the action of the quantum channel $\mathcal{N}(\cdot)$ implementing the noisy transmission of state ρ. From an informational standpoint, $S(\rho, \mathcal{N})$ quantifies the information exchanged between the quantum system initially in state ρ and the environment during the evolution by $\mathcal{N}(\cdot)$. As such, $S(\rho, \mathcal{N})$ in particular limits the amount of information an eavesdropper could acquire about the system in a quantum cryptographic protocol.

When the environment starts in a pure state (it is always feasible, possibly via a purification step), after the action of the noise channel $\mathcal{N}(\cdot)$ on ρ the environment terminates in a mixed state ρ'_E expressible [32,30] with the matrix representation $\rho'_E = [W_{k\ell}]$ of matrix elements

$$W_{k\ell} = \mathrm{tr}(\Lambda_k \rho \Lambda_\ell^\dagger) , \tag{14}$$

for $k, \ell = 1$ to K. The entropy exchange is then equivalent to the final entropy of the environment, i.e. $S(\rho, \mathcal{N}) = S(\rho'_E)$. When the input state ρ is expressed as a 2×2 matrix function of the three components of the Bloch vector $\vec{r} = [r_x, r_y, r_z]^\top$, then by using Eqs. (5)–(8) in Eq. (14) one obtains the final state of the environment as

$$\rho'_E = \frac{1}{2} \begin{bmatrix} p[2-\gamma(1-r_z)] & p\sqrt{\gamma}(r_x - ir_y) & 2\sqrt{p(1-p)}\sqrt{1-\gamma} & \sqrt{p(1-p)}\sqrt{\gamma(1-\gamma)}(r_x + ir_y) \\ p\sqrt{\gamma}(r_x + ir_y) & p\gamma(1-r_z) & \sqrt{p(1-p)}\sqrt{\gamma(1-\gamma)}(r_x + ir_y) & 0 \\ 2\sqrt{p(1-p)}\sqrt{1-\gamma} & \sqrt{p(1-p)}\sqrt{\gamma(1-\gamma)}(r_x - ir_y) & (1-p)[2-\gamma(1+r_z)] & (1-p)\sqrt{\gamma}(r_x + ir_y) \\ \sqrt{p(1-p)}\sqrt{\gamma(1-\gamma)}(r_x - ir_y) & 0 & (1-p)\sqrt{\gamma}(r_x - ir_y) & (1-p)\gamma(1+r_z) \end{bmatrix} . \tag{15}$$

Then to evaluate the entropy $S(\rho'_E)$, the four eigenvalues of ρ'_E of Eq. (15) are to be determined. Exact analytical expressions can be found but they are too bulky to be written down here, in general form. However, for the damping $\gamma = 0$ when there is no noise, the state ρ'_E of Eq. (15) reduces to

$$\rho'_E = \begin{bmatrix} p & 0 & \sqrt{p(1-p)} & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{p(1-p)} & 0 & 1-p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \tag{16}$$

having the four eigenvalues 0 with multiplicity 3 and 1, so that at no noise the entropy exchange $S(\rho'_E)$ vanishes as expected. Also, when the input state ρ for the qubit is pure, with the environment starting in a pure state, the joint qubit-environment system undergoes a unitary evolution where their joint state remains pure. It results that at the end of the interaction with the environment materializing the effect of noise on the qubit, the final reduced state ρ' of the noisy qubit and the final reduced state ρ'_E of the environment have the same entropy. In other words, when the input state ρ is pure, the entropy exchange $S(\rho'_E)$ coincides with the output entropy $S(\rho')$, ensuring that the entropy exchange $S(\rho'_E)$ can also experience the same three regimes of variation (decreasing, antiresonant, increasing) as the output entropy $S(\rho')$ when the noise temperature T increases.

In addition, to investigate the same possibilities of variation of the entropy exchange $S(\rho'_E)$ also with a mixed (non-pure) input state ρ, we turn to the restricted class where ρ is characterized by the Bloch vector $\vec{r} = [0, 0, r_z]^\top$ with $r_z \in [-1, 1]$, leading in Eq. (15) to the final state of the environment

$$\rho'_E = \frac{1}{2} \begin{bmatrix} p[2-\gamma(1-r_z)] & 0 & 2\sqrt{p(1-p)}\sqrt{1-\gamma} & 0 \\ 0 & p\gamma(1-r_z) & 0 & 0 \\ 2\sqrt{p(1-p)}\sqrt{1-\gamma} & 0 & (1-p)[2-\gamma(1+r_z)] & 0 \\ 0 & 0 & 0 & (1-p)\gamma(1+r_z) \end{bmatrix} . \tag{17}$$

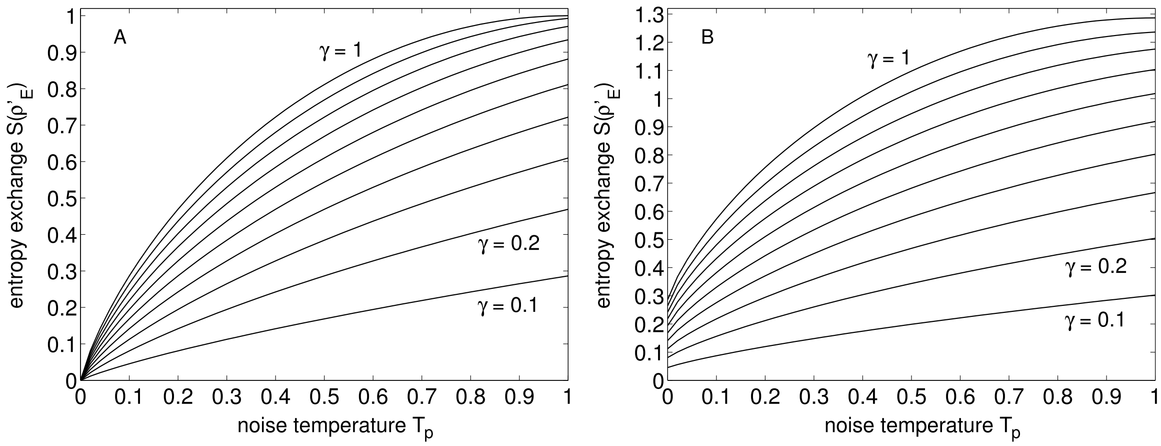

Fig. 5. Increasing entropy exchange $S(\rho_E')$ as a function of the noise temperature T_p, for the damping factor γ increasing from $\gamma = 0$ to $\gamma = 1$ by step 0.1, for a pure input state $\rho = |0\rangle\langle 0|$ of $\vec{r} = [0, 0, 1]^\top$ (panel A), and for a mixed input state ρ of $\vec{r} = [0, 0, 0.9]^\top$ (panel B).

To evaluate the entropy $S(\rho_E')$, the four eigenvalues of ρ_E' of Eq. (17) follow as

$$\lambda_1 = \frac{1}{4}\left[(2p-1)\gamma r_z + 2 - \gamma - \sqrt{\Gamma}\right] , \tag{18}$$

$$\lambda_2 = \frac{1}{4}\left[(2p-1)\gamma r_z + 2 - \gamma + \sqrt{\Gamma}\right] , \tag{19}$$

$$\lambda_3 = \frac{1}{2}(1-p)\gamma(1+r_z) , \tag{20}$$

$$\lambda_4 = \frac{1}{2}p\gamma(1-r_z) , \tag{21}$$

with

$$\Gamma = \gamma^2 r_z^2 + 2(2p-1)\gamma(2-\gamma)r_z + (2p-1)^2\gamma^2 + 4(1-\gamma) . \tag{22}$$

In the special case when $r_z = -1$, the four eigenvalues of Eqs. (18)–(21) reduce to $p\gamma$, $1 - p\gamma$ and 0 with multiplicity 2; while when $r_z = 1$, they reduce to $(1-p)\gamma$, $1-(1-p)\gamma$ and 0 with multiplicity 2. This is an instance of the configuration of ρ pure addressed above, where the entropy exchange $S(\rho_E')$ exactly coincides with the output entropy $S(\rho')$, so that $S(\rho_E')$ also experiences the same three regimes of variation as $S(\rho')$ when the noise temperature T increases. We illustrate in the sequel in Figs. 5–7, other conditions with mixed input states ρ, also governed by Eqs. (18)–(21), where the entropy exchange $S(\rho_E')$ no longer coincides with the output entropy $S(\rho')$ but where the three regimes of variation of $S(\rho_E')$ are still observed as the noise temperature T increases.

Fig. 5 can be viewed as the more standard behavior, where the entropy exchange $S(\rho_E')$ representing the entropy in the environment, increases as the noise temperature T increases. This can occur with a pure input state ρ as well as with a mixed input state ρ, as illustrated in Fig. 5.

Comparatively, Figs. 6–7 demonstrate the possibility of less standard behaviors, where the entropy exchange $S(\rho_E')$ resonates or decreases when the noise temperature T increases, this with pure and with mixed input states ρ. Fig. 6 shows resonant variations, where $S(\rho_E')$ culminates at a maximum at some finite critical value of the noise temperature.

Fig. 7 shows decreasing variations, where the entropy exchange $S(\rho_E')$ is steadily reduced as the noise temperature T increases.

4. Coherent information

From the output entropy $S(\rho')$ and the entropy exchange $S(\rho_E')$ one defines the coherent quantum information [33] as

$$I_c(\rho, \mathcal{N}) = S(\rho') - S(\rho_E') , \tag{23}$$

The coherent information $I_c(\rho, \mathcal{N})$ is not necessarily non-negative, and is closely related to the quantum capacity of the noisy channel $\mathcal{N}(\cdot)$ [33]. For a pure input state ρ we have seen that $S(\rho') = S(\rho_E')$ implying $I_c(\rho, \mathcal{N}) = 0$ for any noise parameters (γ, T). However, for a mixed input state ρ, the coherent information $I_c(\rho, \mathcal{N})$ of Eq. (23) is in general non-vanishing. Moreover, it can be verified from Eq. (23) that $I_c(\rho, \mathcal{N})$ can experience, depending upon the conditions on ρ and γ, three regimes of variation (decreasing, antiresonant, increasing) as the noise temperature T is raised.

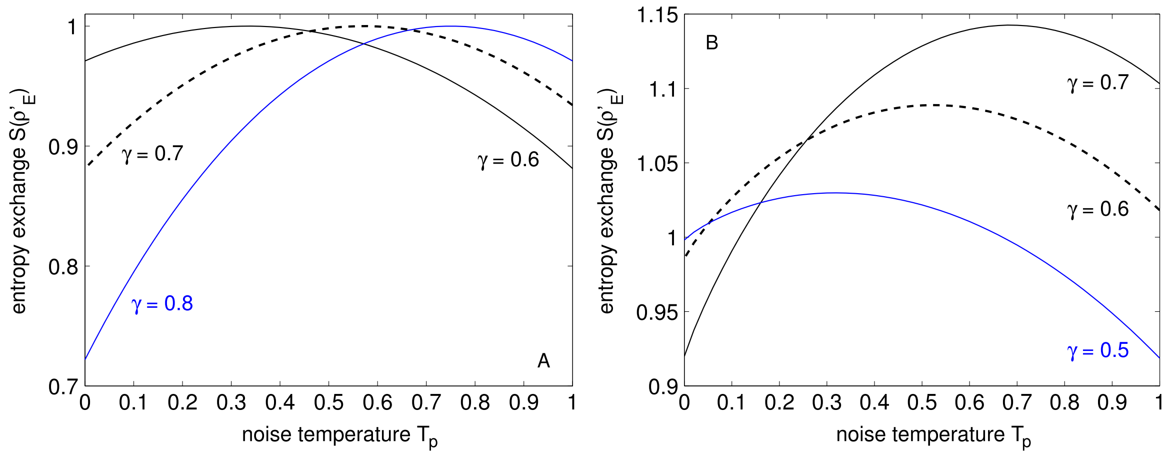

Fig. 6. Resonant entropy exchange $S(\rho'_E)$ as a function of the noise temperature T_p, for damping factors γ between 0.5 and 0.8, for a pure input state $\rho = |1\rangle\langle 1|$ of $\vec{r} = [0, 0, -1]^\top$ (panel A), and for a mixed input state ρ of $\vec{r} = [0, 0, -0.9]^\top$ (panel B).

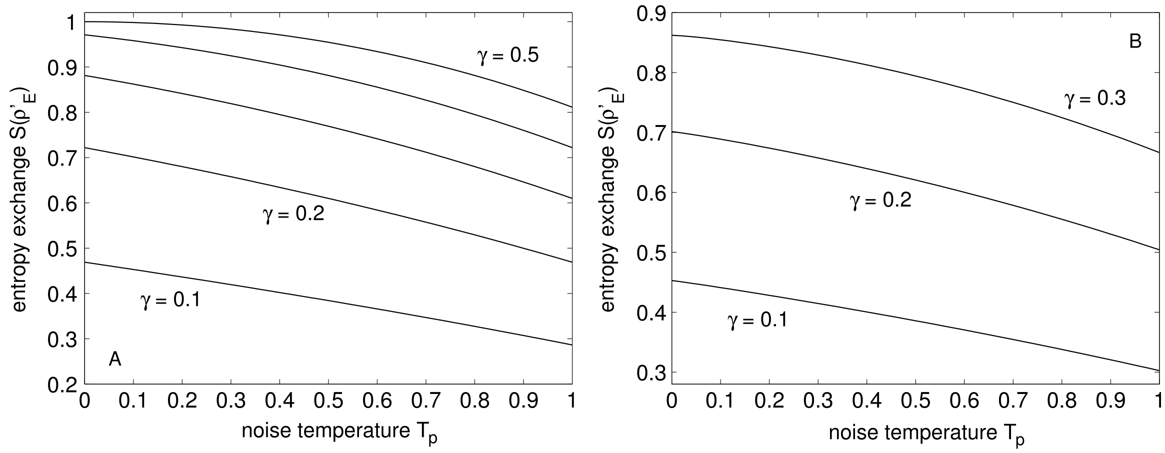

Fig. 7. Decreasing entropy exchange $S(\rho'_E)$ as a function of the noise temperature T_p, for the damping factor γ increasing from $\gamma = 0.1$ to $\gamma = 0.5$ by step 0.1, for a pure input state $\rho = |1\rangle\langle 1|$ of $\vec{r} = [0, 0, -1]^\top$ (panel A), and for a mixed input state ρ of $\vec{r} = [0, 0, -0.9]^\top$ (panel B).

One can also define the quantum mutual information [33] as

$$I(\rho, \mathcal{N}) = S(\rho) + S(\rho') - S(\rho'_E) = S(\rho) + I_c(\rho, \mathcal{N}) , \tag{24}$$

which is non-negative, and quantifies the mutual information between the input quantum state ρ and output quantum state $\rho' = \mathcal{N}(\rho)$. This $I(\rho, \mathcal{N})$ vanishes for pure input states ρ, just like $S(\rho)$ and $I_c(\rho, \mathcal{N})$. For mixed input states ρ, from the present analysis we verify that $I(\rho, \mathcal{N})$ also, depending upon the conditions on ρ and γ, can experience three regimes of variation (decreasing, antiresonant, increasing) as the noise temperature T is raised.

Decreasing quantum informations $I_c(\rho, \mathcal{N})$ and $I(\rho, \mathcal{N})$ is the standard behavior which can be expected as the noise temperature T is raised. This expresses the expected property of an informational efficiency of the transmission channel which monotonically degrades as the level of noise increases. On the contrary, nonmonotonic antiresonant variations and increasing variations for $I_c(\rho, \mathcal{N})$ and $I(\rho, \mathcal{N})$ represent an unconventional behavior, reminiscent of stochastic resonance, stemming from the sophisticated action of the quantum thermal noise, and revealing the possibility, under some conditions, of improving the informational efficiency of the transmission channel by increasing the level of noise through increasing the noise temperature.

One can also define the information loss [33] as

$$L(\rho, \mathcal{N}) = S(\rho) + S(\rho'_E) - S(\rho') = S(\rho) - I_c(\rho, \mathcal{N}) , \tag{25}$$

which is non-negative, and quantifies the mutual information between the input and the environment. The loss $L(\rho, \mathcal{N})$ vanishes for pure input states ρ, just like $S(\rho)$ and $I_c(\rho, \mathcal{N})$. For mixed input states ρ, from the present analysis we verify that $L(\rho, \mathcal{N})$ also, depending upon the conditions on ρ and γ, can experience three regimes of variation (increasing, resonant, decreasing) as the noise temperature T is raised.

One can also define the information noise [33] as

$$N(\rho, \mathcal{N}) = S(\rho'_E) + S(\rho') - S(\rho) , \tag{26}$$

which is non-negative, and quantifies the mutual information between the output and the environment. The information noise $N(\rho, \mathcal{N})$ does not generally vanish on pure input states ρ in the presence of thermal noise. From the present analysis we verify that $N(\rho, \mathcal{N})$ also, depending upon the conditions on ρ and γ, can experience three regimes of variation (increasing, resonant, decreasing) as the noise temperature T is raised.

Increasing information loss $L(\rho, \mathcal{N})$ and noise $N(\rho, \mathcal{N})$ is the standard expected behavior as the noise temperature T is raised, expressing degradation of the transmission channel as the level of noise increases. On the contrary, nonmonotonic resonant and decreasing variations for $L(\rho, \mathcal{N})$ and $N(\rho, \mathcal{N})$ is an unconventional behavior, manifesting the possibility of some improvement of the transmission by increasing the level of thermal noise.

5. Holevo information

The previous informational quantities characterize the behavior of the channel in transmitting a single generic input state. A further fundamental informational quantity characterizes signaling over the channel with a statistical ensemble of input states, as would occur for instance for communication of random symbols with quantum encoding. Accordingly, we consider that at the channel input a quantum state ρ_j is selected with probability p_j from a set of a number J of such quantum states. The noisy channel delivers the output states $\rho'_j = \mathcal{N}(\rho_j)$ for $j = 1$ to J. A fundamental quantity to assess the informational performance is the Holevo information, defined from the von Neumann entropy as [30,34]

$$\chi(\rho') = S(\rho') - \sum_{j=1}^{J} p_j S(\rho'_j) , \tag{27}$$

with the average output state $\rho' = \sum_{j=1}^{J} p_j \rho'_j$.

The Holevo information $\chi(\rho')$ of Eq. (27) forms a lower bound to the compression rate of a lossless coding of a sequence of independent quantum states ρ'_j [35,36]. Also, $\chi(\rho')$ is an upper bound to the input–output mutual information for classical information transmission via successive independent uses of the quantum channel. Moreover, $\chi(\rho')$ forms an achievable rate for classical information transmission, usually reachable by encoding with long blocks of successive independent input states ρ_j. As such, $\chi(\rho')$ represents the maximum rate of the quantum channel for classical information transmission via successive independent channel uses [34,37,38]. The interesting point here is to realize that, as the noise temperature increases, the Holevo information $\chi(\rho')$ of Eq. (27) does not necessarily monotonically decreases. On the contrary, $\chi(\rho')$ can experience the three regimes of variation that were shown in Section 2 accessible to the von Neumann entropy of a noisy quantum state.

The Holevo information $\chi(\rho')$ of Eq. (27) depends on the J input states ρ_j and on their probabilities p_j, together offering a large range of possible configurations. For illustration of the nontrivial variations of the Holevo information $\chi(\rho')$ of Eq. (27) with the temperature T of the thermal noise, we consider a transmission protocol with $J = 2$ pure input states ρ_0 and ρ_1 chosen equiprobable with $p_0 = p_1 = 1/2$. Fig. 8 shows configurations with the two states (ρ_0, ρ_1) yielding a Holevo information $\chi(\rho')$ experiencing a monotonic decay as the noise temperature T increases.

With a decreasing Holevo information $\chi(\rho')$, Fig. 8 illustrates the standard expected behavior of a performance of the quantum channel for information transmission which monotonically degrades has the noise temperature T increases. By contrast, Fig. 9 reveals the possibility of configurations with the two states (ρ_0, ρ_1) yielding a Holevo information $\chi(\rho')$ which does not monotonically decay as the noise temperature T increases.

Fig. 9 (dotted line) displays a configuration of the coding states (ρ_0, ρ_1) leading to an antiresonant variation of the Holevo information $\chi(\rho')$. Here there exists a finite critical temperature around $T_p \approx 0.7$ in Fig. 9 where the thermal noise is specially detrimental to the transmission, manifested by a minimum of the Holevo information $\chi(\rho')$. In such configurations, operating the channel at lower, but also at higher, temperatures is more efficient for information transmission as assessed by $\chi(\rho')$. This antiresonant variation of the performance $\chi(\rho')$ for information transmission, is identifiable with an effect of stochastic antiresonance, where a finite level of noise exists that minimizes the performance and where operating at smaller, but also at larger, noise levels turns out to be more favorable. Such stochastic antiresonance was also observed in other situations, quantum [23,39,40] or classical [41–44], yet with other performance measures differing from the informational quantities considered here.

Fig. 9 (solid line) also displays a configuration of the coding states (ρ_0, ρ_1) where the Holevo information $\chi(\rho')$ monotonically increases as the noise temperature T grows. This is a nonstandard behavior, where enhancing the noise temperature yields better performance in information transmission. In practice however, the temperature will have to be limited before it can cause damage to the transmission system. The possibility of such regime where increasing the noise temperature is always beneficial to the performance in information processing, has also been observed for quantum state detection [20,21] or estimation [22,23]. It manifests another aspect by which the quantum decoherence is not necessarily detrimental but can prove beneficial.

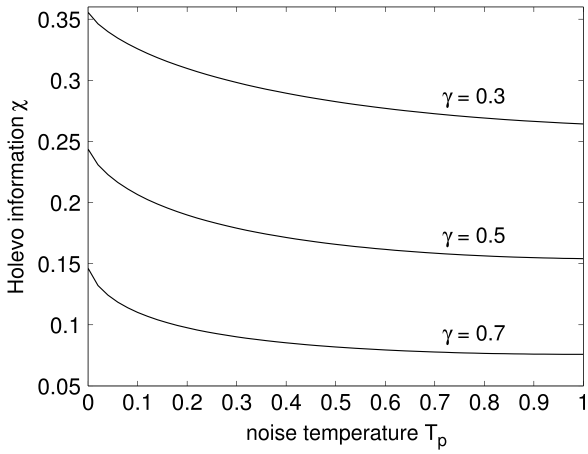

Fig. 8. Holevo information $\chi(\rho')$ from Eq. (27) as a function of the noise temperature T_p, at three values of the damping factor γ, for two pure input states with Bloch vectors $\vec{r}_0 = [0, 0, 1]^\top$ and $\vec{r}_1 = [1, 0, 0]^\top$ and probabilities $p_0 = p_1 = 1/2$.

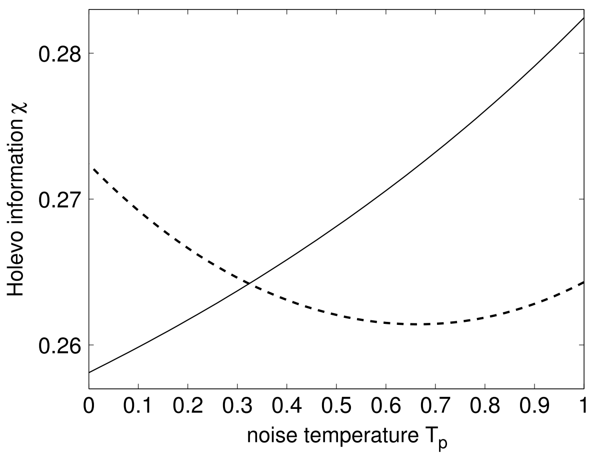

Fig. 9. Holevo information $\chi(\rho')$ from Eq. (27) as a function of the noise temperature T_p, for two pure input states with Bloch vectors $\vec{r}_0$ and $\vec{r}_1$ and probabilities $p_0 = p_1 = 1/2$. Dashed line: $\vec{r}_0 = [0, 0, -1]^\top$ and $\vec{r}_1 = [1, 0, 0]^\top$ at damping $\gamma = 0.3$. Solid line: $\vec{r}_0 = [0, 0, -1]^\top$ and $\vec{r}_1 = [\sqrt{0.84}, 0, -0.4]^\top$ at damping $\gamma = 0.1$.

6. Discussion

In this report we have considered informational quantities characterizing the qubit. These informational quantities have been analyzed in the presence of quantum thermal noise modeling the decoherence process caused by interaction with an uncontrolled environment represented as a heat bath at arbitrary temperature. It has been specifically observed that the informational quantities do not necessarily degrade monotonically as the noise temperature increases. On the contrary, they can experience nontrivial and nonmonotonic variations where higher noise temperatures can prove more favorable to the informational contents. Such nontrivial variations have been observed for the von Neumann entropy $S(\rho')$ of a noisy qubit state ρ'. Other informational quantities related to the von Neumann entropy have also been examined, with the entropy exchange or entropy generated in the environment $S(\rho, \mathcal{N}) = S(\rho'_E)$, with the coherent information $I_c(\rho, \mathcal{N})$, with the quantum mutual information $I(\rho, \mathcal{N})$, with the information loss $L(\rho, \mathcal{N})$, and information noise $N(\rho, \mathcal{N})$. All of them were also found capable of experiencing nontrivial and nonmonotonic regimes of variation as the noise temperature increases. In addition, when a statistical ensemble of qubit states is used for information communication over a noisy quantum channel, the fundamental quantity formed by the Holevo information $\chi(\rho')$ was also found capable of experiencing similar nontrivial regimes. In particular, this identifies configurations where information transmission over the noisy channel can be improved by operating at higher noise temperatures. Regimes of antiresonance for the performance measures were also observed, identifying other configurations where specific finite values of the noise temperature are maximally detrimental.

Concerning the physical accessibly of the conditions explored in Figs. 1–9 and where interesting noise effects are reported, the following remarks can be made. For the thermal relaxation times T_1 discussed just after Eq. (9), typical values, for instance

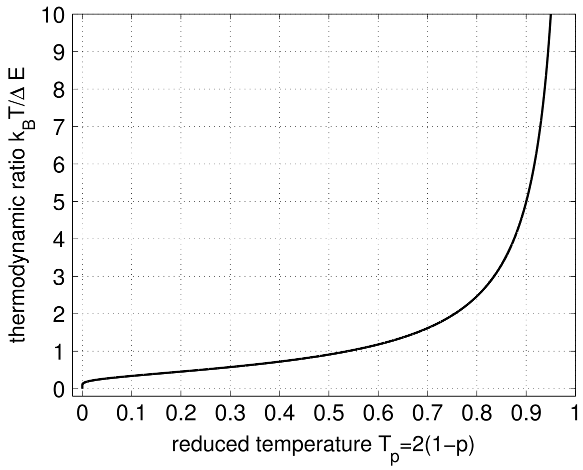

Fig. 10. Thermodynamic ratio $k_BT/\Delta E = 1/\ln(-1+2/T_p)$ deduced from Eq. (10), as a function of the reduced temperature $T_p = 2(1-p)$ of the quantum thermal noise.

in nuclear magnetic resonance, are in the order of $T_1 \sim 100$ ms. On this basis, it is physically feasible to gain access to interaction times t that can range from $t \ll T_1$ up to $t \gg T_1$, allowing in this way to obtain a damping factor $\gamma = 1 - e^{-t/T_1}$ that can cover the interval [0, 1] and span the values of γ tested in Figs. 1–9. For the useful ranges of the reduced temperature $T_p = 2(1-p)$ tested in Figs. 1–9 and where interesting noise effects occur, one has the faculty to invert Eq. (10) to obtain the thermodynamic ratio $k_BT/\Delta E = 1/\ln(-1+2/T_p)$ plotted in Fig. 10, with $\Delta E = E_1 - E_0 > 0$ the transition energy between the ground state $|0\rangle$ and excited state $|1\rangle$ of the qubit.

From Fig. 10 one can observe that the interesting values of T_p in [0, 1] concerned by Figs. 1–9, can be practically covered with a ratio $k_BT/\Delta E$ varying from 0.1 to 10; and $k_BT/\Delta E$ below 0.1 would physically corresponds to $T_p \approx 0$ while $k_BT/\Delta E$ above 10 would corresponds to $T_p \approx 1$. So $k_BT/\Delta E$ in [0.1, 10] sets the physical domain of conditions giving access to the interesting noise effects reported in Figs. 1–9. At room temperature $T \approx 300$ K one has the thermodynamic temperature $k_BT \approx 1/40$ eV, translating for the quantum system into a transition energy $\Delta E \in [1/400, 1/4]$ eV. Such ranges of energy are typically involved in photosynthetic processes of living plants, for which noise-assisted phenomena at the quantum level have been reported [45,46] and which, although distinct, may be related to the stochastic resonance effects reported here. Around room temperature, the stochastic resonance effects reported here with quantum thermal noise, might therefore possibly apply to photosynthetic processes, although this remains to be further explored. Current technological systems developed for quantum computing typically operate with much lower transition energies ΔE. They are much fragile to thermal fluctuations and have usually to be operated well below room temperature. For instance with liquid helium at $T \approx 3$ K we are at a factor of 10^{-2} below room temperature, associated with transition energies $\Delta E \in [1/400 \times 10^{-2}, 1/4 \times 10^{-2}]$ eV which come closer to transition energies of technological quantum systems. Consistently, for such current technological quantum systems, the present noise effects will typically occur at temperatures well below room temperature.

The quantities analyzed here are based on the von Neumann entropy and carry informational significance. Such significance could be extended by considering generalized versions based on generalized, nonadditive or nonextensive quantum entropies [47–50], these having found useful applications for quantum information especially in relation to quantum correlation by entanglement [51–53]. It could therefore be interesting to examine if such generalized informational quantities can also, in the presence of quantum noise, experience nontrivial variations escaping monotonic degradation at increasing noise level.

The nontrivial regimes of variation reported here for informational quantities upon increasing the noise temperature, demonstrate sophisticated properties of quantum decoherence. Increased decoherence does not always translate into poorer informational performance. The present study identifies and analyzes new situations where such counterintuitive behavior of decoherence takes place. Such results are relevant to contribute to deeper understanding and control of decoherence which are essential steps needed for the advancement of quantum information and quantum technologies.

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