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Physics Letters A 351 (2006) 231–237
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Noise-aided SNR amplification by parallel arrays of sensors with saturation

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Received 17 August 2005; received in revised form 25 October 2005; accepted 26 October 2005

Available online 9 November 2005

Communicated by C.R. Doering

Abstract

We consider sensor devices with saturation in their response, in charge of the transmission of a sinusoidal signal buried in Gaussian white noise, with a performance assessed by an input–output gain in the signal-to-noise ratio. We show that such saturating devices can always be tuned to achieve a signal-to-noise ratio gain larger than unity. When replicated to form parallel arrays, further improvement of the gain can be obtained with independent noise injected on the sensors. This provides smart arrays of simple nonlinear sensors capable of acting as noise-aided amplifiers, and where the highest gains in signal-to-noise ratio are always obtained at a nonzero level of the added noises.

Keywords: Nonlinear sensors; Saturation; Arrays; Noise; Stochastic resonance; Signal-to-noise ratio gain

1. Introduction

Nonlinear processes where fluctuations and noise play a beneficial role are currently an active area of research. Such phenomena especially exhibit very interesting potentialities for signal and information processing. Stochastic resonance [1,2] can be presented as designating such a class of phenomena, where the action of noise can improve some processing performed on a signal. For stochastic resonance in isolated nonlinear systems, a common mechanism of improvement can be described as a displacement by noise of the operating zone, initially ill-positioned, of a nonlinear system, towards a region more favorable to the signal. Since more recently, a distinct mechanism for stochastic resonance is being investigated, which arises when nonlinear systems are replicated into parallel arrays [3–9]. In the array, each individual system is subjected to an additional independent noise. As a consequence, each system responds differently to a common input applied to the array. When the responses are collected or averaged over the array, it turns out that the global response of the array with added noise can be more efficient, in regard to some processing performed on the input, than a single system with no added noise. This form of stochastic resonance in arrays was introduced and studied under the name of suprathreshold stochastic resonance in [3,4], where it was applied to the transmission by threshold comparators of a noise-free input with arbitrary (not necessarily subthreshold) amplitude. This form of stochastic resonance was latter applied to process a signal-noise mixture as the input, through injection of additional independent noise in the array [8,9]. So far, stochastic resonance in parallel arrays of nonlinearities has essentially been reported and investigated for threshold nonlinearities, including some neuron models [10–13], and shown applicable to several distinct signal processing tasks.

In the present Letter we will concentrate on the improvement of the signal-to-noise ratio (SNR) of a sinusoidal signal buried in Gaussian white noise. Improving the SNR of a sinusoid in noise is often desirable in many areas of experimental sciences and technologies. The sinusoid can be in itself the (fixed) signal of interest, or it can be a carrier conveying useful information through (slow) modulation of some of its parameters. Moreover, the SNR of a sinusoid in noise and its evolution through nonlinear transformation, is often taken as a reference in the studies of the various forms and modalities of stochastic resonance. We will show that nonlinear sensor devices, linear for small inputs and saturating at large inputs (a common behavior for sensors [14]), are capable of an amplification of the

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doi:10.1016/j.physleta.2005.10.106

SNR of a sinusoid in Gaussian white noise. In addition, we will demonstrate that when these saturating devices are associated into parallel arrays, further improvement of the SNR can occur, through the action of independent noises injected into the array. Our results extend the phenomenon of stochastic resonance in parallel arrays to threshold-free nonlinearities taking the form of saturating sensors. They also establish a novel class of SNR-amplifying systems, under the form of parallel arrays of saturations aided by noise.

2. Evaluation of an SNR gain

Consider the signal-plus-noise mixture $x(t) = s(t) + \xi(t)$, where $s(t)$ is deterministic with period T_s, and $\xi(t)$ is a stationary white noise, independent of $s(t)$, with cumulative distribution function $F_\xi(u)$ and probability density function $f_\xi(u) = dF_\xi(u)/du$. The input signal $x(t) = s(t) + \xi(t)$ is applied onto a parallel array of N identical sensors, conforming to the architecture also considered in [3,10,15]. Each sensor of the array is endowed with the same input–output static or memoryless characteristic $g(\cdot)$. A noise $\eta_i(t)$, independent of $x(t)$, can be added to $x(t)$ at each sensor i, so as to produce the output

$$y_i(t) = g\big[x(t) + \eta_i(t)\big], \quad i = 1, 2, \ldots, N. \tag{1}$$

The N noises $\eta_i(t)$ are white, mutually independent and identically distributed (i.i.d.) with cumulative distribution $F_\eta(u)$ and probability density $f_\eta(u) = dF_\eta(u)/du$. The response $y(t)$ of the array is obtained by averaging the outputs of all the sensors, as

$$y(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t). \tag{2}$$

The transmission of $s(t)$ by the array is assessed by the output SNR which is standard in stochastic resonance studies [2,16]. When $s(t)$ is deterministic with period T_s, the output signal $y(t)$ generically is a cyclostationary random signal, endowed with a power spectrum containing spectral lines at integer multiples of $1/T_s$, emerging out of a continuous noise background [16]. The output SNR $\mathcal{R}_{\mathrm{out}}$ is defined as the power contained in the output spectral line at the fundamental $1/T_s$ divided by the power contained in the noise background in a small frequency band ΔB around $1/T_s$.

For the output signal $y(t)$ of Eq. (2), the power contained in the output spectral line at the frequency $1/T_s$ is given [16] by $|\bar{Y}_1|^2$, where $\bar{Y}_1$ is the Fourier coefficient at the fundamental of the T_s-periodic nonstationary output expectation $\mathrm{E}[y(t)]$, i.e.

$$\bar{Y}_1 = \left\langle \mathrm{E}\big[y(t)\big]\exp\left(-\imath\frac{2\pi}{T_s}t\right)\right\rangle, \tag{3}$$

with the time average defined as

$$\langle\cdots\rangle = \frac{1}{T_s}\int_0^{T_s}\cdots\,dt. \tag{4}$$

The magnitude of the continuous noise background in the output spectrum is measured [16] by the stationarized output variance $\langle\mathrm{var}[y(t)]\rangle$, with the nonstationary variance given by $\mathrm{var}[y(t)] = \mathrm{E}[y^2(t)] - \mathrm{E}[y(t)]^2$ at a fixed time t.

The output SNR $\mathcal{R}_{\mathrm{out}}$, at the fundamental frequency $1/T_s$, follows as

$$\mathcal{R}_{\mathrm{out}} = \frac{|\langle\mathrm{E}[y(t)]\exp(-\imath 2\pi t/T_s)\rangle|^2}{\langle\mathrm{var}[y(t)]\rangle\Delta t\Delta B}, \tag{5}$$

where Δt is the time resolution of the measurement (i.e., the signal sampling period in a discrete time implementation). The white noise assumption, throughout, models broadband physical noises with a correlation duration much smaller than the other relevant time scales, i.e. T_s and Δt [16].

At time t, for a fixed given value x of the input $x(t)$, one has, according to Eq. (2), the conditional expectations

$$\mathrm{E}\big[y(t)|x\big] = \mathrm{E}\big[y_i(t)|x\big] \tag{6}$$

and

$$\mathrm{E}\big[y^2(t)|x\big] = \frac{1}{N}\mathrm{E}\big[y_i^2(t)|x\big] + \frac{N-1}{N}\mathrm{E}^2\big[y_i(t)|x\big], \tag{7}$$

which are both independent of i since the $\eta_i(t)$ are i.i.d. The large array limit $N = \infty$, when needed, will be simply accessible by letting $\mathrm{E}[y^2(t)|x] = \mathrm{E}^2[y_i(t)|x]$ in Eq. (7).

Next, since $x(t) = s(t) + \xi(t)$, the probability density for $x(t)$ is $f_\xi(x - s(t))$, and one has

$$\mathrm{E}\big[y(t)\big] = \int_{-\infty}^{+\infty}\mathrm{E}\big[y(t)|x\big]f_\xi\big(x - s(t)\big)\,dx, \tag{8}$$

and

$$\mathrm{E}\big[y^2(t)\big] = \int_{-\infty}^{+\infty}\mathrm{E}\big[y^2(t)|x\big]f_\xi\big(x - s(t)\big)\,dx. \tag{9}$$

Because of Eq. (1), one has for any i,

$$\mathrm{E}\big[y_i(t)|x\big] = \int_{-\infty}^{+\infty} g(x+u)f_\eta(u)\,du \tag{10}$$

and

$$\mathrm{E}\big[y_i^2(t)|x\big] = \int_{-\infty}^{+\infty} g^2(x+u)f_\eta(u)\,du. \tag{11}$$

Owing to its practical importance, we will consider in the sequel the case of a sinusoidal input

$$s(t) = A\sin(2\pi t/T_s) \tag{12}$$

buried in zero-mean Gaussian noise $\xi(t)$ with variance σ_ξ^2 (the present theory being however valid for any T_s-periodic $s(t)$ with any probability density for $\xi(t)$).

An input SNR $\mathcal{R}_{\mathrm{in}}$ for $x(t)$, defined in a similar way as $\mathcal{R}_{\mathrm{out}}$ of Eq. (5), is then

$$\mathcal{R}_{\mathrm{in}} = \frac{A^2/4}{\sigma_\xi^2\Delta t\Delta B}. \tag{13}$$

The resulting input–output SNR gain follows as

$$G = \frac{\mathcal{R}_{\text{out}}}{\mathcal{R}_{\text{in}}} = \frac{|\langle \mathrm{E}[y(t)] \exp(-\imath 2\pi t/T_s)\rangle|^2}{\langle \mathrm{var}[y(t)]\rangle} \frac{\sigma_\xi^2}{A^2/4}. \tag{14}$$

In the following, we will consider for $g(\cdot)$ the hard saturation defined as

$$g(u) = \begin{cases} -\lambda & \text{for } u \leqslant -\lambda, \\ u & \text{for } -\lambda < u < \lambda, \\ \lambda & \text{for } u \geqslant \lambda. \end{cases} \tag{15}$$

The "clipping" parameter $\lambda > 0$ will especially be used as an adjustable parameter in order to optimize the SNR gain G. Such threshold-free nonlinearities with saturation as in Eq. (15), have already been considered in the context of stochastic resonance in parallel arrays [17,18], but they were not investigated as optimizable SNR amplifiers for a noisy input consisting of a sinusoid in Gaussian noise.

The characteristic of Eq. (15) allows an explicit evaluation of the integrals (10)–(11) as

$$\begin{aligned} \mathrm{E}\big[y_i(t)|x\big] = {} & \lambda + (-\lambda - x)F_\eta(-\lambda - x) - (\lambda - x)F_\eta(\lambda - x) \\ & - G_\eta(-\lambda - x) + G_\eta(\lambda - x), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathrm{E}\big[y_i^2(t)|x\big] = {} & \lambda^2 + (\lambda^2 - x^2)\big[F_\eta(-\lambda - x) - F_\eta(\lambda - x)\big] \\ & - 2x\big[G_\eta(-\lambda - x) - G_\eta(\lambda - x)\big] \\ & - H_\eta(-\lambda - x) + H_\eta(\lambda - x), \end{aligned} \tag{17}$$

with the functions $G_\eta(u) = \int_{-\infty}^{u} v f_\eta(v)\,dv$ and $H_\eta(u) = \int_{-\infty}^{u} v^2 f_\eta(v)\,dv$.

The expressions of Eqs. (16)–(17) are then plugged into Eqs. (6)–(7) so as to provide expressions for the conditional expectations $\mathrm{E}[y(t)|x]$ and $\mathrm{E}[y(t)^2|x]$.

To proceed, since the η_i's can be considered as purposely added noises for the operation of the array, rather than noises imposed by the physical world, we choose their probability density $f_\eta(u)$ uniform over $[-a, a]$. This allows, with the characteristic of Eq. (15) associated to Eqs. (16)–(17), an explicit analytical evaluation of the integrals (8)–(9) as detailed in Appendix A. An explicit evaluation then follows for the SNR gain G of Eq. (14).

3. Improvement by noise of the SNR gain

From the derivation of Section 2 and Appendix A, the SNR gain G of Eq. (14) realized by the array, is known, in particular for any value of the clipping parameter λ of the saturating nonlinearity $g(\cdot)$ of Eq. (15). For a fixed given value of λ, and a fixed input noise level σ_ξ, the evolution of the SNR gain G can be studied as a function of the rms amplitude σ_η of the added array noises $\eta_i(t)$. For illustration, Fig. 1 presents such an evolution, in typical conditions with a zero-mean Gaussian input noise $\xi(t)$.

Fig. 1 shows that, in genuine arrays of size $N > 1$, the added array noises $\eta_i(t)$ can produce an improvement of the SNR gain G. An optimal nonzero value of the level σ_η of the array noises $\eta_i(t)$ raises the gain G to a maximum, which

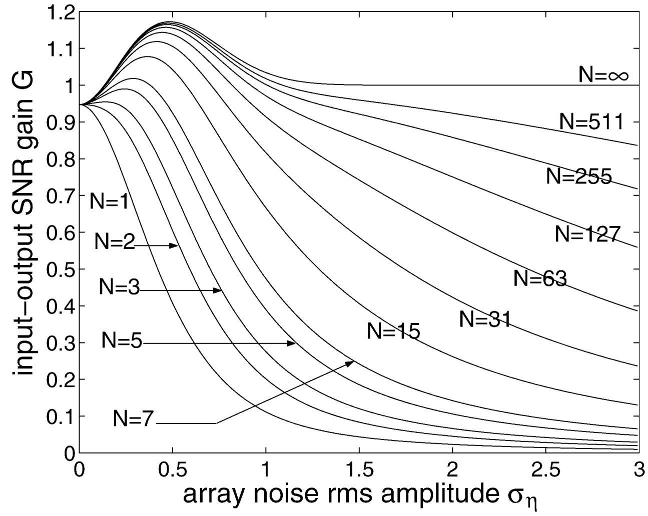

Fig. 1. Input–output SNR gain G of Eq. (14), as a function of the rms amplitude $\sigma_\eta = a/\sqrt{3}$ of the uniform array noises $\eta_i(t)$, with $\lambda = 0.3$ and $\sigma_\xi = 0.5$ (in units of $A = 1$).

is always higher than the value of G in the absence of the added noises $\eta_i(t)$, provided that $N > 1$. This is a phenomenon of noise-aided transmission, or stochastic resonance, in parallel nonlinear arrays, which was also reported with measures of performance other than G and with other nonlinearities [3–9]. Threshold nonlinearities in arrays were shown to lend themselves to such a phenomenon of noise-aided transmission, named on this occasion "suprathreshold stochastic resonance" [3,4]. The results of Fig. 1 prove that noise-aided transmission in arrays does not necessarily require threshold nonlinearities. It can occur with simple saturating nonlinearities as Eq. (15), which are also easily implementable as electronic devices for instance.

In addition, Fig. 1 shows that the input–output SNR gain G achieved by the array of saturations, can be larger than unity. This amplification of the SNR especially occurs in Fig. 1 for the sinusoid $s(t)$ of Eq. (12) added to Gaussian noise $\xi(t)$. It is to note that such an SNR amplification is impossible with a linear device, static or dynamic, whatever its complexity or high order: a linear device, in the frequency domain, multiplies both the coherent spectral line at $1/T_s$ and the noise background around $1/T_s$ by the same factor, the squared modulus of its transfer function at $1/T_s$, and therefore leaves the SNR unchanged. Also, an amplification $G > 1$ has never been obtained with a threshold nonlinearity for a sinusoid in Gaussian noise. The SNR amplification $G > 1$ in Fig. 1 obtained with the saturating nonlinearity $g(\cdot)$ of Eq. (15), occurs in a strong clipping regime of $g(\cdot)$, with $\lambda = 0.3$ for the transmission of the sinusoid $s(t)$ of Eq. (12) with amplitude $A = 1$. This strong clipping has the ability to reduce the input noise $\xi(t)$ more than it reduces the signal $s(t)$, whence the improved SNR. Alternatively, a large clipping parameter $\lambda \to \infty$ would lead to a linear transmission by $g(\cdot)$ of Eq. (15), yielding the array output $y(t) = s(t) + \xi(t) + N^{-1}\sum_{i=1}^{N} \eta_i(t)$, a purely additive signal–noise mixture with no possibility of SNR amplification $G > 1$. Amplification of the SNR is possible only through a truely non-

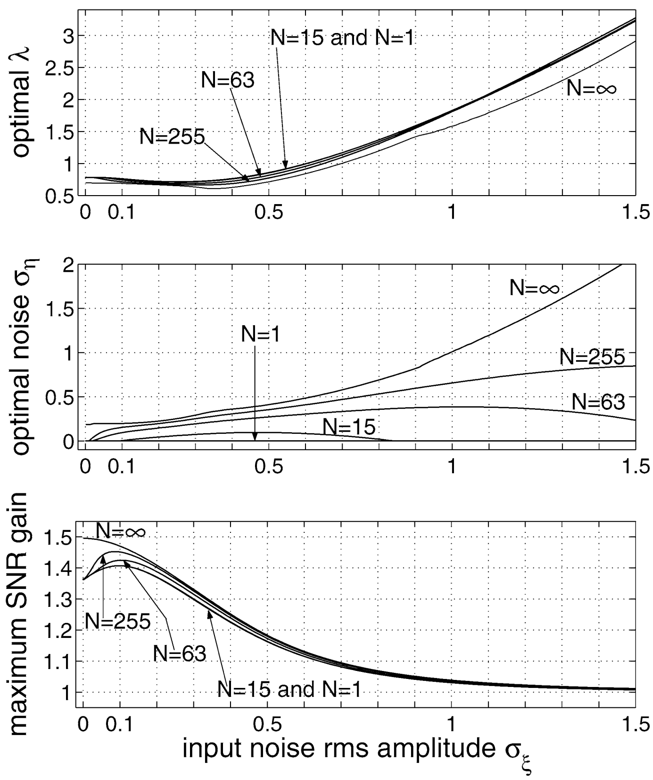

Fig. 2. Optimal value λ_{opt} of the clipping parameter λ (top), optimal value $\sigma_{\eta,\mathrm{opt}}$ of the rms amplitude σ_η of the array noises $\eta_i(t)$ (middle), and maximum of the input–output SNR gain G at $(\lambda_{\mathrm{opt}}, \sigma_{\eta,\mathrm{opt}})$ (bottom), as a function of the rms amplitude σ_ξ (in units of $A = 1$) of the zero-mean Gaussian input noise $\xi(t)$. The curves of λ_{opt} and of maximum G at $N = 1$ and $N = 15$ are distinct but close, that they cannot be visibly separated on the graphs.

linear action by $g(\cdot)$ of Eq. (15). A natural question which arises at this point, is to examine how to tune at its best λ the saturating nonlinearity $g(\cdot)$ of Eq. (15), so as to maximize the SNR gain G.

4. Optimization of the SNR gain

With the expression of the SNR gain G from Section 2, we have, for each input noise level σ_ξ, determined both the optimal value λ_{opt} of the clipping parameter λ and the optimal level $\sigma_{\eta,\mathrm{opt}}$ of the added array noises $\eta_i(t)$, that jointly maximize the SNR gain G of Eq. (14). These results are presented in Fig. 2 for the sinusoid $s(t)$ in Gaussian noise $\xi(t)$ with different array sizes N.

The results of Fig. 2 demonstrate several interesting properties afforded by these saturating nonlinearities.

At $N = 1$, i.e. for isolated nonlinearities, the results of Fig. 2 reveal that at the optimal tuning λ_{opt}, the SNR gain G is already always strictly above unity. This means that for a sinusoidal signal $s(t)$, especially in Gaussian noise $\xi(t)$ as in Fig. 2, the saturating nonlinearity $g(\cdot)$ of Eq. (15) when used in isolation, can always be tuned to achieve an SNR gain $G > 1$. This is an interesting property, since again SNR amplification cannot be obtained with a linear system; and with other types of simple nonlinear system, like a hard-threshold nonlinearity, SNR amplification has never been obtained simultaneously with a sinusoid and Gaussian noise [19,20]. By contrast, SNR amplification is easily obtainable with a saturating nonlinearity. This faculty can be attributed to the clipping effect implemented by a single saturating nonlinearity, which is able to reduce the noise more than the signal. The theoretical expressions of Section 2 and Appendix A make it easy to verify that the SNR amplification $G > 1$ is preserved with non-Gaussian input noise $\xi(t)$. This property of SNR amplification with an isolated saturating nonlinearity was also reported in [21]. Here, we extend the investigation by incorporating the possibility of examining the impact of added noises $\eta_i(t)$ and replication of the saturating nonlinearity into arrays. From Fig. 2, at $N = 1$, it is visible that the maximum SNR gain G always occurs at a zero level of the added noise $\eta_1(t)$: noise addition brings no improvement to the transmission by a single saturating nonlinearity optimally tuned. Yet, this is no longer the case in arrays with $N > 1$.

At $N > 1$, i.e. for genuine arrays, the results of Fig. 2 reveal that at the optimal tuning λ_{opt}, the SNR gain G is always strictly above unity, and also strictly above the gain G achieved at $N = 1$. The maximum gain realized by the optimized array, increases as its size N grows. An important property of the array is that the maximum gain is in general obtained for a nonzero value $\sigma_{\eta,\mathrm{opt}}$ of the level σ_η of the added array noises $\eta_i(t)$. This is the main finding of the present study, that association of saturating nonlinearities into arrays with added noises can improve the SNR amplification G above unity, with a maximum for the gain G which generally occurs for a nonzero level of the added noises. For any given input noise level σ_ξ, the best SNR amplification is always achieved by large arrays with $N \to \infty$ and occurs at a nonzero level of the added array noises $\eta_i(t)$. This property revealed by Fig. 2 in representative conditions (Gaussian input noise $\xi(t)$ with sinusoidal $s(t)$), is robustly preserved in other conditions, as it can be verified with the theoretical expressions for the gain G derived in Section 2 and Appendix A. Beyond the practical interest of arrays of simple nonlinearities like Eq. (15) to act as SNR amplifiers, there is an important conceptual significance to the present results: They demonstrate a situation of signal processing where the optimal configuration of the processing system that achives the best performance, is always associated with a nonzero optimal amount of added noise.

5. Discussion

The issue of the amplification of the SNR of a periodic signal in noise has often been addressed in stochastic resonance studies [19,20,22–24]. Input–output SNR gains larger than unity have been reported, separately for periodic nonsinusoidal signals in Gaussian noise, and for a sinusoidal signal in non-Gaussian noise. For the important case of a sinusoid in Gaussian noise that we address here, a few papers have also reported an SNR gain larger than unity [23–25]. In this respect, for the sinusoid in Gaussian noise, the maximum gain we report here in Fig. 2 are found higher than those reported in [23–25], revealing a superior efficacy of the present arrays of saturating devices for serving as SNR amplifiers.

We also emphasize that the present arrays are *static* nonlinear systems, which do not impose, by themselves, frequency limitations. As a consequence, their SNR amplification will take place in the same way, in principle, whatever the frequency of the sinusoidal signal. This is a notable difference afforded by static systems, compared to dynamic systems as those considered in [23,24] which may introduce frequency limitations through their specific time constants. This may authorize modulation schemes of the sinusoid, for instance a slowly varying frequency, or an epoch-wise fixed frequency which might switch at an appropriate rhythm between two predefined values, and SNR amplification may still be expected for such a modulated sinusoid. Frequency limitations may arise in practice (i) if the white noise assumption breaks down, when the correlation duration of the physical noise ceases to be negligible compared to the other time constants of the process, (ii) or from the physical implementation of the static nonlinearities, for instance through the use of operational amplifiers coming with they own cut-off frequencies. These aspects remain open for further investigation.

The essential features that result from the observations in this article can be summarized as follows.

(i) Isolated saturating nonlinearities can act as SNR amplifiers for a sinusoid in Gaussian noise, a property which is not afforded by linear systems or by other simple nonlinearities like hard thresholds. Power-law nonlinearities were shown in [25] to also exhibit this faculty of SNR amplification, although the maximum gains achieved are slightly lower, and from a practical standpoint they are more complex to implement compared to the saturating nonlinearities studied here. Other nonlinearities offering still better positioning in terms of maximum SNR gain and simplicity of practical implementation may exist, but are today not known as such.

(ii) The possibility of improving the operation of a nonlinear device through replication into a parallel array with added noises, is not restricted to threshold devices. This seems to be a general property shared by many nonlinear devices, not critically dependent on the presence of a threshold, but related to the action of the added noises which enhance the variability and richness of representation of an input by several distinct nonlinear outputs averaged over the array.

(iii) The beneficial effect of added noise in nonlinear systems is sometimes interpreted as a linearization by noise of the response. A linear system would at best yield an SNR gain $G = 1$. Clearly, the present nonlinear arrays with an optimal amount of added noises are not made equivalent, for their action on the SNR, to such a linear system, since they are capable of an SNR amplification $G > 1$. In this respect, the arrays act here as nonlinear systems performing better than linear systems for a given information-processing task, thanks to the beneficial action of noise.

(iv) At this stage of the studies on stochastic resonance, the picture which emerges is that, among the systems reported to realize an SNR amplification $G > 1$ of a sinusoid in Gaussian noise through addition of noise, those systems achieving the highest gains $G > 1$ are the present arrays of clipping devices operating with a nonzero optimal amount of added noise.

These specifically interesting properties arise at the intersection of nonlinearity, noise and array structure. These ingredients are also present in neuronal processes, which are very efficient for information processing, through detailed modalities largely remaining to be understood. All these elements constitute strong motivation to further investigate nonlinear systems assembled in arrays and aided by noise for efficient information processing.

Appendix A

In this appendix we detail the analytical evaluation of the integrals (8)–(9), when the array is used with the nonlinearity of Eq. (15) and the noises $\eta_i(t)$ are zero-mean uniform over $[-a, a]$. In this case, we have the cumulative distribution function

$$F_\eta(u) = \begin{cases} 0 & \text{for } u \leqslant -a, \\ \frac{u+a}{2a} & \text{for } -a < u < a, \\ 1 & \text{for } u \geqslant a, \end{cases} \tag{A.1}$$

and

$$G_\eta(u) = \int_{-\infty}^{u} v f_\eta(v)\,dv = \begin{cases} 0 & \text{for } u \leqslant -a, \\ \frac{u^2-a^2}{4a} & \text{for } -a < u < a, \\ 0 & \text{for } u \geqslant a, \end{cases} \tag{A.2}$$

and

$$H_\eta(u) = \int_{-\infty}^{u} v^2 f_\eta(v)\,dv = \begin{cases} 0 & \text{for } u \leqslant -a, \\ \frac{u^3+a^3}{6a} & \text{for } -a < u < a, \\ a^2/3 & \text{for } u \geqslant a. \end{cases} \tag{A.3}$$

Eqs. (A.1)–(A.3) when plugged into Eqs. (16)–(17) provide expressions for the conditional expectations $\mathrm{E}[y_i(t)|x]$ and $\mathrm{E}[y_i^2(t)|x]$, and then through Eqs. (6)–(7), expressions for $\mathrm{E}[y(t)|x]$ and $\mathrm{E}[y^2(t)|x]$. Next, these two last expressions have to be integrated according to Eqs. (8)–(9), and for this purpose we introduce the four functions $G_\xi(u) = \int_{-\infty}^{u} v f_\xi(v)\,dv$, $H_\xi(u) = \int_{-\infty}^{u} v^2 f_\xi(v)\,dv$, $K_\xi(u) = \int_{-\infty}^{u} v^3 f_\xi(v)\,dv$ and $L_\xi(u) = \int_{-\infty}^{u} v^4 f_\xi(v)\,dv$.

A.1. Evaluation of $\mathrm{E}[y(t)]$

The integral of Eq. (8) comes out as

$$\mathrm{E}\big[y(t)\big] = \lambda + I_1(\lambda) - I_1(-\lambda), \tag{A.4}$$

with the function $I_1(\lambda) = I_{11}(\lambda) + I_{12}(\lambda)$ and

$$I_{11}(\lambda) = -(\lambda + s)F_\xi(u) - G_\xi(u)\big|_{u=-\lambda-s-a} \tag{A.5}$$

(the function of the right-hand side is evaluated at $u = -\lambda - s - a$) and

$$I_{12}(\lambda) = \frac{1}{4a}\big[(\lambda+s-a)^2 F_\xi(u) + 2(\lambda+s-a)G_\xi(u) + H_\xi(u)\big]_{u=-\lambda-s-a}^{u=-\lambda-s+a} \tag{A.6}$$

(in the right-hand side the difference is taken of the value of the function at $u=-\lambda-s+a$ minus its value at $u=-\lambda-s-a$). Throughout this Appendix A we write s for $s(t)$.

A.2. Evaluation of $\mathrm{E}[y^2(t)]$

For the integral of Eq. (9), we first have

$$\int_{-\infty}^{+\infty} \mathrm{E}\big[y_i^2(t)|x\big] f_\xi(x-s)\,dx = \lambda^2 + I_2(\lambda) - I_2(-\lambda), \tag{A.7}$$

with the function $I_2(\lambda) = I_{21}(\lambda) + I_{22}(\lambda)$ and

$$I_{21}(\lambda) = \big(\lambda^2 - s^2 - a^2/3\big)F_\xi(u) - 2sG_\xi(u) - H_\xi(u)\big|_{u=-\lambda-s-a} \tag{A.8}$$

and

$$I_{22}(\lambda) = \frac{1}{6a}\big[(-2\lambda+s-a)(\lambda+s-a)^2 F_\xi(u) - 3\big[\lambda^2-(s-a)^2\big]G_\xi(u) + 3(s-a)H_\xi(u) + K_\xi(u)\big]_{u=-\lambda-s-a}^{u=-\lambda-s+a}. \tag{A.9}$$

We next have

$$\int_{-\infty}^{+\infty} \mathrm{E}^2\big[y_i(t)|x\big] f_\xi(x-s)\,dx = \lambda^2 + 2\lambda\big[I_1(\lambda) - I_1(-\lambda)\big] + I_3(\lambda) + I_3(-\lambda) - 2I_4, \tag{A.10}$$

with the function $I_3(\lambda) = I_{31}(\lambda) + I_{32}(\lambda)$ and

$$I_{31}(\lambda) = (\lambda+s)^2 F_\xi(u) + 2(\lambda+s)G_\xi(u) + H_\xi(u)\big|_{u=-\lambda-s-a} \tag{A.11}$$

and

$$I_{32}(\lambda) = \frac{1}{16a^2}\big[(\lambda+s-a)^4 F_\xi(u) + 4(\lambda+s-a)^3 G_\xi(u) + 6(\lambda+s-a)^2 H_\xi(u) + 4(\lambda+s-a)K_\xi(u) + L_\xi(u)\big]_{u=-\lambda-s-a}^{u=-\lambda-s+a}. \tag{A.12}$$

We introduce

$$I_{41} = \big(s^2-\lambda^2\big)F_\xi(u) + 2sG_\xi(u) + H_\xi(u)\big|_{u=-\lambda-s-a}, \tag{A.13}$$

and

$$I_{42}(u) = \frac{1}{4a}\big[(\lambda+s-a)^2(\lambda-s)F_\xi(u) + (\lambda+s-a)(\lambda-3s+a)G_\xi(u) - (\lambda+3s-2a)H_\xi(u) - K_\xi(u)\big]. \tag{A.14}$$

Now, if $a \leqslant \lambda$, one has

$$I_4 = I_{41} + \big[I_{42}(u)\big]_{u=-\lambda-s-a}^{u=-\lambda-s+a}, \tag{A.15}$$

else if $a > \lambda$, one has

$$I_4 = I_{41} + \big[I_{42}(u)\big]_{u=-\lambda-s-a}^{u=\lambda-s-a} + \big[I_{43}(u)\big]_{u=\lambda-s-a}^{u=-\lambda-s+a}, \tag{A.16}$$

with

$$I_{43}(u) = \frac{1}{16a^2}\big[\big[\lambda^2-(s-a)^2\big]^2 F_\xi(u) - 4(s-a)\big[\lambda^2-(s-a)^2\big]G_\xi(u) - 2\big[\lambda^2-3(s-a)^2\big]H_\xi(u) + 4(s-a)K_\xi(u) + L_\xi(u)\big]. \tag{A.17}$$

This completes the analytical evaluation of the integral of Eq. (9) for $\mathrm{E}[y^2(t)]$, with an arbitrary density $f_\xi(u)$.

A.3. Gaussian input noise

When the input noise $\xi(t)$ has the Gaussian density

$$f_\xi(u) = \frac{1}{\sigma_\xi\sqrt{2\pi}}\exp\left(-\frac{u^2}{2\sigma_\xi^2}\right), \tag{A.18}$$

the cumulative distribution function is

$$F_\xi(u) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{u}{\sqrt{2}\sigma_\xi}\right), \tag{A.19}$$

and it follows that $G_\xi(u) = -\sigma_\xi^2 f_\xi(u)$, $H_\xi(u) = uG_\xi(u) + \sigma_\xi^2 F_\xi(u)$, $K_\xi(u) = (u^2+2\sigma_\xi^2)G_\xi(u)$ and $L_\xi(u) = u^3 G_\xi(u) + 3\sigma_\xi^2 H_\xi(u)$.

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