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Stochastic antiresonance in qubit phase estimation with quantum thermal noise

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A B S T R A C T

We consider the fundamental quantum information processing task consisting in estimating the phase of a qubit. Following quantum measurement, the estimation performance is evaluated by the classical Fisher information which determines the best performance limiting any estimator and achievable by the maximum likelihood estimator. Estimation is analyzed in the presence of decoherence represented by a quantum thermal noise at arbitrary temperature. As the noise temperature is increased, we show the possibility of nontrivial behaviors of decoherence, with an estimation performance which does not necessarily degrade uniformly, but can experience nonmonotonic evolutions. Regimes are found where higher noise temperatures turn more favorable to estimation. Such behaviors are related to stochastic resonance or antiresonance effects, where noise reveals beneficial to information processing.

1. Introduction

The development of quantum information, quantum computation and quantum technologies critically depends on the capability of mastering quantum noise or decoherence. Noise commonly acts as a nuisance impairing information processing. However, in specific circumstances, it has been realized that noise can reveal beneficial to information processing. Such possibility has been explored and analyzed in relation to the phenomenon of stochastic resonance and useful-noise effects [1–4]. For information processing, stochastic resonance can be described as a phenomenon by which a nonzero optimal amount of noise maximizes the performance. Over the recent years, stochastic resonance has been reported in a large variety of processes, often nonlinear processes coupling signal and noise [1–4]. Most studies on stochastic resonance have essentially been accomplished in the classical (non-quantum) domain. Fewer and more recent studies have investigated stochastic resonance in the quantum domain. For instance, quantum forms of stochastic resonance have been reported for noise-assisted transitions in bistable systems [5–7], or for transmission of information over noisy quantum channels, binary [8–11] or of other types [12–16]. Quantum useful-noise effects comparable to stochastic resonance have been shown capable of assisting transport phenomena and energy harvesting in photosynthetic complexes, or capable of enhancing magneto-reception occurring in plant growth or in animal orientation and navigation [17,18].

In the present article, we specifically concentrate on a fundamental information processing task which is parameter estimation, which deals with efficient exploitation of measurement to infer values for physical quantities of interest. For classical estimation, stochastic resonance as useful-noise effects was addressed for instance in [19–23]. Comparatively, few studies have addressed stochastic resonance for quantum estimation. Very recently, [24] reported the possibility of useful-noise effect in the estimation of the norm of the Bloch vector of a qubit state. We extend here such analyses of stochastic resonance for quantum estimation, applied here to a fundamental task in quantum metrology which is the estimation of the phase of a qubit. The effect of an uncontrolled environment inducing decoherence is represented by a thermal bath, as in [24], with a temperature T assessing the level of noise. Several regimes are shown to exist where enhancement of the level of noise reveals beneficial to the performance in phase estimation. In particular, a regime exists where the action of noise takes the form of an antiresonance, with a minimum of the estimation performance occurring at a finite noise level, with smaller noise levels (which is natural) but also larger noise levels (which is counterintuitive) being more favorable to estimation. A comparable type of quantum stochastic antiresonance was recently reported in [25,26], with a performance metric constituted by entanglement preservation, which is shown minimized for a definite noise level specially harmful to entanglement. Comparable stochastic antiresonance has on some occasions been observed with classical systems [27–30],

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but very rarely with quantum systems. It is the first time here to our knowledge that it is reported in the context of quantum phase estimation assisted by noise, as we investigate, to contribute to a broader appreciation of the action of quantum decoherence.

2. Phase estimation on a noisy qubit

We consider the fundamental problem of quantum metrology which is the estimation of the phase ξ of a qubit, with relevance for instance to atomic clocks, interferometry, magnetometry [31]. A qubit with two-dimensional Hilbert space $\mathcal{H}_2$ is prepared in an initial quantum state represented by the density operator ρ_0 and it experiences the transformation $\rho_0 \to \mathsf{U}_\xi \rho_0 \mathsf{U}_\xi^\dagger$ defined by the unitary operator $\mathsf{U}_\xi = \exp(-i\xi\vec{n}\cdot\vec{\sigma}/2)$, where $\vec{n} = [n_x, n_y, n_z]^\top$ is a unit vector of $\mathbb{R}^3$. In Bloch representation [32], the qubit is prepared in the initial state $\rho_0 = (\mathrm{I}_2 + \vec{r}_0\cdot\vec{\sigma})/2$, with I_2 the identity on $\mathcal{H}_2$, and $\vec{\sigma}$ a formal vector assembling the three Pauli operators $[\sigma_x, \sigma_y, \sigma_z] = \vec{\sigma}$. The coordinates of ρ_0 are specified by the Bloch vector $\vec{r}_0 \in \mathbb{R}^3$, with norm $\|\vec{r}_0\| = 1$ for a pure state and $\|\vec{r}_0\| < 1$ for a mixed state. The transformed state $\mathsf{U}_\xi\rho_0\mathsf{U}_\xi^\dagger = \rho_1(\xi)$ is specified by a Bloch vector $\vec{r}_1(\xi)$ formed by $\vec{r}_0$ rotated by the angle ξ around the axis $\vec{n}$ in $\mathbb{R}^3$. The rotated state $\rho_1(\xi)$, before it becomes accessible to measurement for estimating ξ, is affected by a quantum noise. The action of a quantum noise [32,33] is generally representable by a completely positive trace-preserving superoperator $\mathcal{N}(\cdot)$ producing the ξ-dependent noisy quantum state $\rho_\xi = \mathcal{N}(\rho_1)$. This is equivalent to a Bloch vector $\vec{r}_\xi$ specifying ρ_ξ supplied by the affine transformation [32,34]

$$\vec{r}_\xi = A\vec{r}_1(\xi) + \vec{c}\,, \tag{1}$$

with A a 3×3 real matrix and $\vec{c}$ a real vector of $\mathbb{R}^3$ characterizing the quantum noise.

A quantum measurement is then implemented on the noisy qubit in state ρ_ξ in order to estimate the unknown value of the phase ξ. From the outcomes of the measurement, having the status of realizations of a classical random variable, an estimator $\widehat{\xi}$ is devised for the phase ξ. After classical estimation theory [35,36], any estimator $\widehat{\xi}$ for ξ is endowed with a mean-squared error $\langle(\widehat{\xi}-\xi)^2\rangle$ lower bounded by the Cramér–Rao bound involving the reciprocal of the classical Fisher information $F_c(\xi)$. The larger the Fisher information $F_c(\xi)$, the more efficient the estimation can be. The maximum likelihood estimator [36] is known to achieve the best performance dictated by the Cramér–Rao bound and Fisher information $F_c(\xi)$, at least in the asymptotic regime of a large number of independent measurements. The classical Fisher information $F_c(\xi)$ stands in this way as a fundamental metric quantifying the best achievable performance in estimation. It is therefore relevant to identify the conditions of optimality maximizing the Fisher information $F_c(\xi)$. In this respect, there exists a general upper bound [37,38] formed by the quantum Fisher information $F_q(\xi)$ which limits the classical Fisher information $F_c(\xi)$ by imposing $F_c(\xi) \le F_q(\xi)$. For estimation of the phase ξ of a noisy qubit in a state ρ_ξ specified by the Bloch vector $\vec{r}_\xi$ of Eq. (1), the quantum Fisher information $F_q(\xi)$ is expressible as [39]

$$F_q(\xi) = \frac{\big[(A\vec{r}_1+\vec{c})A(\vec{n}\times\vec{r}_1)\big]^2}{1-(A\vec{r}_1+\vec{c})^2} + \big[A(\vec{n}\times\vec{r}_1)\big]^2\,. \tag{2}$$

The quantum Fisher information $F_q(\xi)$ of Eq. (2) is intrinsic to the quantum state ρ_ξ and its relation to the parameter ξ, and does not refer to any specific measurement protocol. By contrast, the classical Fisher information $F_c(\xi)$ characterizes an explicit measurement protocol which is required for effective estimation. A general quantum measurement on a qubit is represented by a generalized measurement [32] defined by K measurement operators $\mathsf{M}_k = b_k\mathrm{I}_2 + \vec{a}_k\cdot\vec{\sigma}$ which are positive operators on $\mathcal{H}_2$ with $(\vec{a}_k, b_k)$ real satisfying $\sum_{k=1}^K \vec{a}_k = \vec{0}$ and $\sum_{k=1}^K b_k = 1$, so as to realize $\sum_{k=1}^K \mathsf{M}_k = \mathrm{I}_2$. Especially, $\|\vec{a}_k\| \le b_k \le 1-\|\vec{a}_k\|$ is required for all k to ensure $0 \le \mathsf{M}_k \le \mathrm{I}_2$. For estimating the phase ξ, when such a generalized measurement is applied to the qubit in the state ρ_ξ from Eq. (1), the classical Fisher information $F_c(\xi)$ results as [39]

$$F_c(\xi) = \sum_{k=1}^K \frac{\big[\vec{a}_k A(\vec{n}\times\vec{r}_1)\big]^2}{b_k + \vec{a}_k(A\vec{r}_1+\vec{c})}\,. \tag{3}$$

For a qubit, the most accessible measurement consists in measuring a spin observable $\Omega = \vec{\omega}\cdot\vec{\sigma}$ with eigenvalues $\pm\|\vec{\omega}\| = \pm 1$. This is equivalent to implementing a von Neumann projective measurement defined by the $K=2$ measurement operators $\mathsf{M}_\pm = (\mathrm{I}_2 \pm \vec{\omega}\cdot\vec{\sigma})/2$, with $\|\vec{\omega}\| = 1$, forming two projectors on two orthogonal directions in $\mathcal{H}_2$. In this circumstance, the classical Fisher information $F_c(\xi)$ of Eq. (3) reduces to

$$F_c(\xi) = \frac{\big[\vec{\omega}A\big(\vec{n}\times\vec{r}_1\big)\big]^2}{1-\big[\vec{\omega}\big(A\vec{r}_1+\vec{c}\big)\big]^2}\,. \tag{4}$$

Due to the great practical importance of measuring a spin observable, in the sequel we will essentially concentrate on estimation by means of such type of von Neumann measurements.

An important quantum noise relevant to the qubit we specifically examine here, is the generalized amplitude damping noise or quantum thermal noise [32,40], which describes the interaction of the qubit with an uncontrolled environment represented as a thermal bath at temperature T. It is characterized in Eq. (1), in the orthonormal basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ of $\mathbb{R}^3$, by the 3×3 diagonal matrix $A = \mathrm{diag}[\sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma]$ and vector $\vec{c} = [0, 0, (2p-1)\gamma]^\top$. The damping factor $\gamma \in [0,1]$ characterizes the coupling of the qubit with the thermal bath. At long interaction times, $\gamma \to 1$, and the noisy qubit relaxes to the equilibrium mixed state $\rho_\infty = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ of Bloch vector $\vec{r}_\infty = \vec{c}$. At equilibrium, the qubit has probabilities p of being measured in the ground state $|0\rangle$ and $1-p$ of being measured in the excited state $|1\rangle$. With the energies E_0 and $E_1 > E_0$ respectively for the states $|0\rangle$ and $|1\rangle$, the equilibrium probabilities are governed by the Boltzmann distribution

$$p = \frac{1}{1+\exp[-(E_1-E_0)/(k_BT)]}\,. \tag{5}$$

In this way, in the quantum thermal noise, the probability p is determined by the temperature T of the bath via Eq. (5). From Eq. (5), the probability p is a decreasing function of the temperature T. At $T=0$ the probability is $p=1$ for the ground state $|0\rangle$, while at $T\to\infty$ there is equiprobability with $p\to 1/2$ for the ground state $|0\rangle$ and the excited state $|1\rangle$. Therefore, from Eq. (5), when the temperature T monotonically increases from 0 to ∞, then the probability p monotonically decreases from 1 to 1/2. This variational behavior is preserved for any energy difference $E_1 - E_0 > 0$ in Eq. (5). For the sake of definiteness, in figures where illustrations are presented we shall take $E_1 - E_0 = 1$ in units where $k_B = 1$. These are illustrative conditions for display, but which do not affect the significance of the analysis.

With the quantum thermal noise, from [40] the quantum Fisher information $F_q(\xi)$ of Eq. (2) is maximized by a pure initial state ρ_0 specified by a unit Bloch vector $\vec{r}_0$ orthogonal to the rotation axis $\vec{n}$, with a maximum achievable Fisher information $F_q^{\max} = 1-\gamma$. The quantum thermal noise, in addition to its important practical relevance, is interesting because it has recently been shown [24] to lend itself to useful-noise effect or stochastic resonance, where an increase in the level of noise can induce improvement in some information processing tasks. In [24] some possibility of improvement by noise was reported with the quantum

Fisher information for assessing the performance in estimating the norm of a qubit Bloch vector affected by quantum thermal noise. To extend the result of [24], we will investigate here the possibility of comparable improvement by noise for the important task of qubit phase estimation. To assess an effective estimation scenario, we will analyze the evolution of the classical Fisher information $F_c(\xi)$ of Eq. (4) with the level of the thermal noise, and look for the possibility of beneficial noise conditions.

3. Performance evolution with noise

For analysis of the classical Fisher information $F_c(\xi)$ of Eq. (4), due to the rotational symmetry around $\vec{e}_z$ of the thermal noise, with no loss of generality it is always possible to choose in $\mathbb{R}^3$ the basis vector $\vec{e}_x$ orthogonal to the rotation axis $\vec{n}$. As a result $\vec{n}$ is in the plane $(\vec{e}_y, \vec{e}_z)$, with coordinates $\vec{n} = [0, n_y, n_z]^\top$ satisfying $n_y^2 + n_z^2 = 1$, and it is enough to consider $n_y \in [0, 1]$ due to the symmetry of the situation. For the orthonormal basis $\{\vec{n}, \vec{n}_\perp \equiv \vec{e}_x, \vec{n}'_\perp = \vec{n} \times \vec{n}_\perp\}$ of $\mathbb{R}^3$ tied to the rotation axis $\vec{n}$, one therefore has $\vec{n}'_\perp = \vec{n} \times \vec{e}_x = [0, n_z, -n_y]^\top$. From [40] it is known that optimizing the measurement requires an initial Bloch vector $\vec{r}_0$ orthogonal to the rotation axis $\vec{n}$, and also a vector $\vec{\omega}$ in Eq. (4) orthogonal to $\vec{n}$, which we choose to parametrize as $\vec{\omega} = \cos(\phi)\vec{n}_\perp + \sin(\phi)\vec{n}'_\perp = [\cos(\phi), \sin(\phi)n_z, -\sin(\phi)n_y]^\top$. Since measurement of the observable $\Omega = \vec{\omega} \cdot \vec{\sigma}$ implements two projectors defined by the two unit vectors $\pm\vec{\omega}$ of $\mathbb{R}^3$, it is enough to consider $\phi \in [0, \pi[$. An initial Bloch vector $\vec{r}_0$ orthogonal to the rotation axis $\vec{n}$ is in the plane $(\vec{n}_\perp, \vec{n}'_\perp)$. It is always possible to place it as $\vec{r}_0 = \vec{n}_\perp = \vec{e}_x$ in order to estimate the rotation angle ξ around $\vec{n}$, since any other initial angle of $\vec{r}_0$ in the plane $(\vec{n}_\perp, \vec{n}'_\perp)$ would amount to an unimportant known shift of the origin for defining ξ. The rotated state $\rho_1(\xi)$ results with a unit Bloch vector $\vec{r}_1(\xi)$ which is $\vec{r}_0$ rotated by ξ in the plane $(\vec{n}_\perp, \vec{n}'_\perp)$ orthogonal to the rotation axis $\vec{n}$, leading to $\vec{r}_1(\xi) = \cos(\xi)\vec{n}_\perp + \sin(\xi)\vec{n}'_\perp = [\cos(\xi), \sin(\xi)n_z, -\sin(\xi)n_y]^\top$. The configuration of the vectors in $\mathbb{R}^3$ is depicted in Fig. 1.

For the purpose of computing the Fisher information $F_c(\xi)$ of Eq. (4), we thus have the generic parametrization $\vec{n} = [0, n_y, n_z]^\top$, also $\vec{\omega} = [\cos(\phi), \sin(\phi)n_z, -\sin(\phi)n_y]^\top$ and $\vec{r}_1(\xi) = [\cos(\xi), \sin(\xi)n_z, -\sin(\xi)n_y]^\top$ with the vector coordinates referring to the orthonormal basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ of $\mathbb{R}^3$ where the noise is characterized by $(A, \vec{c})$. For $F_c(\xi)$ of Eq. (4), it then follows that

$$\vec{\omega}A\big(\vec{n} \times \vec{r}_1\big) = \sqrt{1-\gamma}\Big(\sin(\phi-\xi) - \big(1-\sqrt{1-\gamma}\big)\sin(\phi)\cos(\xi)n_y^2\Big) , \tag{6}$$

and

$$\vec{\omega}A\vec{r}_1 = \sqrt{1-\gamma}\Big(\cos(\phi-\xi) - \big(1-\sqrt{1-\gamma}\big)\sin(\phi)\sin(\xi)n_y^2\Big) , \tag{7}$$

while

$$\vec{\omega}\vec{c} = \omega_z c_z = -\sin(\phi)n_y(2p-1)\gamma . \tag{8}$$

By replacing Eqs. (6)–(8) in Eq. (4), we obtain the Fisher information $F_c(\xi)$ as a function of all the relevant variables of the estimation problem in generic conditions.

We are specially interested by studying the influence of the temperature T on the Fisher information $F_c(\xi)$. Increasing the temperature T of the bath acting as a thermal noise intuitively amounts to increasing the detrimental level of the noise; we want to examine if this always translates into a degradation of the Fisher information $F_c(\xi)$. The temperature T acts on the probability p via Eq. (5); in turn p acts on $F_c(\xi)$ via the inner product $\vec{\omega}\vec{c}$ of Eq. (8). Right away Eq. (8) identifies two special configurations where the

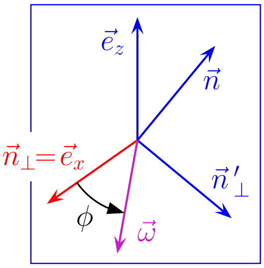

Fig. 1. The unit vectors in $\mathbb{R}^3$ controlling the quantum estimation. Vectors $\vec{e}_z$, $\vec{n}$ and $\vec{n}'_\perp = \vec{n} \times \vec{n}_\perp$ are in the same plane orthogonal to $\vec{n}_\perp$, while $\vec{\omega}$ rotates in the plane $(\vec{n}_\perp, \vec{n}'_\perp)$ according to the angle ϕ.

Fisher information $F_c(\xi)$ is found independent of the temperature T. When $\sin(\phi) = 0$, i.e. for a measurement vector $\vec{\omega} = \vec{e}_x$, then it results in Eq. (4) that

$$F_c(\xi) = \frac{(1-\gamma)\sin^2(\xi)}{1-(1-\gamma)\cos^2(\xi)} , \tag{9}$$

which is maximized at $1-\gamma$ for $\xi = \pm\pi/2$ and minimized at 0 for $\xi = 0$ or π. Similarly, when $n_y = 0$, i.e. for a rotation axis $\vec{n} = \vec{e}_z$, then it results in Eq. (4) that

$$F_c(\xi) = \frac{(1-\gamma)\sin^2(\phi-\xi)}{1-(1-\gamma)\cos^2(\phi-\xi)} , \tag{10}$$

which is maximized at $1-\gamma$ for $\phi - \xi = \pm\pi/2$ and minimized at 0 for $\phi - \xi = 0$ or π.

In the general case when $\sin(\phi)n_y \neq 0$, the temperature T influences the Fisher information $F_c(\xi)$ only through the term $(\vec{\omega}A\vec{r}_1 + \vec{\omega}\vec{c})^2$ in the denominator of Eq. (4) via $\vec{\omega}\vec{c}$ of Eq. (8). Moreover, when T is varied, $F_c(\xi)$ and $(\vec{\omega}A\vec{r}_1 + \vec{\omega}\vec{c})^2$ both vary in the same direction as a function of T. In particular, the minimum of $F_c(\xi)$ occurs at the minimum of $(\vec{\omega}A\vec{r}_1 + \vec{\omega}\vec{c})^2$. Based on Eq. (8), the term $(\vec{\omega}A\vec{r}_1 + \vec{\omega}\vec{c})^2$ is a $\cup$-shaped parabola in p. Therefore $F_c(\xi)$ is also expected to display a $\cup$-shaped evolution with p, however limited to the allowed interval $p \in [1/2, 1]$ attainable when $T \in [0, \infty[$ in Eq. (5). The minimum value possible for $(\vec{\omega}A\vec{r}_1 + \vec{\omega}\vec{c})^2$ is zero, produced by $\vec{\omega}\vec{c} = -\vec{\omega}A\vec{r}_1$, and this would occur at a critical value p_c for the probability p which from Eq. (8) follows as

$$p_c = \frac{1}{2} + \frac{1}{2}\alpha_c , \tag{11}$$

with the scalar parameter $\alpha_c = \vec{\omega}A\vec{r}_1/[\sin(\phi)n_y\gamma]$ which is known from Eq. (7) as

$$\alpha_c = \frac{\sqrt{1-\gamma}}{\gamma}\left[\frac{\cos(\phi-\xi)}{n_y\sin(\phi)} - \big(1-\sqrt{1-\gamma}\big)n_y\sin(\xi)\right] . \tag{12}$$

It is therefore critical, for the evolution of $F_c(\xi)$ with p (and subsequently with T), to locate the position of p_c of Eq. (11) in relation to the allowed interval $[1/2, 1] \ni p$; this is equivalent to locating α_c of Eq. (12) in relation to the interval $[0, 1]$. It results that there exist three accessible regimes, which lead to three qualitatively distinct evolutions of the Fisher information $F_c(\xi)$ with the temperature $T \in [0, \infty[$ of the thermal bath, and that we will analyze in the sequel.

At first, it is useful to identify the two extreme values of the Fisher information $F_c(\xi)$ at the two extreme temperatures $T = 0$ and $T = \infty$. From Eqs. (4) and (8), since at $T = 0$ one has $p = 1$, it follows

$$F_c(\xi; T=0) = \frac{\big[\vec{\omega}A\big(\vec{n} \times \vec{r}_1\big)\big]^2}{1 - \big[\vec{\omega}A\vec{r}_1 - \sin(\phi)n_y\gamma\big]^2} , \tag{13}$$

while at $T = \infty$ since $p = 1/2$ it follows

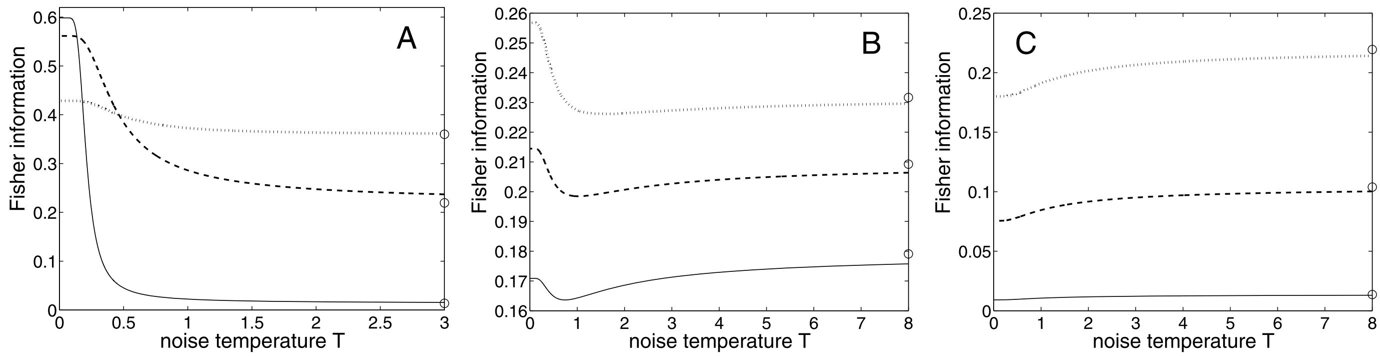

Fig. 2. Fisher information F_c from Eq. (4) as a function of the noise temperature T at $n_y = 1$ and $\phi = \pi/2$. The circles ($\circ$) are the asymptotic values $F_c(\xi; T = \infty)$ of Eq. (14). (A) Decreasing F_c with $\gamma = 0.4$, at $\xi = 0$ (dotted line), $\xi = -0.25\pi$ (dashed line), $\xi = -0.45\pi$ (solid line). (B) Antiresonant F_c with $\gamma = 0.5$, at $\xi = 0.1\pi$ (dotted line), $\xi = 0.15\pi$ (dashed line), $\xi = 0.2\pi$ (solid line). (C) Increasing F_c with $\gamma = 0.4$, at $\xi = 0.25\pi$ (dotted line), $\xi = 0.35\pi$ (dashed line), $\xi = 0.45\pi$ (solid line).

$$F_c(\xi; T = \infty) = \frac{\left[\vec{\omega} A\left(\vec{n} \times \vec{r}_1\right)\right]^2}{1 - \left[\vec{\omega} A \vec{r}_1\right]^2} . \tag{14}$$

The two extreme values for F_c in Eqs. (13)–(14) differ by the term $-\sin(\phi) n_y \gamma$ in the denominator; therefore in general the further away is $-\sin(\phi) n_y \gamma$ from zero the larger will be the difference between $F_c(\xi; T = 0)$ and $F_c(\xi; T = \infty)$. In particular, for γ fixed by the thermal noise, $\sin(\phi) n_y = \pm 1$ maximizes this difference; such configuration may not be reachable in practice since n_y is imposed by the rotation axis $\vec{n}$, yet it forms a useful reference expressing the maximal excursion of F_c. The excursion between the extreme values $F_c(\xi; T = 0)$ and $F_c(\xi; T = \infty)$ as the temperature T increases from 0 to ∞ can however take place according to three distinct regimes, as anticipated above.

3.1. F_c decreasing with T

In Eq. (11) for $p_c \le 1/2$ (i.e. for $\alpha_c \le 0$) then the zero of $(\vec{\omega} A \vec{r}_1 + \vec{\omega}\vec{c})^2$ at p_c occurs before the interval $[1/2, 1] \ni p$, so that for any $p \in [1/2, 1]$ the parabola $(\vec{\omega} A \vec{r}_1 + \vec{\omega}\vec{c})^2$, or equivalently the Fisher information $F_c(\xi)$ of Eq. (4), increases with an increasing $p \in [1/2, 1]$. This is equivalent to a decreasing Fisher information $F_c(\xi)$ as the temperature T grows from 0 to ∞. This is somehow the expected natural behavior: as the temperature T of the thermal noise increases, the performance in estimation quantified by $F_c(\xi)$ steadily degrades.

Such a regime of decreasing F_c is obtained by any set of conditions ensuring $\alpha_c \le 0$ in Eq. (12). This can be realized in various configurations of the rotation axis via n_y, of the measurement observable via ϕ, of the thermal noise via the damping γ, for estimating the phase ξ. Some illustrative conditions of this type are shown in Fig. 2(A).

3.2. F_c antiresonant with T

In Eq. (11) for $p_c \in]1/2, 1[$ (i.e. for $\alpha_c \in]0, 1[$) then the zero of $(\vec{\omega} A \vec{r}_1 + \vec{\omega}\vec{c})^2$ at p_c occurs inside the interval $[1/2, 1] \ni p$, so that for any $p \in [1/2, 1]$ the parabola $(\vec{\omega} A \vec{r}_1 + \vec{\omega}\vec{c})^2$, or equivalently the Fisher information $F_c(\xi)$ of Eq. (4), undergoes a $\cup$-shaped evolution with an increasing $p \in [1/2, 1]$ passing through a minimum at $p = p_c$. This is equivalent to also a $\cup$-shaped evolution of the Fisher information $F_c(\xi)$ as the temperature T increases from 0 to ∞, with $F_c(\xi)$ passing through a minimum at the critical temperature T_c related to p_c via Eq. (5). Such a regime of antiresonant $\cup$-shaped F_c is obtained by any set of conditions ensuring $\alpha_c \in]0, 1[$ in Eq. (12). Some illustrative conditions of this type are shown in Fig. 2(B).

Such antiresonant evolutions of F_c as in Fig. 2(B) are reminiscent of a stochastic resonance effect, where a relevant measure of performance for some definite information processing task undergoes a nonmonotonic evolution as the level of noise increases, instead of a monotonic degradation [2–4]. Most often, the nonmonotonic evolution observed in stochastic resonance occurs as a peak where the performance culminates at a maximum for a nonzero optimal amount of noise. By contrast, here as in some other studies, the nonmonotonic evolution occurs as a dip where the performance is minimized by a nonzero amount of noise specially harmful to the process. This manifests a nontrivial action of the noise, capable of occurring in classical [27–30] as well as in quantum [25,26] processes, although it is the first time here that such a stochastic antiresonance is reported in the Fisher information $F_c(\xi)$ assessing the performance in quantum phase estimation. This reveals some sophisticated aspects in the action of quantum noise or decoherence, which is not necessarily uniformly more detrimental as its amount increases. Antiresonant $\cup$-shaped evolutions of F_c as in Fig. 2(B) indicate that configurations exist where the conditions at $T = 0$ and at $T \to \infty$ are more favorable for estimation than at intermediate temperatures T. According to the analysis above of such antiresonant $\cup$-shaped evolutions, at large temperatures $T \to \infty$ in Fig. 2(B), the Fisher information $F_c(\xi)$ follows the same increasing trend up to the asymptotic limit $F_c(\xi; T = \infty)$ of Eq. (14) materialized by the circles ($\circ$) in Fig. 2(B). In particular, as visible in Fig. 2(B), configurations exist where $F_c(\xi; T = \infty)$ is larger than $F_c(\xi; T = 0)$. This is achieved, based on Eqs. (13)–(14), when $|\vec{\omega} A \vec{r}_1 - \sin(\phi) n_y \gamma| < |\vec{\omega} A \vec{r}_1|$ which occurs when $\sin(\phi) n_y \gamma$ lies between 0 and $2\vec{\omega} A \vec{r}_1$. In such configurations, in principle, large temperatures $T \to \infty$ come out as the most favorable for estimation; however, in practice, this has to be mitigated by the necessity to limit the temperature T before it can cause damage to the quantum system.

3.3. F_c increasing with T

Finally, in Eq. (11) for $p_c \ge 1$ (i.e. for $\alpha_c \ge 1$) then the zero of $(\vec{\omega} A \vec{r}_1 + \vec{\omega}\vec{c})^2$ at p_c occurs after the interval $[1/2, 1] \ni p$, so that for any $p \in [1/2, 1]$ the parabola $(\vec{\omega} A \vec{r}_1 + \vec{\omega}\vec{c})^2$, or equivalently the Fisher information $F_c(\xi)$ of Eq. (4), decreases with an increasing $p \in [1/2, 1]$. This is equivalent to an increasing Fisher information $F_c(\xi)$ as the temperature T rises from 0 to ∞.

Such a regime of increasing F_c is obtained by any set of conditions ensuring $\alpha_c \ge 1$ in Eq. (12). Some illustrative conditions of this type are shown in Fig. 2(C). For such increasing evolutions, the asymptotic value $F_c(\xi; T = \infty)$ materialized by the circles ($\circ$) in Fig. 2(C) is always larger than $F_c(\xi; T = 0)$, and the performance in estimation steadily improves as the temperature T increases. This is another counterintuitive behavior obtainable in definite conditions with quantum noise or decoherence, where the largest amount thereof turns out to be most favorable to the in-

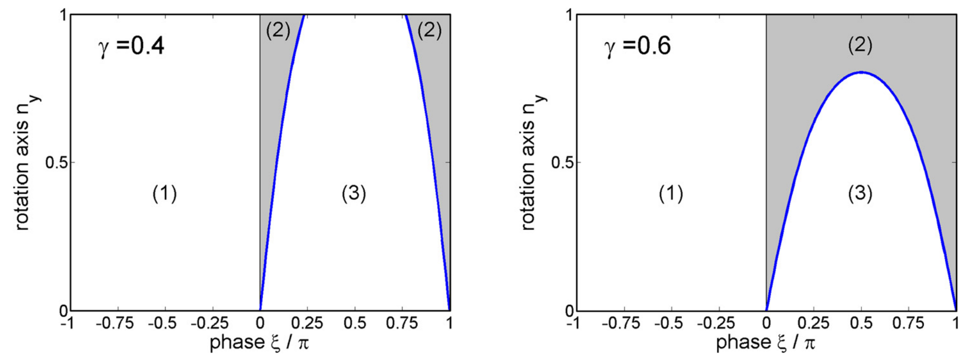

Fig. 3. For $(\xi, n_y) \in [-\pi, \pi] \times [0, 1]$, the three domains of evolution of the Fisher information $F_c(\xi; \phi = \pi/2)$ of Eq. (15) with the temperature T, as controlled by α_c of Eq. (16). Domain (1) is a decreasing $F_c(\xi)$ when $\alpha_c \le 0$; domain (2) in gray is an antiresonant $F_c(\xi)$ when $\alpha_c \in]0, 1[$; domain (3) is an increasing $F_c(\xi)$ when $\alpha_c \ge 1$. Two panels show the damping $\gamma = 0.4$ and $\gamma = 0.6$.

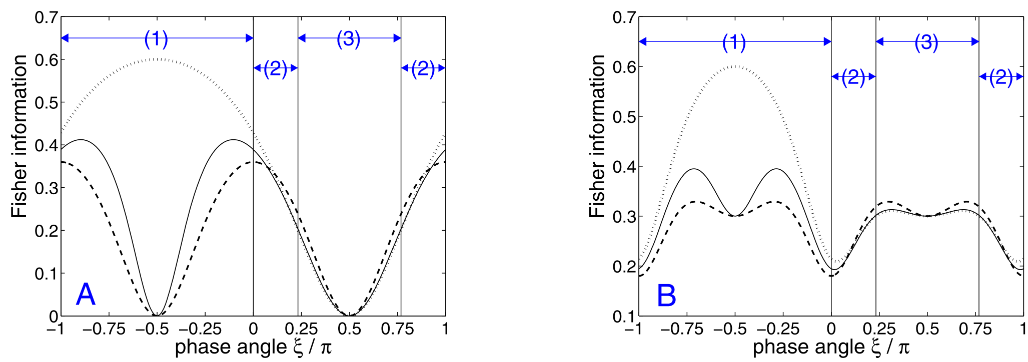

Fig. 4. As a function of the phase angle ξ, for damping $\gamma = 0.4$, at three different temperatures T, the dotted curves are at $T = 0$ with $p = 1$, the solid curves at $T = 0.6$ with $p \approx 0.841$ deduced from Eq. (5), the dashed curves at $T = \infty$ with $p = 1/2$. The vertical lines separate the three regimes of evolution for $F_c(\xi)$ with T (see text). (A) Fisher information $F_c(\xi; \phi = \pi/2, n_y = 1)$ from Eq. (17). (B) Fisher information $F_c(\xi) = [F_c(\xi; \phi = 0) + F_c(\xi; \phi = \pi/2)]/2$ from Eqs. (9) and (17) at $n_y = 1$, for the generalized measurement with $K = 4$ operators.

formation processing task. In practice here also the temperature will have to be limited before it can cause damage to the quantum system being estimated.

4. Parameter-independent characterization

A significant aspect in the estimation task is that the value of the Fisher information $F_c(\xi)$ and the occurrence of its three accessible regimes are usually dependent on the value or at least the range of the unknown phase ξ to be estimated. This is a common property for parameter estimation, where the performance and its conditions of optimality may depend on the value or range of the parameter to be estimated, and this is true also for quantum estimation [41,40,39]. It is in general helpful to have some prior appreciation of this dependence with ξ, in order to gain better control on the estimation process operating in the presence of an unknown ξ. Below, we demonstrate that the nonmonotonic regimes of evolution of the Fisher information with the temperature T are robustly preserved in relevant conditions concerning the parameter ξ. Here we have a dependence in ξ of $F_c(\xi)$ from Eqs. (6)–(8) placed in Eq. (4), and a dependence in ξ of α_c in Eq. (12) to control the three regimes. This dependence in ξ can be illustrated with a measurement vector $\vec{\omega}$ placed at $\phi = \pi/2$, so as to yield with Eqs. (6)–(8) in Eq. (4),

$$F_c(\xi; \phi = \pi/2) = \frac{(1-\gamma)\cos^2(\xi)\big[1 - \big(1 - \sqrt{1-\gamma}\,\big)n_y^2\big]^2}{1 - \Big(\sqrt{1-\gamma}\,\sin(\xi)\big[1 - \big(1-\sqrt{1-\gamma}\,\big)n_y^2\big] - \gamma n_y(2p-1)\Big)^2}, \tag{15}$$

to be contrasted with Eq. (9) valid at $\phi = 0$. And in this case $\phi = \pi/2$, the three regimes are controlled in Eqs. (11)–(12) by

$$\alpha_c = \frac{\sqrt{1-\gamma}}{\gamma} \sin(\xi) \left[\frac{1}{n_y} - \big(1 - \sqrt{1-\gamma}\,\big) n_y \right], \tag{16}$$

relative to the interval $[0, 1]$. Both Eqs. (15) and (16) explicitly manifest the dependence in ξ. This can be visualized in Fig. 3 which displays in the plane (ξ, n_y) the three domains of evolution of the Fisher information $F_c(\xi)$ with the temperature T, as controlled by α_c of Eq. (16), this at two values of the damping γ.

Fig. 3 demonstrates how the three regimes of evolution of the Fisher information $F_c(\xi)$ with the temperature T are preserved over broad conditions, especially involving the unknown phase ξ, and how they are crossed by varying the conditions. In addition, Fig. 4(A) represents the Fisher information $F_c(\xi)$ as a function of the phase angle $\xi \in [-\pi, \pi]$ in the rotation around an axis $\vec{n}$ at $n_y = 1$, when Eq. (15) reduces to

$$F_c(\xi; \phi = \pi/2, n_y = 1) = \frac{(1-\gamma)^2 \cos^2(\xi)}{1 - \big[(1-\gamma)\sin(\xi) - \gamma(2p-1)\big]^2}. \tag{17}$$

The thermal noise in Fig. 4(A) has a damping $\gamma = 0.4$ and three temperatures T are tested. In these conditions of Fig. 4(A), the three regimes are governed from Eq. (16) by $\alpha_c = (\gamma^{-1} - 1)\sin(\xi) = 1.5\sin(\xi)$. For ξ a critical value results as $\xi_c = \arcsin(1/1.5) \approx 0.23\pi$ realizing $\alpha_c = 1$. For $\xi \in [-\pi, 0]$, then $\alpha_c \le 0$ sets the regime of a decreasing $F_c(\xi)$ with increasing T, marked as the region (1) in Fig. 4(A). For $\xi \in [\alpha_c, \pi - \alpha_c]$, then $\alpha_c \ge 1$ sets the regime of an increasing $F_c(\xi)$ with increasing T, marked as the region (3) in Fig. 4(A). For ξ elsewhere, then

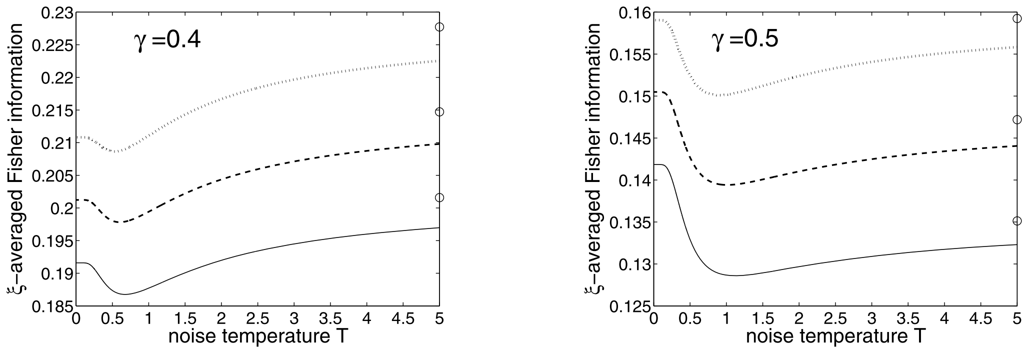

Fig. 5. Fisher information $F_c(\xi;\phi=\pi/2)$ from Eq. (15) after averaging over the phase ξ, as a function of the noise temperature T; with $n_y=1$ (solid lines), $n_y=0.95$ (dashed lines), $n_y=0.9$ (dotted lines). The circles (○) are the asymptotic values at $T=\infty$. Two panels show the damping $\gamma=0.4$ and $\gamma=0.5$.

$\alpha_c \in]0,1[$ sets the regime of an antiresonant $F_c(\xi)$ with T, marked as the region (2) in Fig. 4(A).

Fig. 4(A) as well as Fig. 3 manifest the dependence with ξ of the values of the Fisher information $F_c(\xi)$ and of its three regimes of evolution. In general, Eq. (6) is also

$$\vec{\omega}A(\vec{n}\times\vec{r}_1)=\sqrt{1-\gamma}\Big(-\cos(\phi)\sin(\xi)+\big[1-\big(1-\sqrt{1-\gamma}\big)n_y^2\big]\sin(\phi)\cos(\xi)\Big), \qquad (18)$$

so that for any n_y, i.e. any rotation axis $\vec{n}$, and any orientation ϕ of the measurement vector $\vec{\omega}$, there always exists a phase ξ realizing $\vec{\omega}A(\vec{n}\times\vec{r}_1)=0$ and in this way achieving a vanishing Fisher information $F_c(\xi)$ in Eq. (4). This is accomplished from Eq. (18) by ξ solution to $\tan(\xi)=\big[1-\big(1-\sqrt{1-\gamma}\big)n_y^2\big]\tan(\phi)$ which always exists; moreover for $\xi\in]-\pi,\pi]$ there always exist two such solutions separated by π. These two solutions generally form two zeros of $F_c(\xi)$, except in the special configuration ($\phi=\pi/2, n_y=1, p=1$) where the two solutions are $\xi=\pm\pi/2$ but since the denominator of $F_c(\xi)$ reduced to Eq. (17) also vanishes at $\xi=-\pi/2$, only $\xi=\pi/2$ forms a zero of $F_c(\xi)$ here. Such a vanishing Fisher information means that, for any rotation axis $\vec{n}$ and any measurement vector $\vec{\omega}$, there always exist generally two (seldom, one) values of $\xi\in]-\pi,\pi]$ where the measurement is completely inoperative for estimating such ξ. For instance, at $\phi=\pi/2$, the Fisher information $F_c(\xi)$ of Eq. (15) generally goes to zero at $\xi=\pm\pi/2$, except $\xi=-\pi/2$ in the special configuration ($n_y=1, p=1$) from Eq. (17) where it goes to $1-\gamma$, as visible in Fig. 4(A). Yet, as we next explain, there is a possibility of avoiding such inoperative conditions where the Fisher information $F_c(\xi)$ vanishes.

As understandable from Fig. 1, in the plane orthogonal to the rotation axis $\vec{n}$, the measurement vector $\vec{\omega}$ with orientation ϕ somehow has to track the unknown rotation angle ξ for estimation. As illustrated in Fig. 4(A), it is usually better for efficient estimation to have ϕ and ξ separated by $\pm\pi/2$ rather than by 0 or π; but since ξ is unknown, any fixed ϕ will usually lead to an estimation performance $F_c(\xi)$ varying with ξ; and as explained above, for any ϕ there always exist values of ξ where $F_c(\xi)$ vanishes. For better tracking of ξ, instead of using a single unit vector $\vec{\omega}$ defining a von Neumann measurement, we have the faculty to use two such unit vectors $\vec{\omega}_1$ and $\vec{\omega}_2$ to define a generalized measurement with $K=4$ measurement operators $(\mathrm{I}_2\pm\vec{\omega}_1\cdot\vec{\sigma})/4$ and $(\mathrm{I}_2\pm\vec{\omega}_2\cdot\vec{\sigma})/4$. Based on Eq. (3), the overall Fisher information of such a generalized measurement is simply the average $F_c(\xi)=[F_c(\xi;\vec{\omega}_1)+F_c(\xi;\vec{\omega}_2)]/2$, where $F_c(\xi;\vec{\omega}_1)$ and $F_c(\xi;\vec{\omega}_2)$ are the two individual Fisher informations given by Eq. (4) at $\vec{\omega}_1$ and $\vec{\omega}_2$. Moreover, for efficient tracking of ξ it is interesting to take $\vec{\omega}_1$ and $\vec{\omega}_2$ at right angle in the plane orthogonal to the rotation axis $\vec{n}$, ensuring that $F_c(\xi;\vec{\omega}_1)$ and $F_c(\xi;\vec{\omega}_2)$, and thus the overall $F_c(\xi)$, do not vanish at the same ξ so as to keep the estimation operative for any ξ. Placing $\vec{\omega}_1$ at $\phi=0$ and $\vec{\omega}_2$ at $\phi=\pi/2$ yields in Eq. (3) the overall Fisher information $F_c(\xi)=[F_c(\xi;\phi=0)+F_c(\xi;\phi=\pi/2)]/2$ issued from Eq. (9) and Eq. (15), and this $F_c(\xi)$ is depicted in Fig. 4(B) in some illustrative conditions.

As expected, the Fisher information $F_c(\xi)$ of Fig. 4(B) achieved by the generalized measurement with $K=4$ operators, never vanishes (as opposed to that of Fig. 4(A)), and in this way maintains operative efficiency for estimation for any phase ξ. Also, since $F_c(\xi;\phi=0)$ from Eq. (9) is independent of the noise temperature T, this Fisher information $F_c(\xi)$ of Fig. 4(B) inherits from $F_c(\xi;\phi=\pi/2)$ of Fig. 4(A) the same dependence with the noise temperature T. In this respect, $F_c(\xi)$ of Fig. 4(B) has access to the three regimes of dependence with T, as shown in Fig. 4(B).

It is even possible to completely eliminate the dependence in ξ of the characterization of the performance in estimation. This can be obtained by performing an averaging of the Fisher information $F_c(\xi)$ over the phase angle ξ considered uniform over $]-\pi,\pi]$. This characterizes an average performance, over repeated estimation experiments involving values of ξ uniformly covering $]-\pi,\pi]$. Fig. 5 shows such a ξ-averaged Fisher information resulting from $F_c(\xi)$ of Eq. (15). In general, depending on the rotation axis $\vec{n}$, on the damping γ and on the measurement vector $\vec{\omega}$, the three regimes of evolution with the noise temperature T are still accessible for such a ξ-averaged Fisher information. Fig. 5 specifically shows conditions of an antiresonant regime, with the ξ-averaged Fisher information which becomes minimal at a critical noise temperature T_c, with T_c occurring around 1 in Fig. 5, although the precise value of T_c is usually dependent on $\vec{n}$ and on γ as visible in Fig. 5. Around such a critical temperature T_c, the thermal noise is maximally detrimental with an estimation efficacy which is at a minimum. It is therefore preferable, when such T_c is identified, to operate the process at lower or at higher temperatures, whenever possible.

In definite conditions fixing the rotation axis $\vec{n}$, the damping γ and the measurement vector $\vec{\omega}$, based on such a ξ-averaged Fisher information as in Fig. 5 providing a meaningful performance metric independent of the phase ξ to be estimated, we are provided with an intrinsic appreciation of the effect of the noise temperature T, bearing no dependence on the unknown phase ξ. This may serve to devise an optimized setting for the estimation process, having a general usefulness not tied to some specific value or range of the phase ξ, possibly by increasing the noise temperature if appropriate. For instance, the setting of Fig. 5 incites to avoid operating around the critical temperature $T_c\approx 1$ and to adjust to a higher (or lower) temperature if accessible.

The nonmonotonic evolution of the ξ-averaged performance is also preserved with the generalized measurement of $K=4$ operators examined in Fig. 4(B). This is illustrated in Fig. 6 which represents the result of averaging over ξ the Fisher information

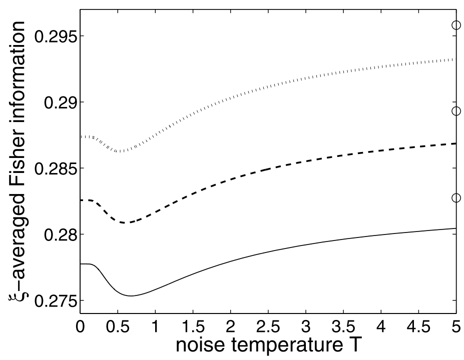

Fig. 6. Fisher information $F_c(\xi) = [F_c(\xi;\phi=0) + F_c(\xi;\phi=\pi/2)]/2$ from Eqs. (9), (15) and Fig. 4(B) after averaging over the phase ξ, as a function of the noise temperature T; with damping $\gamma = 0.4$ and rotation axis $n_y = 1$ (solid line), $n_y = 0.95$ (dashed line), $n_y = 0.9$ (dotted line). The circles (∘) are the asymptotic values at $T = \infty$.

$F_c(\xi) = [F_c(\xi;\phi=0) + F_c(\xi;\phi=\pi/2)]/2$ from Eq. (9) and Eq. (15) also displayed in Fig. 4(B) at $n_y = 1$. The conditions of Fig. 6, for the rotation axis $\vec{n}$ and noise damping γ, specifically illustrate the antiresonant regime where high noise temperatures are generally preferable.

The conditions of Fig. 6, as well as those of Fig. 5, demonstrate that the performance assessed by the ξ-averaged Fisher information, which is made independent of the unknown parameter ξ, still has access to the nonmonotonic regimes of evolution with the noise temperature T. This illustrates the possibility of nontrivial behavior of the quantum noise or decoherence, which is not uniformly more detrimental as its amount increases. And based on this recognition, one may have the option to avoid detrimental temperatures and enhance the performance by adjusting to higher (or lower) temperatures.

5. Discussion

We have considered the fundamental information processing task consisting in estimating the phase ξ of a qubit state. After implementation of a quantum measurement, the estimation performance is evaluated by the classical Fisher information $F_c(\xi)$ which determines the best performance limiting any estimator and achievable by the maximum likelihood estimator. The estimation process was analyzed in the presence of decoherence represented by a quantum thermal noise at an arbitrary temperature T. We have shown the possibility of nontrivial behaviors of decoherence, manifested by an estimation performance $F_c(\xi)$ which does not necessarily degrade uniformly as the noise temperature T increases. Especially, two unusual regimes of evolution of the performance have been shown possible. In definite conditions, there exists a finite noise temperature specially detrimental to estimation where the performance antiresonates at a minimum, with smaller or larger noise which is always preferable. Such regime of antiresonant evolution points out the existence of finite temperature values or ranges that should be avoided for efficient estimation. In other conditions, it is found that increasing the noise temperature always improves the performance for estimation. Uncovering such counterintuitive possibilities demonstrates some sophisticated role of decoherence, which can turn beneficial to information processing, leading to unusual means of enhancing performance by increasing decoherence.

To summarize the conditions enabling such nonmonotonic evolutions of the performance, these can be grounded in the form of the Fisher information $F_c(\xi)$ of Eq. (4) with its built-in dependence with the noise parameter $\vec{c}$ carrying the influence of the temperature T. From Eq. (4), the evolution of the Fisher information $F_c(\xi)$ with T is controlled by the scalar $[\vec{\omega}(A\vec{r}_1 + \vec{c})]^2$ occurring in the denominator. If this scalar can increase with the temperature T then so will the Fisher information $F_c(\xi)$. When T increases from 0 to ∞, then p monotonically decreases from 1 to 1/2 and $\|\vec{c}\|$ monotonically decreases from γ to 0. When $\vec{\omega}A\vec{r}_1(\xi)$ and $\vec{\omega}\vec{c}$ have the same sign, then a decrease of $\|\vec{c}\|$ decreases $[\vec{\omega}(A\vec{r}_1 + \vec{c})]^2$ and therefore decreases $F_c(\xi)$. This is the common regime where the performance $F_c(\xi)$ decreases as the noise temperature T is raised. On the contrary, when $\vec{\omega}A\vec{r}_1(\xi)$ and $\vec{\omega}\vec{c}$ have opposite signs, then a decrease of $\|\vec{c}\|$ increases $[\vec{\omega}(A\vec{r}_1 + \vec{c})]^2$ and therefore increases $F_c(\xi)$. This is the unusual regime where the performance $F_c(\xi)$ increases as the noise temperature T is raised. Geometrically, $\vec{\omega}A\vec{r}_1(\xi)$ and $\vec{\omega}\vec{c}$ have opposite signs when one of the two vectors $A\vec{r}_1(\xi)$ or $\vec{c}$ makes an acute angle with $\vec{\omega}$ and the other an obtuse angle. This geometric configuration depends in conjunction on the phase angle ξ and rotation axis $\vec{n}$ via $\vec{r}_1(\xi)$, on the quantum measurement via $\vec{\omega}$, on the noise damping γ via A, and it can be controlled by the temperature via $\vec{c}$. A detailed analysis of such conditions has been worked out here, showing the possibility of favorable outcome of raising the temperature.

In the general noise model of Eq. (1), the matrix A is always contractive, ensuring $\|A\vec{r}_1\| \le \|\vec{r}_1\|$ for any $\vec{r}_1$, otherwise for pure states with $\|\vec{r}_1\| = 1$ there could exist some $\vec{r}_1$ yielding $\|A\vec{r}_1\| > 1$ which is not allowed. A quantum noise with $\vec{c} \equiv \vec{0}$ in Eq. (1) is a unital noise, such as the depolarizing noise or Pauli noises [32,34]. As the level of a unital noise increases, the contraction gets more pronounced, and in such a process the Fisher information $F_c(\xi)$ of Eq. (4) never increases. Only a nonunital noise with $\vec{c} \not\equiv \vec{0}$ as the thermal noise, can lead to an increase of $F_c(\xi)$ of Eq. (4) by increasing the noise, as it occurs by raising the temperature T here. Therefore, when measuring a spin observable $\Omega = \vec{\omega} \cdot \vec{\sigma}$ as in Eq. (4), a unital noise cannot lead to a nonmonotonic evolution of the Fisher information $F_c(\xi)$ in Eq. (4), but $F_c(\xi)$ will monotonically decrease as the noise level increases. A nonunital noise is required, as the thermal noise, for nonmonotonic evolutions of $F_c(\xi)$ in Eq. (4) and stochastic resonance or antiresonance effects. In essence the effect of increasing Fisher information can be described in geometric terms. The rotated Bloch vector $\vec{r}_1(\xi)$ conveying the unknown phase ξ, after the geometric transformation $\vec{r}_1(\xi) \to A\vec{r}_1(\xi) + \vec{c} = \vec{r}_\xi$ by the noise, ends up in a position relative to the spin-measurement vector $\vec{\omega}$ more favorable to estimation.

Beyond spin observables, with generalized measurements as in Eq. (3), the geometric situation gets more involved, with possibly broader noise conditions to entail stochastic resonance or antiresonance effects. In this way, further exploration of information processing with beneficial action of quantum noise or decoherence can be extended in several directions. For instance, extensions can be made to other quantum noise models, or for other information processing tasks. This can be accomplished on quantum systems of larger dimension, although the geometric analysis tractable here in Bloch representation, may be more difficult to handle in dimension higher than that of the two-dimensional qubit, which remains a fundamental reference for quantum information.

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