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Optimized probing states for qubit phase estimation with general quantum noise

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We exploit the theory of quantum estimation to investigate quantum state estimation in the presence of noise. The quantum Fisher information is used to assess the estimation performance. For the qubit in Bloch representation, general expressions are derived for the quantum score and then for the quantum Fisher information. From this latter expression, it is proved that the Fisher information always increases with the purity of the measured qubit state. An arbitrary quantum noise affecting the qubit is taken into account for its impact on the Fisher information. The task is then specified to estimating the phase of a qubit in a rotation around an arbitrary axis, equivalent to estimating the phase of an arbitrary single-qubit quantum gate. The analysis enables determination of the optimal probing states best resistant to the noise, and proves that they always are pure states but need to be specifically matched to the noise. This optimization is worked out for several noise models important to the qubit. An adaptive scheme and a Bayesian approach are presented to handle phase-dependent solutions.

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I. INTRODUCTION

Quantum estimation is a fundamental process for efficient extraction of information from physical measurement on a quantum system. The theoretical grounds of quantum estimation were laid in [1,2], though new developments continue to appear [3]. A generic form of a quantum estimation problem is when a quantum state, generally represented by a density operator ρ_ξ, is dependent upon an unknown parameter ξ, and from measurement performed on ρ_ξ one has to efficiently infer a value for ξ. Due to the probabilistic nature of quantum measurement, quantum estimation is naturally assessed in a statistical framework. Fundamental limits exist which bound the efficiency of quantum estimation, and which can be expressed by means of the Fisher information derived from the statistical score function. A classical form of the Fisher information limits the statistical exploitation of the classical random variables occurring as the outcomes of a quantum measurement; in turn, a quantum form of the Fisher information puts a bound, of purely quantum origin, on the classical Fisher information, as analyzed in [4,5]. We will focus here on the Fisher information for performance analysis in quantum estimation, and review its two forms.

In quantum estimation, as analyzed for instance in [1,2,4,5], quantum states are usually represented by generic density operators; however, an important and realistic condition is to envisage that the quantum states accessible to measurement for estimation, carry the mark of some quantum noise inherent to the environment. One considers in this way that the quantum state dependent on the unknown parameter ξ, is subsequently affected by a specified quantum noise, before it becomes accessible to measurement so as to infer a value for ξ. This replaces estimation from a noise-free quantum system which is somehow implicit in the standard approach, by estimation from a noisy quantum system with an explicit handle on the noise and its impact in the estimation task. One can especially address the issue of optimizing the initial ξ-dependent state prior to the action of noise, so as to maximize the estimation performance after the action of a specified noise.

Such an approach of estimation from a noisy quantum system has been considered recently in [6–8], with the quantum Fisher information to assess the performance which is maximized through selection of the initial states best resistant to the noise, for several noise models. The references [6,7] consider estimation of the phase of a qubit acquired in a rotation around the Oz axis, when the qubit is affected by a phase-flip noise in [6] and then extended to bit-flip and to phase-bit-flip noises in [7]. A further extension to quantum systems of higher dimension is performed in [8], for estimation on phase-shifted pure single-mode Gaussian states of light affected by a phase noise. In the present study we generalize the work of [6,7] for estimation from a noisy qubit, with the quantum Fisher information to assess the performance in estimation. We consider estimation of the phase of a qubit acquired in a rotation around an arbitrary given axis, this being equivalent to estimation of the phase of an arbitrary single-qubit quantum gate. We consider also the possibility of an arbitrary quantum noise affecting the qubit. In such general conditions, we exploit the Bloch representation for qubit states to derive explicit expressions for the quantum score and the quantum Fisher information. This outcome then enables a characterization of the optimal initial qubit states best resistant to the noise, upon estimation from a noisy qubit affected by an arbitrary given noise. Several noise models important to the qubit are analyzed for illustration.

II. QUANTUM STATE ESTIMATION

In this section we review the theory of quantum state estimation, for completeness of the present study and to serve as a guideline for its subsequent application to the noisy qubit.

A. Optimal performance in estimation

A quantum system, with complex Hilbert space $\mathcal{H}_N$ of dimension N, has its state represented by the density operator ρ_ξ dependent upon an unknown real scalar parameter ξ. To estimate the value of ξ, a measurement is performed on the quantum system, generally by means of a positive operator-valued measure (POVM) with elements $\{M_x, x \in \mathcal{X}\}$ returning a set of possible outcomes $x \in \mathcal{X}$. A number M of copies of the quantum system prepared in state ρ_ξ are successively measured so as to provide M independent measurement outcomes $(x_1, x_2, \dots, x_M)^\top = \vec{x}$, from which an estimator is constructed as the mapping $\widehat{\xi} = \widehat{\xi}(\vec{x})$ to deduce a value $\widehat{\xi}$ for the unknown parameter ξ. Due to the generally probabilistic nature of quantum measurement, the generic outcome x represents a classical random variable.

The performance in estimation [9] can be assessed by the mean-squared error $E[(\widehat{\xi}(\vec{x}) - \xi)^2]$, related to the bias $b(\widehat{\xi}) = E(\widehat{\xi}) - \xi$ and variance $\mathrm{var}(\widehat{\xi})$ of estimator $\widehat{\xi}(\vec{x})$ by $E[(\widehat{\xi} - \xi)^2] = b^2(\widehat{\xi}) + \mathrm{var}(\widehat{\xi})$. Especially, for any estimator $\widehat{\xi}(\vec{x})$, the mean-squared error is bounded below by the Cramér-Rao inequality

$$E[(\widehat{\xi} - \xi)^2] \geqslant b^2 + \frac{(1 + \partial_\xi b)^2}{M F_c(\xi)}, \tag{1}$$

with $F_c(\xi)$ the classical Fisher information, and where for the derivative we use throughout the shorthand notation $\partial_\xi b \equiv \partial b / \partial \xi$. The right side of Eq. (1) is the Cramér-Rao bound, which especially depends on the estimator through its bias $b(\widehat{\xi})$. In general, a larger Fisher information $F_c(\xi)$ is more favorable to estimation as associated to a lower Cramér-Rao bound. The important class of unbiased estimators $\widehat{\xi}(\vec{x})$ with $b(\widehat{\xi}) = 0$, is ruled by an intrinsic form of the Cramér-Rao inequality

$$E[(\widehat{\xi} - \xi)^2] = \mathrm{var}(\widehat{\xi}) \geqslant \frac{1}{M F_c(\xi)}, \tag{2}$$

with an intrinsic Cramér-Rao bound $1/[M F_c(\xi)]$ common to all unbiased estimators.

From a given dataset $\vec{x}$, an optimal strategy for estimation is then to seek an estimator $\widehat{\xi}(\vec{x})$ that saturates the Cramér-Rao inequality. There is no general methodology to construct such an optimal estimator. However, asymptotically in the limit of a large number M of independent data points, the maximum likelihood method achieves such an optimum and provides an unbiased estimator saturating the Cramér-Rao inequality of Eq. (2).

The classical Fisher information $F_c(\xi)$ acting in Eqs. (1) and (2) is defined as the variance of the (classical) score $V(x;\xi)$. In turn, the score $V(x;\xi)$ is the random variable defined by

$$V(x;\xi) = \partial_\xi \ln p(x;\xi) = \frac{\partial_\xi p(x;\xi)}{p(x;\xi)}, \tag{3}$$

where $p(x;\xi)$ is the probability density of the data point x bearing dependence on the unknown parameter ξ. The score has vanishing mean $E[V(x;\xi)] = 0$, since

$$\begin{aligned} E[V(x;\xi)] &= \int_{\mathcal{X}} \frac{\partial_\xi p(x;\xi)}{p(x;\xi)} p(x;\xi) dx \\ &= \partial_\xi \int_{\mathcal{X}} p(x;\xi) dx = \partial_\xi 1 = 0. \end{aligned} \tag{4}$$

The variance of the score $\mathrm{var}[V(x;\xi)] = E[V^2(x;\xi)]$ defines the classical Fisher information

$$F_c(\xi) = E[V^2(x;\xi)] = \int_{\mathcal{X}} \frac{1}{p(x;\xi)} [\partial_\xi p(x;\xi)]^2 dx. \tag{5}$$

For the whole dataset of M independent random variables $\vec{x} = (x_1, x_2, \dots, x_M)^\top$ the probability density factorizes as $p(\vec{x}) = p(x_1)p(x_2)\cdots p(x_M)$ so the score $V(\vec{x},\xi)$ is additive; and since the variance of a sum of independent random variables is also additive, the Fisher information of the whole set of the M identically distributed data points $\vec{x}$ is M times the Fisher information $F_c(\xi)$ of a single data point of Eq. (5), i.e., $M F_c(\xi)$ at the denominator in Eqs. (1)–(2). The Cramér-Rao inequality in Eqs. (1) and (2) follows from the Cauchy-Schwarz inequality

$$\begin{aligned} &\{E([V - E(V)][\widehat{\xi} - E(\widehat{\xi})])\}^2 \\ &\quad \leqslant E([V - E(V)]^2) E([\widehat{\xi} - E(\widehat{\xi})]^2). \end{aligned} \tag{6}$$

The right side of Eq. (6) is $\mathrm{var}[V(\vec{x},\xi)]\mathrm{var}(\widehat{\xi}) = M F_c(\xi)\mathrm{var}(\widehat{\xi})$; since $E(V) = 0$ the left side of Eq. (6) is $[E(V\widehat{\xi})]^2$, and via Eq. (3) it evaluates (for $\widehat{\xi}$ independent of ξ) to $[\partial_\xi E(\widehat{\xi})]^2 = (1 + \partial_\xi b)^2$ so as to yield Eq. (1).

This is an analysis of the performance in estimation based on the classical random variables $(x_1, x_2, \dots, x_M)$ independent and identically distributed according to $p(x;\xi)$. Additional specifications arise when it is considered that the probability distribution $p(x;\xi)$ is determined by a quantum measurement of the quantum state ρ_ξ that introduces the dependence on the unknown parameter ξ.

Upon measuring with measurement operator M_x, the resulting probability is given by the trace $p(x;\xi) = \mathrm{tr}(M_x \rho_\xi)$. The derivative present in the Fisher information $F_c(\xi)$ of Eq. (5) goes as $\partial_\xi p(x;\xi) = \mathrm{tr}(M_x \partial_\xi \rho_\xi)$, since the measurement operator M_x, much like the classical estimator $\widehat{\xi}(\vec{x})$, ought to be independent of the unknown parameter ξ. The derivative is now transferred to a density operator, and a useful way to handle the derivative of a density operator is through the introduction of another operator L_ξ, known as the symmetrized logarithmic derivative, and defined by the equation (yet to be solved) [1,2]

$$\partial_\xi \rho_\xi \equiv \frac{\partial \rho_\xi}{\partial \xi} = \frac{1}{2}(L_\xi \rho_\xi + \rho_\xi L_\xi). \tag{7}$$

Since ρ_ξ is a positive hence Hermitian operator verifying $\rho_\xi^\dagger = \rho_\xi$, hermiticity is also transferred to its derivative to yield $(\partial_\xi \rho_\xi)^\dagger = (\rho_\xi L_\xi^\dagger + L_\xi^\dagger \rho_\xi)/2 = \partial_\xi \rho_\xi$, for any ξ, so that by identification with Eq. (7) one deduces $L_\xi^\dagger = L_\xi$. Hence, Eq. (7) defines a Hermitian operator L_ξ, which is matched to the density operator ρ_ξ, and for this reason generally bears dependence on the unknown parameter ξ. (This is, in particular, preferable to the alternative of handling the derivative through $\partial_\xi \rho_\xi = L_\xi \rho_\xi$, which would not set a Hermitian L_ξ.)

Also, taking the trace of Eq. (7) yields $\mathrm{tr}(\partial_\xi \rho_\xi) = \mathrm{tr}(\rho_\xi L_\xi)$ by the circular invariance of the trace; this is also $\mathrm{tr}(\rho_\xi L_\xi) = \partial_\xi \, \mathrm{tr}(\rho_\xi) = \partial_\xi 1 = 0$, so that the mean value $\langle L_\xi \rangle = \mathrm{tr}(\rho_\xi L_\xi) = 0$. The quantum observable L_ξ with vanishing mean, is also known as the quantum score, and can be viewed as a quantum analog of the classical score $V(x;\xi)$ of Eq. (3), as it will also further verify in the sequel.

From Eq. (7) one obtains

$$\mathrm{tr}(M_x \partial_\xi \rho_\xi) = \tfrac{1}{2}[\mathrm{tr}(M_x L_\xi \rho_\xi) + \mathrm{tr}(M_x \rho_\xi L_\xi)] \tag{8}$$

$$= \tfrac{1}{2}(\mathrm{tr}(\rho_\xi M_x L_\xi) + \mathrm{tr}[(\rho_\xi M_x L_\xi)^\dagger]) \tag{9}$$

$$= \mathrm{Re}\,\mathrm{tr}(\rho_\xi M_x L_\xi). \tag{10}$$

This real part of the trace thus provides the derivative $\partial_\xi p(x;\xi) = \mathrm{tr}(M_x \partial_\xi \rho_\xi)$ giving access to an expression of the classical Fisher information $F_c(\xi)$ of Eq. (5) as a function of the quantum ingredients determining the probability $p(x;\xi) = \mathrm{tr}(M_x \rho_\xi)$ of the classical data x, and reading

$$F_c(\xi) = \int_{\mathcal{X}} \frac{1}{\mathrm{tr}(\rho_\xi M_x)} [\mathrm{Re}\ \mathrm{tr}(\rho_\xi M_x L_\xi)]^2 dx. \tag{11}$$

For efficient estimation from measurement on the quantum state ρ_ξ, one has then the faculty to choose the measuring POVM $\{M_x, x \in \mathcal{X}\}$ in order to maximize the classical Fisher information $F_c(\xi)$ of Eq. (11). There exists in this respect an upper bound [4] which limits from above the classical Fisher information $F_c(\xi)$ of Eq. (11) for all feasible POVM $\{M_x, x \in \mathcal{X}\}$, established from the Cauchy-Schwarz inequality, as we now show.

Since $\mathrm{tr}(\rho_\xi M_x)$ is real positive, a useful factorization introduces the two operators $A^\dagger = \sqrt{\rho_\xi}\sqrt{M_x}/\sqrt{\mathrm{tr}(\rho_\xi M_x)}$ and $B = \sqrt{M_x} L_\xi \sqrt{\rho_\xi}$ so as to write for Eq. (11),

$$F_c(\xi) = \int_{\mathcal{X}} [\mathrm{Re}\ \mathrm{tr}(A^\dagger B)]^2 dx \leqslant \int_{\mathcal{X}} |\,\mathrm{tr}(A^\dagger B)|^2 dx. \tag{12}$$

The Hilbert-Schmidt inner product $\mathrm{tr}(A^\dagger B)$ of two operators satisfies the Cauchy-Schwarz inequality $|\,\mathrm{tr}(A^\dagger B)|^2 \leqslant \mathrm{tr}(A^\dagger A)\,\mathrm{tr}(B^\dagger B)$. Both $\sqrt{\rho_\xi}$ and $\sqrt{M_x}$ are Hermitian, since the positive operators ρ_ξ and M_x are. One then deduces $\mathrm{tr}(A^\dagger A) = 1$ and $\mathrm{tr}(B^\dagger B) = \mathrm{tr}(M_x L_\xi \rho_\xi L_\xi)$, leading from Eq. (12) to

$$F_c(\xi) \leqslant \int_{\mathcal{X}} |\,\mathrm{tr}(A^\dagger B)|^2 dx \tag{13}$$

$$\leqslant \int_{\mathcal{X}} \mathrm{tr}(M_x L_\xi \rho_\xi L_\xi) dx = \mathrm{tr}(L_\xi \rho_\xi L_\xi), \tag{14}$$

the last equality arising because the elements M_x of any valid POVM satisfy $\int_{\mathcal{X}} M_x dx = \mathbb{1}$ summing to the identity operator $\mathbb{1}$ on $\mathcal{H}_N$. Equation (14) leads then to the targeted inequality

$$F_c(\xi) \leqslant \mathrm{tr}(\rho_\xi L_\xi^2) \equiv F_q(\xi), \tag{15}$$

defining the quantum Fisher information $F_q(\xi) = \mathrm{tr}(\rho_\xi L_\xi^2) = \langle L_\xi^2 \rangle$ limiting the classical Fisher information $F_c(\xi)$, and bearing analogy with the classical counterpart $F_c(\xi) = E[V^2(x;\xi)]$ of Eq. (5), also reinforcing the interpretation of L_ξ as the quantum score. From Eq. (7) one also obtains the alternative expression $F_q(\xi) = \mathrm{tr}(L_\xi \partial_\xi \rho_\xi)$.

Then, for efficient estimation, one would like to find an optimal POVM $\{M_x, x \in \mathcal{X}\}$ saturating the inequality of Eq. (15), i.e., a POVM achieving $F_c(\xi) = F_q(\xi)$. This requires first to achieve equality in Eq. (12) or (13), and a necessary and sufficient condition for this is for $\mathrm{tr}(A^\dagger B)$ to be real. The second requirement is to achieve equality also in Eq. (14), and a necessary and sufficient condition for this is for A and B to be proportional. These two requirements are jointly met by the equivalent necessary and sufficient condition of A and B proportional through a real proportionality coefficient, that we write $A \propto_{\mathbb{R}} B$, so as to saturate the inequality of Eq. (15). In this way, equality in Eq. (15) is achieved if and only if each POVM element M_x realizes [4,5]

$$\sqrt{M_x} L_\xi \sqrt{\rho_\xi} \propto_{\mathbb{R}} \sqrt{M_x}\sqrt{\rho_\xi}. \tag{16}$$

A sufficient condition to satisfy Eq. (16) is to satisfy $\sqrt{M_x} L_\xi \propto_{\mathbb{R}} \sqrt{M_x}$, which becomes a sufficient and necessary condition when ρ_ξ, and hence $\sqrt{\rho_\xi}$, are full-rank operators. Moreover, we have to keep in mind that we are seeking a complete set of POVM elements $\{M_x, x \in \mathcal{X}\}$ that together sum to the identity operator $\mathbb{1}$ on $\mathcal{H}_N$. The known M_x solutions to Eq. (16) are usually expressed in terms of the eigen decomposition of L_ξ. A complete POVM $\{M_x, x \in \mathcal{X}\}$ with general significance known [4,5,10] to satisfy Eq. (16) is the von Neumann measurement made by the rank-one projectors on the eigenstates of L_ξ. In this case, L_ξ and $\sqrt{M_x}$ commute in Eq. (16), and the proportionality coefficient is the (real) eigenvalue in the corresponding eigendirection of (Hermitian) L_ξ. However such POVM, being characterized through L_ξ, generally bear explicit dependence on the unknown parameter ξ. In the context of the estimation problem, such solutions are therefore unacceptable (inaccessible) for a practically feasible quantum measurement. Also, only ξ-independent POVM and estimators are constrained by the inequalities of Eqs. (15) and (1), as it appeared in their derivations; if ξ-dependent solutions were allowed, the best estimator would always be $\widehat{\xi}(\vec{x}) = \xi$, which is clearly unacceptable since ξ is unknown. In general, there is no known solution, and *a fortiori* no known methodology, to obtain an optimal acceptable (ξ-independent) POVM $\{M_x, x \in \mathcal{X}\}$ that would satisfy Eq. (16) and maximize the Fisher information $F_c(\xi)$ at $F_q(\xi)$ in Eq. (15).

Even if one cannot generally define a ξ-independent optimal POVM saturating Eq. (15), one can, however, resort to an adaptive scheme with feedback to iteratively construct such an optimal POVM, whenever a series of identical copies of the state ρ_ξ are accessible for successive measurements, as proposed in [5]. One would start with a rough estimate of ξ from a nonoptimized initial POVM, then use this estimate of ξ to construct the optimal POVM based on the eigen decomposition of L_ξ at this current estimate of ξ, then redo an estimation with this current "optimal" POVM, and iterate over the series of copies accessible for ρ_ξ. In addition, even if it cannot be saturated with a ξ-independent optimal POVM, the inequality of Eq. (15) is nevertheless quite useful to assess the performance of any feasible POVM. For a quantum estimation problem, it is therefore quite desirable to explicitly evaluate the quantum Fisher information $F_q(\xi)$ to serve in the quantitative assessment of the performance via Eq. (15).

B. Computing the quantum Fisher information

For a quantum estimation problem, the explicit computation of the quantum Fisher information $F_q(\xi)$ defined in Eq. (15), requires an explicit characterization of the quantum score L_ξ defined by Eq. (7) for a given density operator ρ_ξ. This computation is now performed in this section, mainly following

[11], and will serve in the guideline for the estimation from a noisy qubit to be addressed in Sec. III.

Equation (7) for the linear operator L_ξ is known as the Lyapunov matrix equation. Its general solution can be expressed as

$$L_\xi = 2\int_0^\infty \exp(-\rho_\xi u)(\partial_\xi \rho_\xi)\exp(-\rho_\xi u)du. \quad (17)$$

From Eq. (17) it is indeed verified that

$$\begin{aligned}\frac{\rho_\xi L_\xi + L_\xi \rho_\xi}{2} &= \int_0^\infty [\rho_\xi \exp(-\rho_\xi u)(\partial_\xi \rho_\xi)\exp(-\rho_\xi u) \\ &\quad + \exp(-\rho_\xi u)(\partial_\xi \rho_\xi)\exp(-\rho_\xi u)\rho_\xi]du \\ &= \int_0^\infty -\frac{d}{du}[\exp(-\rho_\xi u)(\partial_\xi \rho_\xi)\exp(-\rho_\xi u)]du \\ &= -[\exp(-\rho_\xi u)(\partial_\xi \rho_\xi)\exp(-\rho_\xi u)]_{u=0}^\infty = \partial_\xi \rho_\xi,\end{aligned} \quad (18)$$

satisfying Eq. (7).

It is further possible to use the spectral decomposition of ρ_ξ in its orthonormal eigenbasis $\rho_\xi = \sum_{n=1}^N \lambda_n |\lambda_n\rangle\langle\lambda_n|$ to obtain $\exp(-\rho_\xi u) = \sum_{n=1}^N \exp(-\lambda_n u)|\lambda_n\rangle\langle\lambda_n|$, which after replacing in Eq. (17) and integrating in u yields

$$L_\xi = 2\sum_{m,n} \frac{\langle\lambda_m|\partial_\xi \rho_\xi|\lambda_n\rangle}{\lambda_m + \lambda_n}|\lambda_m\rangle\langle\lambda_n|, \quad (19)$$

where the sums include any nonzero eigenvalue. Both the eigenvalues λ_n and eigenvectors $|\lambda_n\rangle$ may depend on the parameter ξ, so that by differentiating the spectral decomposition of ρ_ξ one obtains

$$\partial_\xi \rho_\xi = \sum_{n=1}^N [(\partial_\xi \lambda_n)|\lambda_n\rangle\langle\lambda_n| + \lambda_n|\partial_\xi \lambda_n\rangle\langle\lambda_n| + \lambda_n|\lambda_n\rangle\langle\partial_\xi \lambda_n|]. \quad (20)$$

Also, since $\langle\lambda_n|\lambda_m\rangle = \delta_{nm}$, one has $0 = \partial_\xi\langle\lambda_n|\lambda_m\rangle = \langle\partial_\xi\lambda_n|\lambda_m\rangle + \langle\lambda_n|\partial_\xi\lambda_m\rangle$, so that $\langle\partial_\xi\lambda_n|\lambda_m\rangle = -\langle\lambda_n|\partial_\xi\lambda_m\rangle$; especially $\langle\partial_\xi\lambda_n|\lambda_n\rangle = -\langle\lambda_n|\partial_\xi\lambda_n\rangle = -\langle\partial_\xi\lambda_n|\lambda_n\rangle^*$ so $\langle\partial_\xi\lambda_n|\lambda_n\rangle$ is purely imaginary. This used with Eqs. (19) and (20) leads finally to

$$L_\xi = \sum_n \frac{\partial_\xi \lambda_n}{\lambda_n}|\lambda_n\rangle\langle\lambda_n| + 2\sum_{m,n}\frac{\lambda_n - \lambda_m}{\lambda_n + \lambda_m}\langle\lambda_m|\partial_\xi\lambda_n\rangle|\lambda_m\rangle\langle\lambda_n|. \quad (21)$$

For the quantum Fisher information $F_q(\xi) = \mathrm{tr}(\rho_\xi L_\xi^2) = \mathrm{tr}(L_\xi \partial_\xi \rho_\xi)$ of Eq. (15), one deduces from Eq. (17),

$$F_q(\xi) = 2\int_0^\infty \mathrm{tr}([(\partial_\xi \rho_\xi)\exp(-\rho_\xi u)]^2)du. \quad (22)$$

Or referring to the eigenbasis of ρ_ξ, from Eq. (19),

$$F_q(\xi) = 2\sum_{m,n}\frac{|\langle\lambda_m|\partial_\xi\rho_\xi|\lambda_n\rangle|^2}{\lambda_m + \lambda_n} \quad (23)$$

$$= \sum_n \frac{(\partial_\xi\lambda_n)^2}{\lambda_n} + 2\sum_{m,n}\frac{(\lambda_n - \lambda_m)^2}{\lambda_n + \lambda_m}|\langle\lambda_m|\partial_\xi\lambda_n\rangle|^2. \quad (24)$$

For the special but important case of a pure state $\rho_\xi = |\lambda\rangle\langle\lambda|$, having zero as eigenvalues, the expressions of Eqs. (17) and (19) for L_ξ no longer apply. Instead, one has $\rho_\xi^2 = \rho_\xi$ and therefore $\rho_\xi \partial_\xi \rho_\xi + (\partial_\xi \rho_\xi)\rho_\xi = \partial_\xi \rho_\xi$, and by comparing with Eq. (7) one obtains the solution

$$L_\xi = 2\partial_\xi \rho_\xi = 2(|\partial_\xi\lambda\rangle\langle\lambda| + |\lambda\rangle\langle\partial_\xi\lambda|). \quad (25)$$

The quantum Fisher information $F_q(\xi) = \mathrm{tr}(\rho_\xi L_\xi^2) = \mathrm{tr}(L_\xi \partial_\xi \rho_\xi)$ follows for a pure state as

$$F_q(\xi) = 4(\langle\partial_\xi\lambda|\partial_\xi\lambda\rangle + \langle\partial_\xi\lambda|\lambda\rangle^2). \quad (26)$$

The characterizations of the quantum score L_ξ and quantum Fisher information $F_q(\xi)$ of this section, involving the eigen decomposition of the density operator ρ_ξ, will now serve for the noisy qubit.

III. ESTIMATION ON A NOISY QUBIT

A. Quantum Fisher information for the qubit

In this section we further specify the characterization of Sec. II for the qubit. For this purpose, we specifically exploit the Bloch representation accessible for the qubit. We show that this representation makes possible explicit general expressions on which we shall rely for the analysis of estimation from the noisy qubit. We consider the quantum system as a qubit in $\mathcal{H}_2$, whose state can be represented in the general Bloch sphere representation [12] as

$$\rho_\xi = \frac{1}{2}[\mathbb{1} + \vec{r}(\xi)\vec{\sigma}], \quad (27)$$

with the real three-dimensional Bloch vector $\vec{r} = \vec{r}(\xi)$ carrying the dependence with the unknown parameter ξ, and $\vec{\sigma}$ a vector assembling the three 2×2 (traceless Hermitian unitary) Pauli matrices $[\sigma_x,\sigma_y,\sigma_z] = \vec{\sigma}$. The purity $\mathrm{tr}(\rho_\xi^2) = (1 + \|\vec{r}\|^2)/2$ is controlled by the Euclidean norm $\|\vec{r}\|$, with $\|\vec{r}\| < 1$ for a mixed state and $\|\vec{r}\| = 1$ for a pure state. The qubit state ρ_ξ of Eq. (27) has eigenvalues $\lambda_\pm = (1 \pm \|\vec{r}\|)/2$ and normalized eigenvectors $|\lambda_\pm\rangle$, the two projectors on these eigenvectors having the Bloch representation $|\lambda_\pm\rangle\langle\lambda_\pm| = (\mathbb{1} \pm \vec{r}\,\vec{\sigma}/\|\vec{r}\|)/2$. The state of Eq. (27) differentiates as $\partial_\xi \rho_\xi = \partial_\xi \vec{r}\,\vec{\sigma}/2$.

Equation (19) is also

$$L_\xi = 2\sum_{m,n}\frac{1}{\lambda_m + \lambda_n}|\lambda_m\rangle\langle\lambda_m|\partial_\xi\rho_\xi|\lambda_n\rangle\langle\lambda_n|, \quad (28)$$

for the case of a mixed qubit state ρ_ξ, and can be evaluated in Bloch representation. For example, among the four terms in the sum of Eq. (28) for the qubit, one is

$$\frac{1}{\lambda_- + \lambda_+}|\lambda_-\rangle\langle\lambda_-|\partial_\xi\rho_\xi|\lambda_+\rangle\langle\lambda_+| = \frac{\mathbb{1} - \vec{r}\,\vec{\sigma}/\|\vec{r}\|}{2}\,\frac{\partial_\xi\vec{r}\,\vec{\sigma}}{2}\,\frac{\mathbb{1} + \vec{r}\,\vec{\sigma}/\|\vec{r}\|}{2}. \quad (29)$$

To evaluate expressions such as Eq. (29) involving products of noncommuting operators on $\mathcal{H}_2$, we use the identities $(\vec{a}\,\vec{\sigma})(\vec{b}\,\vec{\sigma}) = (\vec{a}\,\vec{b})\mathbb{1} + i(\vec{a}\times\vec{b})\vec{\sigma}$, and in $\mathbb{R}^3$ for the double cross product $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\,\vec{c})\vec{b} - (\vec{b}\,\vec{c})\vec{a}$. Handling

Eq. (28) in this way for the qubit, we finally obtain the quantum score in Bloch representation

$$L_\xi = -\frac{\vec{r}\,\partial_\xi \vec{r}}{1-\|\vec{r}\,\|^2}\mathbb{1} + \left(\frac{\vec{r}\,\partial_\xi \vec{r}}{1-\|\vec{r}\,\|^2}\vec{r} + \partial_\xi \vec{r}\right)\vec{\sigma}. \tag{30}$$

With a comparable approach in Bloch representation, the quantum Fisher information $F_q(\xi) = \mathrm{tr}(\rho_\xi L_\xi^2) = \mathrm{tr}(L_\xi \partial_\xi \rho_\xi)$ follows as

$$F_q(\xi) = \frac{[\vec{r}(\xi)\,\partial_\xi \vec{r}(\xi)]^2}{1-\|\vec{r}(\xi)\|^2} + [\partial_\xi \vec{r}(\xi)]^2. \tag{31}$$

For the case of a pure qubit state ρ_ξ, differentiating $\|\vec{r}\,\|^2 = \vec{r}\,\vec{r} = 1$ gives $2\vec{r}\,\partial_\xi \vec{r} = 0$, whence $\vec{r}\,\partial_\xi \vec{r} = 0$ indicating $\vec{r}$ orthogonal to $\partial_\xi \vec{r}$. The quantum score $L_\xi = 2\partial_\xi \rho_\xi$ of Eq. (25) in Bloch representation is

$$L_\xi = \partial_\xi \vec{r}\,\vec{\sigma}, \tag{32}$$

and the quantum Fisher information $F_q(\xi) = \mathrm{tr}(L_\xi \partial_\xi \rho_\xi)$ of Eq. (26) is

$$F_q(\xi) = [\partial_\xi \vec{r}(\xi)]^2. \tag{33}$$

Equations (31) and (33) offer a general characterization of the quantum Fisher information $F_q(\xi)$ characterizing the performance in estimating any scalar parameter ξ attached to a qubit state ρ_ξ. These general equations for the qubit expressed in Bloch representation are new here, and stand as a useful basis for performance analysis in qubit state estimation.

A general property which can be deduced is that the quantum Fisher information $F_q(\xi)$ always increases with the purity of the measured qubit state ρ_ξ, for any dependence of ρ_ξ on the parameter ξ, as we now demonstrate. For the Bloch vector $\vec{r}(\xi)$ we separate the magnitude $r(\xi)$ and direction, which both can depend on ξ, by writing $\vec{r}(\xi) = r(\xi)\vec{r}^{\,\mathrm{un}}(\xi)$ with $r(\xi) = \|\vec{r}(\xi)\| \in [0,1]$ and $\vec{r}^{\,\mathrm{un}}(\xi)$ the unitary vector fixing the direction of $\vec{r}(\xi)$ in $\mathbb{R}^3$. Then $\partial_\xi \vec{r}(\xi) = (\partial_\xi r)\vec{r}^{\,\mathrm{un}} + r\partial_\xi \vec{r}^{\,\mathrm{un}}$. Since $\|\vec{r}^{\,\mathrm{un}}\|^2 = (\vec{r}^{\,\mathrm{un}})^2 = 1$, it follows $\partial_\xi(\vec{r}^{\,\mathrm{un}})^2 = 2\vec{r}^{\,\mathrm{un}}\partial_\xi \vec{r}^{\,\mathrm{un}} = 0$, so that $\vec{r}\partial_\xi \vec{r} = r\partial_\xi r$ and $(\partial_\xi \vec{r})^2 = (\partial_\xi r)^2 + r^2(\partial_\xi \vec{r}^{\,\mathrm{un}})^2$, to yield in Eq. (31)

$$F_q(\xi) = \frac{[\partial_\xi r(\xi)]^2}{1-r^2(\xi)} + r^2(\xi)[\partial_\xi \vec{r}^{\,\mathrm{un}}(\xi)]^2. \tag{34}$$

From Eq. (34), we deduce that $F_q(\xi)$ is an increasing function of the magnitude r when r increases over $[0,1]$. So if one seeks to maximize the quantum Fisher information $F_q(\xi)$, whenever feasible it is always favorable to increase $r = \|\vec{r}(\xi)\|$ as much as possible, i.e., increase the purity of the measured qubit state ρ_ξ. This is true for any dependence of ρ_ξ on the parameter ξ. In this respect, qubit states with higher purity are always more efficient for estimation.

As a specificity here, to explore more realistic conditions, we want to analyze the impact of a quantum noise affecting the qubit which is measured for estimation.

B. Noise on the qubit

A quantum noise acting on a qubit affects its state ρ in a way which can be generally represented by a completely positive linear trace-preserving map of the form [12,13]

$$\rho \longrightarrow \mathcal{N}(\rho) = \sum_\ell \Lambda_\ell \rho \Lambda_\ell^\dagger, \tag{35}$$

with the Kraus operators Λ_ℓ (which need not be more than four for the qubit) satisfying $\sum_\ell \Lambda_\ell^\dagger \Lambda_\ell = \mathbb{1}$. Equivalently, Eq. (35) realizes a transformation of the Bloch vector $\vec{r}$ of ρ under the general form [12,14]

$$\vec{r} \to A\vec{r} + \vec{c}, \tag{36}$$

where A is a 3×3 real matrix, and $\vec{c}$ a real vector in $\mathbb{R}^3$, mapping the Bloch ball onto itself. By the polar decomposition [12], one can write $A = US$, where U is a real unitary matrix, and S a real symmetric matrix. The matrix S always has, associated with three real eigenvalues (s_1,s_2,s_3), three eigenvectors $\{\vec{s}_1,\vec{s}_2,\vec{s}_3\}$ forming an orthonormal basis of $\mathbb{R}^3$. The transformation of the Bloch vector $\vec{r}$ in Eq. (36) is thus a deformation by S along the axes $\{\vec{s}_1,\vec{s}_2,\vec{s}_3\}$ followed by an isometry U and a translation by $\vec{c}$.

We consider a noise-free qubit state $\rho_1(\xi)$, with Bloch vector $\vec{r}_1(\xi)$, to introduce the dependence on the unknown parameter ξ. The state $\rho_1(\xi)$ is not directly accessible to measurement, but only after alteration by some quantum noise $\mathcal{N}(\cdot)$. The noisy qubit state accessible to measurement is thus $\mathcal{N}[\rho_1(\xi)] = \rho_\xi$ with Bloch vector

$$\vec{r}(\xi) = A\vec{r}_1(\xi) + \vec{c}, \tag{37}$$

where $(A,\vec{c})$ are given as a characterization of the quantum noise affecting the qubit, and

$$\partial_\xi \vec{r}(\xi) = A\partial_\xi \vec{r}_1(\xi). \tag{38}$$

Equations (37) and (38) used in Eq. (31) or (33) enable one to study the impact of any quantum noise defined by $(A,\vec{c})$, on the performance expressed by the quantum Fisher information $F_q(\xi)$, upon estimation from a noisy qubit.

One can, for instance, predict that, for a Bloch vector $\vec{r}_1(\xi)$ with a given fixed direction $\vec{r}_1^{\,\mathrm{un}}$, increasing the purity of the state ρ_1 by increasing $\|\vec{r}_1(\xi)\|$ may not always be favorable to enhance the quantum Fisher information $F_q(\xi)$. There may exist geometric configurations, when $\vec{r}_1(\xi)$ and $\vec{c}$ form an obtuse angle in $\mathbb{R}^3$, where an increase of the purity $\|\vec{r}_1(\xi)\|$ of the nonmeasurable noise-free state ρ_1 induces in Eq. (37) a decrease of the purity $\|\vec{r}(\xi)\|$ of the measured noisy state ρ_ξ, and hence a decrease of the quantum Fisher information $F_q(\xi)$. Such types of unexpected counterintuitive behaviors might be compared to stochastic resonance or useful-noise effects occurring in classical estimation [15–17] or quantum information processing [18–20].

For further analysis of the estimation process, we now introduce a general family of parametric quantum state ρ_ξ relevant to the qubit and providing an explicit specification of the estimation task.

C. Qubit transformation

An arbitrary unitary transformation on the qubit (an arbitrary single-qubit quantum gate) can be expressed [12] in the form $U = \exp(i\alpha)\exp(-i\xi\vec{n}\,\vec{\sigma}/2)$, and since the overall scalar phase α is unimportant here, we consider the general unitary transformation on the qubit as

$$U_\xi = \exp\left(-i\frac{\xi}{2}\vec{n}\,\vec{\sigma}\right), \tag{39}$$

where $\vec{n} = [n_x, n_y, n_z]^\top$ is a real unit vector of $\mathbb{R}^3$ and ξ introduces the unknown phase parameter to be estimated. The transformation of Eq. (39) acts on an input qubit state ρ_0 with Bloch vector $\vec{r}_0$, which is considered as the probing state or probe that will serve as a support to the estimation process. The probe ρ_0 is transformed by U_ξ of Eq. (39) so as to yield the quantum state $\rho_1(\xi) = U_\xi \rho_0 U_\xi^\dagger$ (especially, the presence of a scalar phase α in U would have had no effect on ρ_1). As a result, the transformed qubit state $\rho_1(\xi)$ is characterized by the Bloch vector $\vec{r}_1(\xi) = R_\xi \vec{r}_0$, where R_ξ is the rotation in $\mathbb{R}^3$ with axis $\vec{n}$ and angle ξ, expressible in matrix form as [21]

$$R_\xi = \exp(\xi N) = I + \sin(\xi)N + [1-\cos(\xi)]N^2, \tag{40}$$

with I the 3×3 identity matrix on $\mathbb{R}^3$ and the matrix

$$N = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}. \tag{41}$$

The qubit transformation of Eqs. (40) and (41) generalizes to an arbitrary rotation axis $\vec{n}$ the transformation considered in [6,7] which was a rotation around the Oz axis. One deduces the derivative $\partial_\xi \vec{r}_1(\xi) = (\partial_\xi R_\xi)\vec{r}_0$, with the matrix

$$\partial_\xi R_\xi = N\,\exp(\xi N) = \cos(\xi)N + \sin(\xi)N^2. \tag{42}$$

For the noisy qubit state $\rho_\xi = \mathcal{N}[\rho_1(\xi)]$ accessible to measurement, come from Eqs. (37) and (38), the Bloch vector

$$\vec{r}(\xi) = A\vec{r}_1(\xi) + \vec{c} = AR_\xi\vec{r}_0 + \vec{c} \tag{43}$$

and

$$\partial_\xi\vec{r}(\xi) = A\partial_\xi\vec{r}_1(\xi) = A(\partial_\xi R_\xi)\vec{r}_0. \tag{44}$$

From Eqs. (43) and (44), one then gets access to a complete characterization of the quantum Fisher information $F_q(\xi)$ of Eq. (31), for estimation of the phase angle ξ in the rotation of the qubit state around any given axis $\vec{n}$ in the presence of an arbitrary quantum noise defined by $(A,\vec{c})$. It is especially possible to study the impact of the input probe ρ_0 defined by Bloch vector $\vec{r}_0$, on the performance expressed by $F_q(\xi)$, upon estimation from a noisy qubit affected by any given quantum noise.

To continue on this path, a general property which can be deduced is that maximization of the quantum Fisher information $F_q(\xi)$ in Eq. (31) necessarily requires an input probe ρ_0 satisfying $\|\vec{r}_0\| = 1$, i.e., a pure probe state ρ_0. This follows from Eqs. (43) and (44) acting in $F_q(\xi)$ of Eq. (31). Especially, in order to maximize $F_q(\xi)$ in Eq. (31), it is always favorable to act on $\vec{r}_0$ to maximize the magnitude of $\vec{r}$ in Eq. (43). The favorable configuration to maximize the magnitude of $\vec{r}$ and $F_q(\xi)$ occurs when $AR_\xi\vec{r}_0$ and $\vec{c}$ in Eq. (43) have a positive inner product or equivalently form an acute angle in $\mathbb{R}^3$, otherwise reversing the input probe with $\vec{r}_0 \to -\vec{r}_0$ realizes this favorable configuration. Then with an $\vec{r}_0$ realizing such favorable configuration in $\mathbb{R}^3$, pushing to $\|\vec{r}_0\| \to 1$ always increases the magnitude of both $\vec{r}$ in Eq. (43) and $\partial_\xi\vec{r}$ in Eq. (44) and also decreases the positive denominator in Eq. (31), raising in this way $F_q(\xi)$ of Eq. (31). The same conclusion of a pure ρ_0 applies for maximization of $F_q(\xi)$ in Eq. (33) with $\vec{r}$ from a pure state ρ_ξ, although the noise usually causes a mixed state ρ_ξ. A pure probe state ρ_0 is thus generally required for maximizing the Fisher information $F_q(\xi)$, for any quantum noise acting on the qubit.

When the increase of the purity $\|\vec{r}_0\|$ of the input probe ρ_0 increases the quantum Fisher information $F_q(\xi)$, at the same time it increases in Eq. (43) the purity $\|\vec{r}\,\|$ of the measured quantum state ρ_ξ. So this is another demonstration that the quantum Fisher information $F_q(\xi)$ always increases with the purity $\|\vec{r}\,\|$ of the measured quantum state ρ_ξ. This is obtained here for a parametric dependence specified by the phase ξ of the qubit, and when, in the presence of a specified quantum noise $(A,\vec{c})$, the purity $\|\vec{r}\,\|$ of the measured quantum state ρ_ξ is varied through the purity $\|\vec{r}_0\|$ of the input probe ρ_0.

To further investigate the performance in estimation and the conditions for maximization of the quantum Fisher information $F_q(\xi)$ in the presence of noise, we shall now adapt the parametrization of the Bloch vectors to enable a concise vision on relevant factors of influence.

D. In the frame of the rotation axis

A convenient parametrization in $\mathbb{R}^3$ refers to an orthonormal basis made of the rotation axis $\vec{n} = [n_x, n_y, n_z]^\top$, of an orthogonal unit vector that we choose in the plane (Ox, Oy) as $\vec{n}_\perp = [n_y, -n_x, 0]^\top/\sqrt{1-n_z^2}$, and of a third orthogonal unit vector $\vec{n}\,'_\perp = \vec{n}\times\vec{n}_\perp = [n_x n_z, n_y n_z, n_z^2 - 1]^\top/\sqrt{1-n_z^2}$. In the orthonormal basis $\{\vec{n}, \vec{n}_\perp, \vec{n}\,'_\perp\}$ of $\mathbb{R}^3$, any input probe state ρ_0 has a Bloch vector $\vec{r}_0$ whose direction in $\mathbb{R}^3$ is defined by the two angles (θ_0, φ_0): one coelevation angle $\theta_0 \in [0,\pi]$ between $\vec{r}_0$ and $\vec{n}$; and in the plane $(\vec{n}_\perp, \vec{n}\,'_\perp)$, one azimuth angle $\varphi_0 \in [0, 2\pi)$. With this parametrization, the rotated state $\rho_1(\xi) = U_\xi\rho_0 U_\xi^\dagger$ has a Bloch vector $\vec{r}_1(\xi) = R_\xi\vec{r}_0$ whose direction is defined in the orthonormal basis $\{\vec{n}, \vec{n}_\perp, \vec{n}\,'_\perp\}$ by the two angles $(\theta_1 = \theta_0, \varphi_1 = \varphi_0 + \xi)$. Also, for the derivative $\partial_\xi\vec{r}_1(\xi) = (\partial_\xi R_\xi)\vec{r}_0$ present in Eq. (44), since in the basis $\{\vec{n}, \vec{n}_\perp, \vec{n}\,'_\perp\}$ of $\mathbb{R}^3$ one has for the rotation of axis $\vec{n}$ and angle ξ the matrix

$$R_\xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\xi) & -\sin(\xi) \\ 0 & \sin(\xi) & \cos(\xi) \end{bmatrix}, \tag{45}$$

then

$$\partial_\xi R_\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(\xi) & -\cos(\xi) \\ 0 & \cos(\xi) & -\sin(\xi) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\xi+\pi/2) & -\sin(\xi+\pi/2) \\ 0 & \sin(\xi+\pi/2) & \cos(\xi+\pi/2) \end{bmatrix}, \tag{46}$$

so that $\partial_\xi\vec{r}_1$ is just $\vec{r}_1$ with an extra angle $\varphi_1 \to \varphi_1 + \pi/2$ and the component along $\vec{n}$ set to zero. Thus $\partial_\xi\vec{r}_1$ is orthogonal to $\vec{r}_1$ and $0 \leqslant \|\partial_\xi\vec{r}_1\| \leqslant \|\vec{r}_1\| \leqslant 1$.

The quantum noise then acts on $\vec{r}_1$ and $\partial_\xi\vec{r}_1$ to produce their noisy versions $\vec{r}$ and $\partial_\xi\vec{r}$ of Eqs. (43) and (44). As a result, the quantum Fisher information of Eq. (31) becomes

$$F_q(\xi) = \frac{[(A\vec{r}_1 + \vec{c}\,)A\partial_\xi\vec{r}_1]^2}{1-(A\vec{r}_1+\vec{c}\,)^2} + (A\partial_\xi\vec{r}_1)^2. \qquad (47)$$

Especially, a pure input probe ρ_0 which is required to maximize the quantum Fisher information $F_q(\xi)$, leads to a pure rotated state ρ_1 with $\|\vec{r}_1\| = 1$ which in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$ has components

$$\vec{r}_1 = [\cos(\theta_1), \sin(\theta_1)\cos(\varphi_1), \sin(\theta_1)\sin(\varphi_1)]^\top \qquad (48)$$

and

$$\partial_\xi\vec{r}_1 = [0, -\sin(\theta_1)\sin(\varphi_1), \sin(\theta_1)\cos(\varphi_1)]^\top. \qquad (49)$$

If there were no noise, i.e., $A = I$ and $\vec{c} = \vec{0}$ in Eq. (47), the quantum Fisher information based on $\vec{r}_1 \perp \partial_\xi\vec{r}_1$ yielding $\vec{r}_1\partial_\xi\vec{r}_1 = 0$ would amount to $F_q(\xi) = [\partial_\xi\vec{r}_1(\xi)]^2 \leqslant 1$, and this is true both for a pure or a mixed state ρ_1. Moreover, reaching the maximum $F_q(\xi) = 1$ would require $\|\partial_\xi\vec{r}_1\| = 1$, which is obtained for a pure state ρ_1 with $\|\vec{r}_1\| = 1$ and $\vec{r}_1$ with no component along $\vec{n}$, i.e., $\vec{r}_1 \perp \vec{n}$, which is accomplished by a pure input probe ρ_0 with $\theta_0 = \pi/2$ also orthogonal to $\vec{n}$. There is thus an intrinsic limit $F_q(\xi) \leqslant 1$ on the quantum Fisher information for qubit phase estimation. With an active quantum noise, $F_q(\xi) = 1$ can be preserved only with a pure ρ_1, in special conditions with $\vec{r}_1 \perp \vec{n}$ and specific noise parameters $(A,\vec{c})$ as we will see in the sequel.

With the parametrization in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$, we gain the advantage that to study the impact of a pure input probe ρ_0 of Bloch vector $\vec{r}_0$ on the quantum Fisher information $F_q(\xi)$ of Eq. (31), it is feasible and enough to study F_q as a function of (θ_1,φ_1) only. This is accomplished essentially through Eq. (47), with Eqs. (48) and (49) to explore the conditions for maximization of $F_q(\theta_1,\varphi_1)$ occurring with a pure input probe ρ_0 equivalent to a pure rotated state ρ_1. For instance, a maximum of $F_q(\theta_1,\varphi_1)$ observed at $(\theta_1^{\rm opt},\varphi_1^{\rm opt})$ reveals that the optimal input probe $\rho_0^{\rm opt}$ is the pure state defined in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$ by $(\theta_0^{\rm opt} = \theta_1^{\rm opt},\varphi_0^{\rm opt} = \varphi_1^{\rm opt} - \xi)$, for estimation of any given phase ξ. In this way, referring to the parametrization (θ_1,φ_1) and (θ_0,φ_0) relative to the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$, enables a general characterization of the maximum Fisher information $F_q(\xi)$ as a function of the input probe ρ_0 in the presence of any phase ξ.

E. Evaluation with different noise models

For estimation of the phase ξ from a noisy qubit, we now consider different relevant quantum noise models, and analyze their impact on the estimation performance assessed by the quantum Fisher information $F_q(\xi)$ from Eq. (47). In each case also, we characterize the optimal (pure) input probe state ρ_0 maximizing the quantum Fisher information $F_q(\xi)$.

1. Pauli noises

A significant class of noise processes relevant to the qubit is the class of Pauli noises [13]. A Pauli noise acts through random applications of the four Pauli operators $\{\sigma_0 \equiv \mathbb{1},\sigma_x,\sigma_y,\sigma_z\}$ which form an orthogonal basis for operators on $\mathcal{H}_2$. In the Kraus representation of Eq. (35), a Pauli noise implements the quantum operation

$$\rho \longrightarrow \mathcal{N}(\rho) = \sum_{\ell=0,x,y,z} p_\ell\sigma_\ell\rho\sigma_\ell^\dagger, \qquad (50)$$

with the $\{p_\ell\}$ a probability distribution, leading for Eq. (36) to

$$\vec{r} \longrightarrow A\vec{r} = \begin{bmatrix} a_{xx} & 0 & 0 \\ 0 & a_{yy} & 0 \\ 0 & 0 & a_{zz} \end{bmatrix}\vec{r}, \qquad (51)$$

with the real scalar coefficients

$$a_{xx} = p_0 + p_x - p_y - p_z, \qquad (52)$$

$$a_{yy} = p_0 - p_x + p_y - p_z, \qquad (53)$$

$$a_{zz} = p_0 - p_x - p_y + p_z, \qquad (54)$$

referring to the original frame (Ox,Oy,Oz) of $\mathbb{R}^3$.

An important instance of a Pauli noise is the depolarizing noise [12,13] for which $a_{xx} = a_{yy} = a_{zz}$. With this maximally symmetric noise, the configuration $\vec{r}_1 \perp \partial_\xi\vec{r}_1$ which always holds, entails $\vec{r} = A\vec{r}_1 \perp \partial_\xi\vec{r} = A\partial_\xi\vec{r}_1$. The depolarizing noise thus preserves the orthogonality of $\vec{r}_1$ and $\partial_\xi\vec{r}_1$, which persists in their noisy versions $\vec{r}$ and $\partial_\xi\vec{r}$. As a result, the quantum Fisher information of Eq. (31) or (47) reduces to $F_q(\xi) = [\partial_\xi\vec{r}(\xi)]^2 = [A\partial_\xi\vec{r}_1(\xi)]^2 = a_{xx}^2\|\partial_\xi\vec{r}_1\|^2$, which for the pure state ρ_1 of Eqs. (48) and (49) is $F_q(\xi) = a_{xx}^2\sin^2(\theta_1) = a_{xx}^2\sin^2(\theta_0)$. A visualization of the corresponding quantum Fisher information $F_q(\xi)$ is provided by Fig. 1 in the plane (θ_1,φ_1).

In the conditions similar to Fig. 1, the variation $F_q(\xi) = a_{xx}^2\sin^2(\theta_1) = a_{xx}^2\sin^2(\theta_0)$ indicates that with the highly symmetric depolarizing noise, the quantum Fisher information $F_q(\xi)$ is invariant with the rotated angle $\varphi_1 = \varphi_0 + \xi$, i.e., invariant with the phase ξ to be estimated on the noisy qubit. However, $F_q(\xi)$ varies with the orientation $\vec{n}$ of the qubit rotation, yet only through the coelevation $\theta_1 = \theta_0$ controllable via the input probe. This indicates that optimization of the Fisher information at its maximum $F_q(\xi) = a_{xx}^2$ can always be

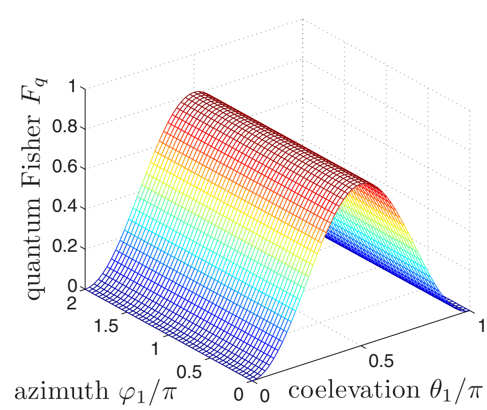

FIG. 1. (Color online) Quantum Fisher information $F_q(\theta_1,\varphi_1)$ from Eq. (47) in the plane of the two angles (θ_1,φ_1) defining in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$ the rotated Bloch vector $\vec{r}_1(\xi) = R_\xi\vec{r}_0$ related to the pure input probe ρ_0 by $(\theta_1 = \theta_0,\varphi_1 = \varphi_0 + \xi)$. The qubit is affected by the depolarizing noise from Eq. (51) with $a_{xx} = a_{yy} = a_{zz} = 0.9$, yielding the uniform maximum $F_q(\theta_1,\varphi_1) = a_{xx}^2 = 0.81$ for $\theta_1^{\rm opt} = \pi/2, \forall\varphi_1$.

achieved by selecting the pure input probe ρ_0 to ensure $\theta_0 = \pi/2$, i.e., a ξ-independent pure input probe ρ_0 characterized by a Bloch vector $\vec{r}_0$ orthogonal to the rotation axis $\vec{n}$ having arbitrary direction.

For Pauli noises less symmetric than the depolarizing noise, the orthogonality of $\vec{r}_1$ and $\partial_\xi \vec{r}_1$ is likely to disappear in their noisy versions $\vec{r} = A\vec{r}_1$ and $\partial_\xi \vec{r} = A\partial_\xi \vec{r}_1$. The variation of the Fisher information $F_q(\xi)$ can be expected to depend also on the angle φ_1 and on the orientation in $\mathbb{R}^3$ of the rotation axis $\vec{n}$ relative to the eigenaxes (Ox,Oy,Oz) of the Pauli noise. However, for any Pauli noise, when the rotation axis $\vec{n}$ is parallel to one of the eigenaxes (Ox,Oy,Oz) of the noise matrix A, the rotated vector $\partial_\xi \vec{r}_1$ in Eq. (49) which is always in the plane orthogonal to $\vec{n}$, has its noisy version $A\partial_\xi \vec{r}_1 = \partial_\xi \vec{r}$ which remains in this plane. Choosing $\vec{r}_1$ of Eq. (48) also in the plane orthogonal to $\vec{n}$, i.e., $\theta_1 = \pi/2$, yields $A\vec{r}_1 = \vec{r}$ which also remains in this plane. This condition $\theta_1 = \pi/2$ locates the maximum of $F_q(\theta_1,\varphi_1)$. This can be proved by differentiating $F_q(\theta_1,\varphi_1)$ from Eq. (47) with respect to θ_1, using from Eqs. (48) and (49),

$$\frac{\partial}{\partial \theta_1}\vec{r}_1 = [-\sin(\theta_1), \cos(\theta_1)\cos(\varphi_1), \cos(\theta_1)\sin(\varphi_1)]^\top \quad (55)$$

and

$$\frac{\partial}{\partial \theta_1}\partial_\xi \vec{r}_1 = [0, -\cos(\theta_1)\sin(\varphi_1), \cos(\theta_1)\cos(\varphi_1)]^\top, \quad (56)$$

and their evaluation at $\theta_1 = \pi/2$ yielding $\partial(\partial_\xi \vec{r}_1)/\partial\theta_1 = \vec{0}$ and $\partial \vec{r}_1/\partial\theta_1 = [-1,0,0]^\top \perp \vec{r}_1$, to show that $\partial F_q(\theta_1,\varphi_1)/\partial\theta_1$ always vanishes in $\theta_1 = \pi/2$, for any φ_1. Furthermore, in Eq. (47), one chooses $\varphi_1 = 0$ or $\pi/2$ so as to place $\partial_\xi \vec{r}_1$ in the direction of the dominant eigendirection of A in the plane orthogonal to $\vec{n}$. This choice minimizes the compression of $\partial_\xi \vec{r}_1$ by A and completes the maximization of $F_q(\theta_1,\varphi_1)$ at the level of the associated dominant squared eigenvalue. For instance, with a rotation axis $\vec{n}$ parallel to Oz, one obtains from Eq. (47)

$$F_q(\theta_1 = \pi/2,\varphi_1) = \frac{(1-a_{xx}^2)a_{yy}^2 + (a_{xx}^2 - a_{yy}^2)\sin^2(\varphi_1)}{1 - a_{xx}^2 + (a_{xx}^2 - a_{yy}^2)\sin^2(\varphi_1)}, \quad (57)$$

reaching the maximum a_{yy}^2 in $\varphi_1^{\rm opt} = 0$ when $a_{yy} > a_{xx}$, or the maximum a_{xx}^2 in $\varphi_1^{\rm opt} = \pi/2$ when $a_{xx} \geqslant a_{yy}$. With the rotation axis $\vec{n}$ parallel to one of the eigenaxes of any Pauli noise, one can thus expect a maximum of the quantum Fisher information $F_q(\theta_1,\varphi_1)$ at $\theta_1^{\rm opt} = \theta_0^{\rm opt} = \pi/2$, i.e., with an optimal input probe ρ_0 orthogonal to $\vec{n}$, and at $\varphi_1^{\rm opt} = 0$ or $\varphi_1^{\rm opt} = \pi/2$ fixing the optimal input probe with an azimuth $\varphi_0^{\rm opt} = \varphi_1^{\rm opt} - \xi$ dependent on the phase ξ.

Another Pauli noise important to the qubit is the bit-flip noise [12,13], characterized in Eq. (50) by $p_y = p_z = 0$, implementing a random application of σ_x to the qubit with probability p_x or no change with probability $1 - p_x$. This leads in Eq. (51) to $a_{xx} = 1$ and $a_{yy} = a_{zz} = 1 - 2p_x$. The reduced symmetry of the bit-flip noise entails, as expected, that the orthogonality of $\vec{r}_1$ and $\partial_\xi \vec{r}_1$ is not generally preserved in their noisy versions $\vec{r} = A\vec{r}_1$ and $\partial_\xi \vec{r} = A\partial_\xi \vec{r}_1$. The Fisher information $F_q(\xi)$ in Eq. (47) then usually depends on φ_1 and θ_1, and on the orientation in $\mathbb{R}^3$ of the rotation axis $\vec{n}$ relative to the eigenaxes (Ox,Oy,Oz) of the Pauli noise. A rotation axis $\vec{n}$ parallel to Oz is considered in Fig. 2(a) for illustration. Figure 2(a) shows that with the bit-flip noise, the quantum Fisher information $F_q(\theta_1,\varphi_1)$ from Eq. (47) bears explicit dependence on both angles $\varphi_1 = \varphi_0 + \xi$ and $\theta_1 = \theta_0$. With $a_{xx} = 1$ of the bit-flip noise, however, there is from Eq. (57) a uniform maximum at $F_q(\theta_1,\varphi_1) = 1$ for $\theta_1^{\rm opt} = \pi/2$, $\forall \varphi_1$. For an arbitrary φ_1, there is an effective compression by the noise of both $\vec{r}_1$ and $\partial_\xi \vec{r}_1$, reducing the magnitude of their noisy versions $\vec{r} = A\vec{r}_1$ and $\partial_\xi \vec{r} = A\partial_\xi \vec{r}_1$; yet, as revealed by the

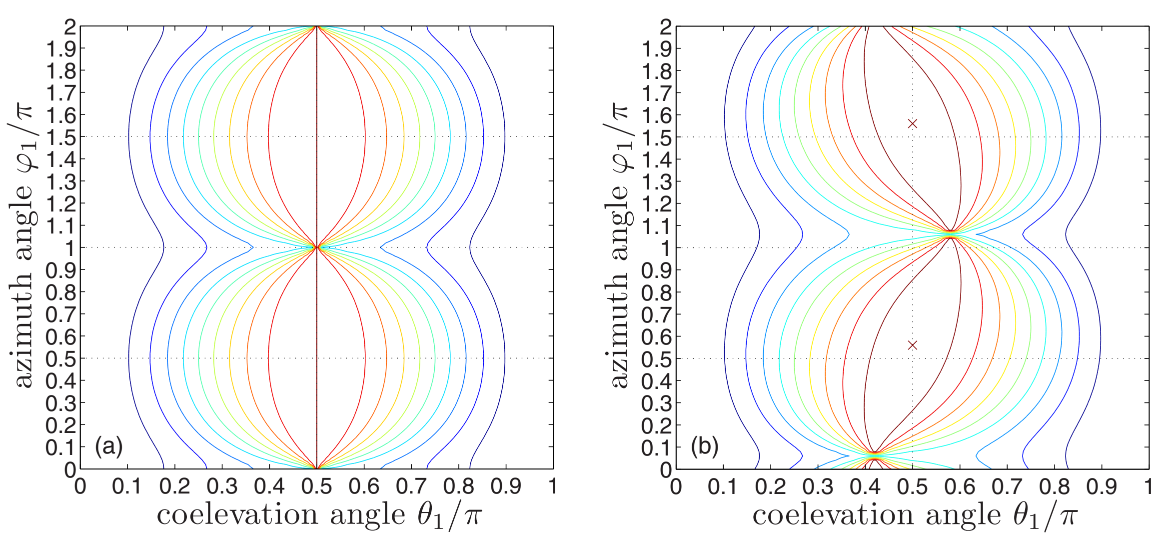

FIG. 2. (Color online) The level curves of the quantum Fisher information $F_q(\theta_1,\varphi_1)$ from Eq. (47) in the plane of the two angles (θ_1,φ_1) defining in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}'_\perp\}$ the rotated Bloch vector $\vec{r}_1(\xi) = R_\xi \vec{r}_0$ related to the pure input probe ρ_0 by ($\theta_1 = \theta_0$, $\varphi_1 = \varphi_0 + \xi$). The qubit is affected by a bit-flip noise with $p_x = 0.2$. In the frame (Ox,Oy,Oz) of $\mathbb{R}^3$ the rotation axis is (a) $\vec{n} = (\theta_n = 0,\varphi_n) \parallel Oz$ with the uniform maximum $F_q(\theta_1,\varphi_1) = 1$ for $\theta_1^{\rm opt} = \pi/2$, $\forall\varphi_1$; (b) $\vec{n} = (\theta_n = 0.3\pi$, $\varphi_n = 0.4\pi)$ with two maxima at $F_q(\theta_1,\varphi_1) = 0.960$ for ($\theta_1^{\rm opt} = \pi/2$, $\varphi_1^{\rm opt} = 0.56\pi$) and ($\theta_1^{\rm opt} = \pi/2$, $\varphi_1^{\rm opt} = 1.56\pi$) at the locations of the two crosses (×).

analysis of the Fisher information of Eq. (47), this does not prevent the preservation of a maximum $F_q(\theta_1 = \pi/2, \varphi_1) = 1$ for any φ_1. The condition $\theta_1^{\text{opt}} = \pi/2 = \theta_0^{\text{opt}}$ is achieved by a pure input probe ρ_0 with a Bloch vector $\vec{r}_0$ orthogonal to the rotation axis $\vec{n}$; this ensures the maximum $F_q(\theta_1 = \pi/2, \varphi_1) = 1$ for any rotated angle $\varphi_1 = \varphi_0 + \xi$, i.e., any phase ξ to be estimated. The same outcome of $F_q(\theta_1 = \pi/2, \varphi_1) = 1$, $\forall \varphi_1 = \varphi_0 + \xi$, achievable by any input probe orthogonal to $\vec{n}$, could be obtained in equivalent configurations with a phase-flip noise when $a_{zz} = 1$, and with a bit-phase-flip noise when $a_{yy} = 1$. In this way, with a rotation axis $\vec{n}$ parallel to an eigenaxis (Ox, Oy, Oz), these three Pauli noises, having one of the diagonal coefficients $a_{jj} = 1$ in Eq. (51) in an eigendirection not coinciding with $\vec{n}$, lead to an optimized input probe ρ_0 independent of the phase ξ and allowing complete immunity from the noise: the maximum $F_q(\theta_1 = \pi/2, \varphi_1) = 1$ materializing that there is no reduction by the noise of the available Fisher information.

For the more general configurations where the rotation axis $\vec{n}$ is not parallel to any of the eigenaxes (Ox, Oy, Oz) of the Pauli noise, the analysis of Eq. (47) and its derivatives in (θ_1, φ_1) especially via Eqs. (55) and (56), shows that the Fisher information $F_q(\theta_1, \varphi_1)$ culminates at a maximum $\leqslant 1$ occurring in $\theta_1^{\text{opt}} = \pi/2$ and specific values of $\varphi_1 = \varphi_1^{\text{opt}}$ dependent on the orientation of $\vec{n}$. An example is presented in Fig. 2(b) for an oblique $\vec{n}$ in the frame (Ox, Oy, Oz). The optimality condition $\theta_1^{\text{opt}} = \pi/2 = \theta_0^{\text{opt}}$ can always be achieved by a pure input probe ρ_0 with $\vec{r}_0$ in the plane orthogonal to the rotation axis $\vec{n}$; yet this has to be complemented, so as to reach the maximum of $F_q(\theta_1, \varphi_1)$, for the input probe by an azimuth $\varphi_0^{\text{opt}} = \varphi_1^{\text{opt}} - \xi$ having a specific tuning dependent on the phase ξ.

The transformation of Eq. (50) always satisfying $\mathcal{N}(\mathbb{1}) = \mathbb{1}$ belongs to the class of unital noise models for the qubit, with specific interesting properties [22,23]. This is associated with $\vec{c} = \vec{0}$ in Eq. (36). Beyond the case of the Pauli noises of Eqs. (50) and (51), any quantum noise with $\vec{c} = \vec{0}$ and $A = US$ non-necessarily diagonal in the original frame $(0x, 0y, Oz)$ is also unital, satisfying $\mathcal{N}(\mathbb{1}) = \mathbb{1}$. For the quantum Fisher information $F_q(\xi)$, the behaviors observed in this section are essentially controlled by the situation of the rotation axis $\vec{n}$ relative to the eigenaxes (Ox, Oy, Oz) of the Pauli noise. With an arbitrary unital noise characterized by $(A = US,\ \vec{c} = \vec{0})$, similar behaviors can be expected for the quantum Fisher information $F_q(\xi)$, but controlled this time by the situation of the rotation axis $\vec{n}$ relative to the eigendirections $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ of S, the isometry U preserving the inner product in $\mathbb{R}^3$ will have no effect on $F_q(\xi)$ of Eq. (31).

2. *Nonunital noises*

The effect of $\vec{c} \neq \vec{0}$ with nonunital noises can also play a significant role when optimizing the input probe ρ_0 through maximization of the quantum Fisher information $F_q(\xi)$. A nonunital noise important to the qubit is the generalized amplitude damping (GAD) noise [12], characterized in Eq. (36) by

$$A = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \tag{58}$$

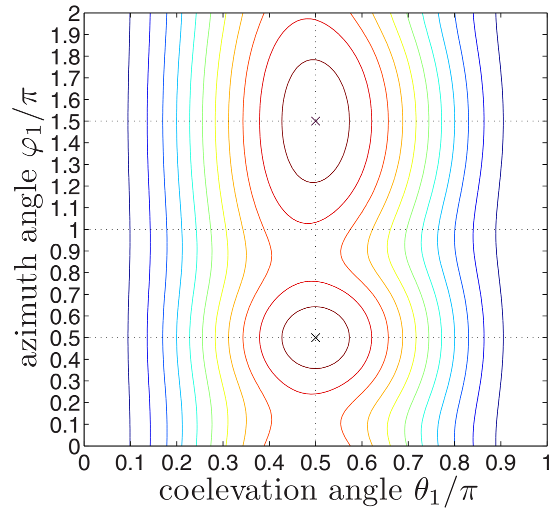

FIG. 3. (Color online) The level curves of the quantum Fisher information $F_q(\theta_1, \varphi_1)$ from Eq. (47) as in Fig. 2, with a rotation axis $\vec{n} = (\theta_n = 0.3\pi,\ \varphi_n = 0.4\pi)$ in the frame (Ox, Oy, Oz) of $\mathbb{R}^3$. The qubit is affected by a GAD noise from Eq. (58) with parameters $(\gamma = 0.42,\ p = 0.12)$. With two maxima at $F_q(\theta_1, \varphi_1) = 0.58$ for $(\theta_1^{\text{opt}} = \pi/2,\ \varphi_1^{\text{opt}} = \pi/2)$ and $(\theta_1^{\text{opt}} = \pi/2,\ \varphi_1^{\text{opt}} = 3\pi/2)$ at the locations of the two crosses $(\times)$.

and $\vec{c} = [0, 0, (2p-1)\gamma]^\top$, with γ and p in $[0,1]$, which can describe the interaction of the qubit with a thermal bath. With the symmetries of the GAD noise, Eq. (47) shows that the quantum Fisher information $F_q(\theta_1, \varphi_1)$ is always maximized at $(\theta_1^{\text{opt}} = \pi/2; \varphi_1^{\text{opt}} = \pi/2, 3\pi/2)$, for any rotation axis $\vec{n}$, with a value of $1 - \gamma$ for the maximum which is therefore independent of $\vec{n}$ and a function only of the noise parameter γ. An illustration is provided in Fig. 3, depicting the landscape of $F_q(\theta_1, \varphi_1)$ in the plane (θ_1, φ_1), which in general changes with both the rotation axis $\vec{n}$ and the GAD noise parameters (γ, p). However, as visible in Fig. 3, there is invariably a maximum of $F_q(\theta_1, \varphi_1) = 1 - \gamma$ in $(\theta_1^{\text{opt}} = \pi/2; \varphi_1^{\text{opt}} = \pi/2, 3\pi/2)$.

For maximizing the quantum Fisher information with the GAD noise, the optimality condition $\theta_1^{\text{opt}} = \pi/2 = \theta_0^{\text{opt}}$ is again achieved by a pure input probe ρ_0 with $\vec{r}_0$ orthogonal to the rotation axis $\vec{n}$. Yet this again has to be complemented for the input probe by an azimuth $\varphi_0^{\text{opt}} = \pi/2 - \xi$ or $\varphi_0^{\text{opt}} = 3\pi/2 - \xi$ having a specific tuning dependent on the phase ξ.

A less symmetrical nonunital noise relevant to the qubit is the squeezed generalized amplitude damping (SGAD) noise [24–26]. Such a noise process describes the interaction of the qubit with a squeezed thermal bath. Squeezing of a thermal bath is obtained by a nonlinear operation capable of introducing correlations between the modes or thermal photons of the bath, with possibilities to counteract the detrimental decohering effect of temperature [24,25,27]. The SGAD quantum noise is characterized in Eq. (36) by

$$A = \begin{bmatrix} a_{xx} & a_{xy} & 0 \\ a_{xy} & a_{yy} & 0 \\ 0 & 0 & a_{zz} \end{bmatrix}, \tag{59}$$

and $\vec{c} = [0, 0, c_z]^\top$.

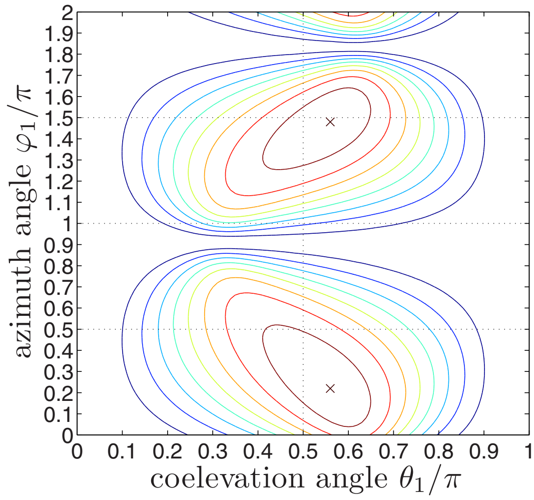

FIG. 4. (Color online) The level curves of the quantum Fisher information $F_q(\theta_1,\varphi_1)$ from Eq. (47) as in Figs. 2 and 3, with a rotation axis $\vec{n} = (\theta_n = 0.3\pi,\ \varphi_n = 0.4\pi)$ in the frame (Ox,Oy,Oz) of $\mathbb{R}^3$. The qubit is affected by a SGAD noise from Eq. (59) with parameters ($a_{xx} = 0.3604$, $a_{xy} = -0.2712$, $a_{yy} = 0.5433$, $a_{zz} = 0.1248$; $c_z = 0.6379$). With two maxima at $F_q(\theta_1,\varphi_1) = 0.436$ for $(\theta_1^{\rm opt} = 0.56\pi,\ \varphi_1^{\rm opt} = 0.22\pi)$ and $(\theta_1^{\rm opt} = 0.56\pi,\ \varphi_1^{\rm opt} = 1.48\pi)$ at the locations of the two crosses (×).

With the SGAD noise, Eq. (47) shows that the quantum Fisher information $F_q(\theta_1,\varphi_1)$ generally takes its maximum at arbitrary locations $(\theta_1^{\rm opt} \neq \pi/2; \varphi_1^{\rm opt})$ specifically determined by the rotation axis $\vec{n}$ and by the SGAD noise parameters. An illustration is provided in Fig. 4, depicting a typical arbitrary (with no simple symmetries) landscape of the quantum Fisher information $F_q(\theta_1,\varphi_1)$ with a SGAD noise.

For maximizing the quantum Fisher information with the SGAD noise, the optimality conditions $(\theta_1^{\rm opt} = \theta_0^{\rm opt},\ \varphi_1^{\rm opt} = \varphi_0^{\rm opt} + \xi)$ as illustrated in Fig. 4, generally point to a pure input probe ρ_0 with a very specific noise-dependent orientation nonorthogonal to the rotation axis $\vec{n}$ and also matched to the phase ξ.

In this way, with approaches similar to those illustrated in Figs. 1–4 on important noises relevant to the qubit, the form of Eq. (47) for the quantum Fisher information $F_q(\xi)$ allows one to determine the conditions of optimality for the input probe ρ_0, with any given quantum noise defined by $(A,\vec{c})$.

F. Bayesian optimization

The evaluation of the quantum Fisher information $F_q(\xi)$ performed in Sec. III E has revealed that, with most types of noise characterized by $(A,\vec{c}\,)$ in Eq. (36) and most rotation axes $\vec{n}$, the optimal pure input probe ρ_0 maximizing the quantum Fisher information $F_q(\xi)$ is usually dependent on the phase ξ to be estimated from the noisy qubit. This is true except with isotropic noises similar to the depolarizing noise, and with unital noises when the rotation axis $\vec{n}$ is parallel to an eigendirection $\{\vec{s}_1,\vec{s}_2,\vec{s}_3\}$ and an eigenvalue $s_j = 1$ in an eigendirection differing form $\vec{n}$, in which cases any pure input probe ρ_0 orthogonal to $\vec{n}$ is optimal for maximizing $F_q(\xi)$.

To cope with a ξ-dependent optimal probe ρ_0, one can resort to an adaptive scheme with feedback whenever a series of experiments can be repeated to estimate a same phase ξ, much like the adaptive approach to optimize a POVM mentioned at the end of Sec. II A. The adaptive scheme is driven by the analysis of Eq. (47) providing the solution $(\theta_1^{\rm opt},\varphi_1^{\rm opt})$ to the maximization of $F_q(\xi)$. A nonoptimized initial (pure) probe ρ_0 defined by $(\theta_0 = \theta_1^{\rm opt},\varphi_0)$ provides a rough estimate $\widehat{\xi}$ of the phase ξ, and this estimate $\widehat{\xi}$ is used to adjust the probe ρ_0 via $\varphi_0 \leftarrow \varphi_1^{\rm opt} - \widehat{\xi}$; the step is

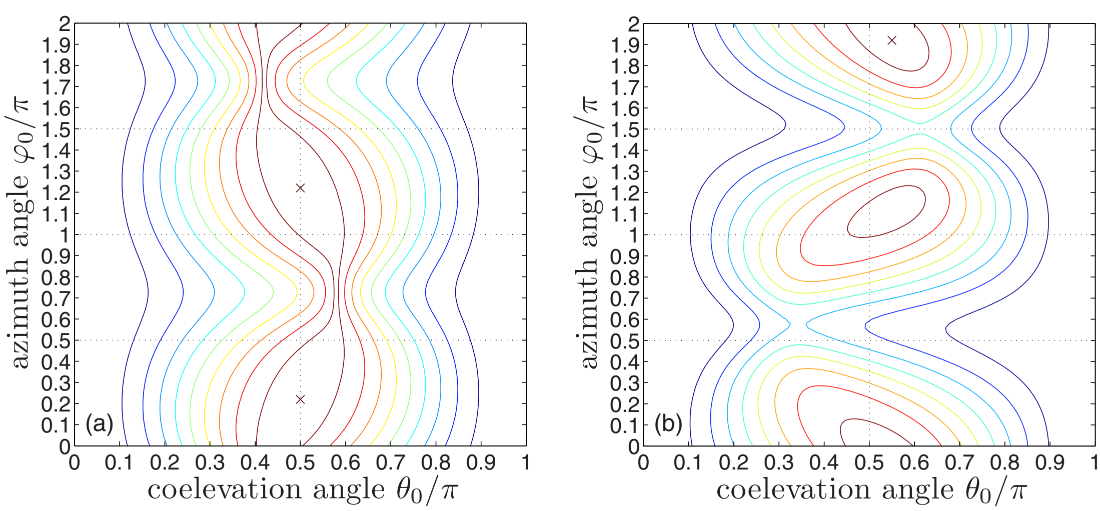

FIG. 5. (Color online) The level curves of the ξ-averaged quantum Fisher information $\overline{F}_q(\theta_0,\varphi_0)$ in the plane of the two angles (θ_0,φ_0) defining in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$ the pure input probe ρ_0. The average is according to a Gaussian prior $p_0(\xi)$ with mean $m_\xi = \pi/3$ and standard deviation $\sigma_\xi = \pi/10$. The rotation axis is $\vec{n} = (\theta_n = 0.3\pi,\ \varphi_n = 0.4\pi)$ in the frame (Ox,Oy,Oz) of $\mathbb{R}^3$. The qubit is affected by (a) a bit-flip noise with $p_x = 0.2$ of Fig. 2(b), with the maximum $\overline{F}_q(\theta_0,\varphi_0) = \overline{F}_q^{\rm max} = 0.955$ for $(\theta_0^{\rm opt} = \pi/2,\ \varphi_0^{\rm opt} = 0.22\pi)$ and $(\theta_0^{\rm opt} = \pi/2,\ \varphi_0^{\rm opt} = 1.22\pi)$ at the locations of the two crosses (×); (b) a SGAD noise with the same parameters as in Fig. 4, with the maximum $\overline{F}_q(\theta_0,\varphi_0) = \overline{F}_q^{\rm max} = 0.425$ for $(\theta_0^{\rm opt} = 0.55\pi,\ \varphi_0^{\rm opt} = 1.92\pi)$ at the location of the cross (×).

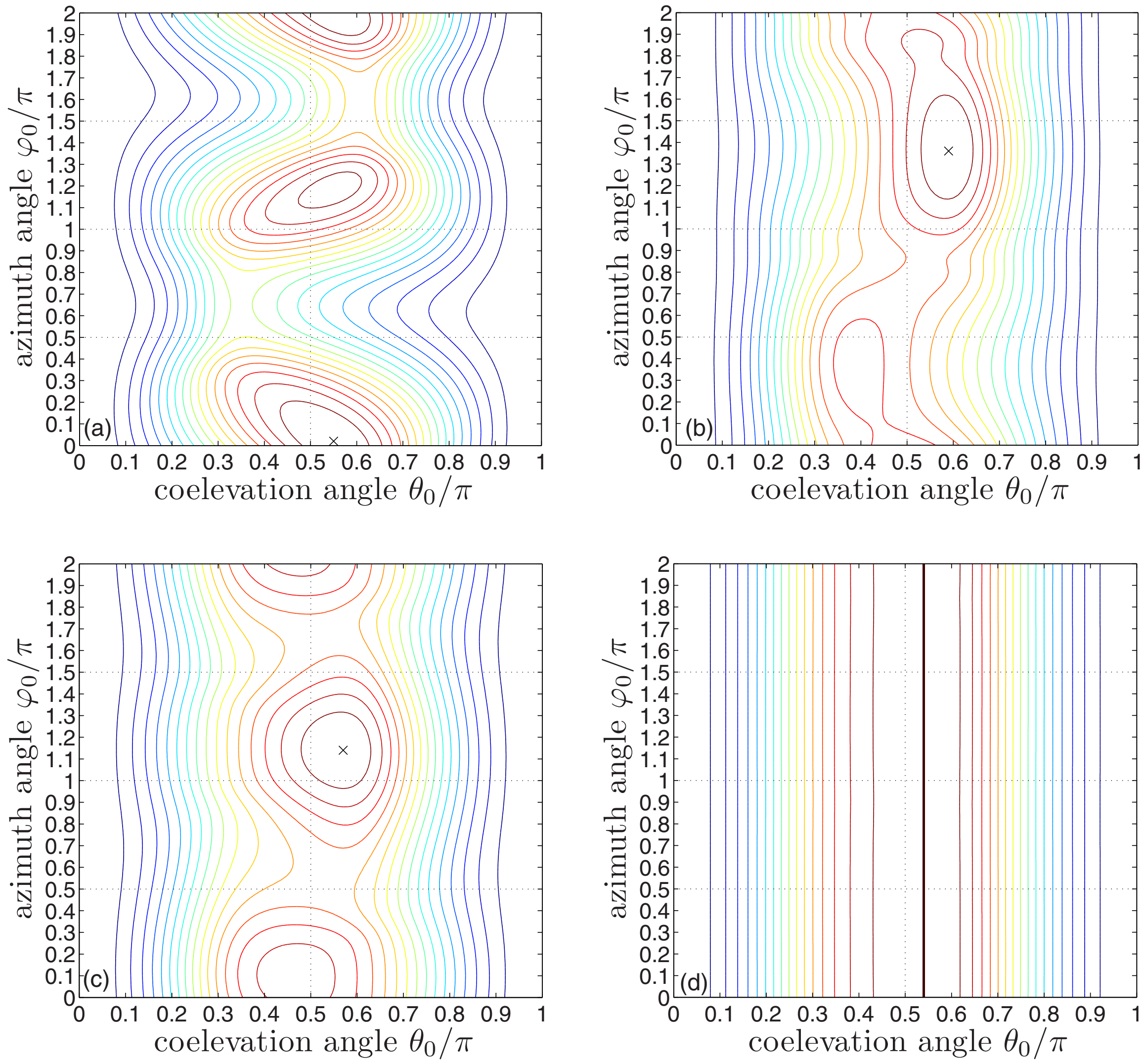

FIG. 6. (Color online) The level curves of the ξ-averaged quantum Fisher information $\overline{F}_q(\theta_0,\varphi_0)$ in the plane of the two angles (θ_0,φ_0) defining in the basis $\{\vec{n},\vec{n}_\perp,\vec{n}\,'_\perp\}$ the pure input probe ρ_0. The average is according to a prior $p_0(\xi)$ uniform over $[0,\xi_{\max}]$. The rotation axis is $\vec{n} = (\theta_n = 0.3\pi,\ \varphi_n = 0.4\pi)$ in the frame (Ox,Oy,Oz) of $\mathbb{R}^3$. The qubit is affected by a SGAD noise with the same parameters as in Fig. 4. (a) $\xi_{\max} = \pi/2$ with the maximum $\overline{F}_q(\theta_0,\varphi_0) = \overline{F}_q^{\max} = 0.415$ for $(\theta_0^{\rm opt} = 0.55\pi,\ \varphi_0^{\rm opt} = 0.02\pi)$ at the location of the cross $(\times)$; (b) $\xi_{\max} = \pi$ with the maximum $\overline{F}_q(\theta_0,\varphi_0) = \overline{F}_q^{\max} = 0.351$ for $(\theta_0^{\rm opt} = 0.59\pi,\ \varphi_0^{\rm opt} = 1.36\pi)$ at the location of the cross $(\times)$; (c) $\xi_{\max} = 3\pi/2$ with the maximum $\overline{F}_q(\theta_0,\varphi_0) = \overline{F}_q^{\max} = 0.348$ for $(\theta_0^{\rm opt} = 0.57\pi,\ \varphi_0^{\rm opt} = 1.14\pi)$ at the location of the cross $(\times)$; (d) $\xi_{\max} = 2\pi$ with the maximum $\overline{F}_q(\theta_0,\varphi_0) = \overline{F}_q^{\max} = 0.293$ for $\theta_0^{\rm opt} = 0.54\pi,\ \forall\varphi_0$.

repeated with the newly adjusted probe, and iterated. Such an adaptive scheme has been experimentally implemented in [6] for the estimation of the rotation angle of a qubit around Oz in the presence of phase-flip noise, and was shown to converge in a few iterations. Our analysis here provides access to $(\theta_1^{\rm opt},\varphi_1^{\rm opt})$ for any rotation axis $\vec{n}$ and any noise $(A,\vec{c})$, and therefore enables application of the adaptive scheme in any conditions.

As an alternative, to obtain an optimal input probe ρ_0 independent of ξ, in general conditions, a Bayesian approach can be adopted. One has to introduce a prior probability density $p_0(\xi)$ quantifying the *a priori* range and values admissible for the unknown phase ξ. The quantum Fisher information $F_q(\xi)$ of Eq. (47) can then be averaged according to the prior $p_0(\xi)$, using Eqs. (48) and (49) with $(\theta_1 = \theta_0,\ \varphi_1 = \varphi_0 + \xi)$. One then obtains the ξ-averaged quantum Fisher information $\overline{F}_q = \overline{F}_q(\theta_0,\varphi_0)$, which is now only a function of the two angles (θ_0,φ_0) defining the pure input probe ρ_0. Finally, maximization of $\overline{F}_q(\theta_0,\varphi_0)$ in the plane (θ_0,φ_0) determines the optimal configuration $(\theta_0^{\rm opt},\varphi_0^{\rm opt})$ for the pure input probe ρ_0, matched to given noise and rotation axis $\vec{n}$. An input probe ρ_0 optimized in this way can be expected to perform well on most occasions, over a series of successive estimations of a large number of independent instances of ξ distributed according to $p_0(\xi)$.

For illustration of this Bayesian approach, Fig. 5 considers the case of a Gaussian prior $p_0(\xi) = \exp[-(\xi - m_\xi)^2/(2\sigma_\xi^2)]/(\sigma_\xi\sqrt{2\pi})$, with mean $m_\xi = \pi/3$ and standard deviation $\sigma_\xi = \pi/10$, and depicts the ξ-averaged quantum Fisher information $\overline{F}_q(\theta_0,\varphi_0)$ corresponding to the qubit affected by the bit-flip noise of Fig. 2(b) or by the SGAD noise of Fig. 4.

Maximization of $\overline{F}_q(\theta_0,\varphi_0)$ in Fig. 5 gives access to the optimal parametrization $(\theta_0^{\rm opt},\varphi_0^{\rm opt})$ for the pure input probe $\rho_0^{\rm opt}$, together with the corresponding maximum average Fisher information $\overline{F}_q(\theta_0^{\rm opt},\varphi_0^{\rm opt}) = \overline{F}_q^{\max}$. The optimal probe $\rho_0^{\rm opt}$ and its performance $\overline{F}_q^{\max}$ are clearly distinct for each noise condition in Fig. 5, expressing again the necessity of an input probe specifically matched to the noise in order to maximize the performance in estimation. There is, however, a rotation invariance of the configuration of the optimum, in the sense that if the prior distribution $p_0(\xi)$ is shifted by an amount $\Delta\xi$, for instance when the Gaussian mean is changed by $m_\xi \to m_\xi + \Delta\xi$, then the optimum is simply displaced as $(\theta_0^{\rm opt},\varphi_0^{\rm opt}) \to (\theta_0^{\rm opt},\varphi_0^{\rm opt} - \Delta\xi)$ while the maximum $\overline{F}_q^{\max}$ remains unchanged. Meanwhile, if the standard deviation σ_ξ of the prior $p_0(\xi)$ increases, this usually entails a decrease of the maximum detection performance $\overline{F}_q^{\max}$. To illustrate this point, Fig. 6 shows the situation of a prior $p_0(\xi)$ uniform over $[0,\xi_{\max}]$, with a SGAD noise.

The four panels of Fig. 6 with the prior $p_0(\xi)$ uniform over $[0,\xi_{\max}]$, show the nontrivial evolution, as $\xi_{\max}$ is increased, of the landscape of the ξ-averaged quantum Fisher information $\overline{F}_q(\theta_0,\varphi_0)$ in the plane of the two angles (θ_0,φ_0) defining the pure input probe ρ_0, with in each condition the possibility of identifying the optimal tuning $(\theta_0^{\rm opt},\varphi_0^{\rm opt})$ achieving the maximum $\overline{F}_q^{\max}$ of the Fisher information $\overline{F}_q(\theta_0,\varphi_0)$. It is also indicated in Fig. 6 that as the dispersion $\xi_{\max}$ increases, expressing less accurate prior information on ξ, then the maximum detection performance assessed by $\overline{F}_q^{\max}$ gradually decreases, consistently since estimation thereof is assisted by lesser prior knowledge. The extreme $\xi_{\max} = 2\pi$ of Fig. 6(d) corresponds to the most dispersed and uninformative prior $p_0(\xi)$, and in this condition any azimuth φ_0 is equivalent to devise the optimal probe $\rho_0^{\rm opt}$. Nevertheless, a specific tuning of the coelevation at $\theta_0^{\rm opt} = 0.54\pi$ is necessary to define the optimal probe $\rho_0^{\rm opt}$ capable of maximizing the average Fisher information at $\overline{F}_q(\theta_0^{\rm opt},\varphi_0) = \overline{F}_q^{\max} = 0.293$, for the SGAD noise of Fig. 6(d). This situation of a prior probability density $p_0(\xi)$ uniform over $[0,2\pi]$ corresponds to no prior knowledge on the unknown phase ξ to be estimated. Yet, even in such circumstance, the present analysis shows the necessity usually of a specific tuning for the input probe $\rho_0^{\rm opt}$, and determines its value, in order to best cope with the quantum noise hindering the estimation.

IV. CONCLUSION

We have reviewed the theory of quantum estimation and applied it for parametric estimation on a noisy qubit. An important step is the exploitation of the Bloch representation of qubit states in order to obtain explicit expressions for the quantum score L_ξ and quantum Fisher information $F_q(\xi)$ according to Eqs. (30)–(33). This served to establish that, for any parametric dependence on ξ of the measured qubit state ρ_ξ, the quantum Fisher information $F_q(\xi)$ always increases with the purity of ρ_ξ. In Bloch representation, an arbitrary quantum noise affecting the qubit has been taken into account, enabling one to describe the impact of any noise on the quantum score and on the quantum Fisher information, and this again for any parametric dependence of ρ_ξ on ξ. The task has then been specified to estimating the phase ξ acquired by a qubit in a rotation around an arbitrary axis $\vec{n}$, equivalent to estimating the phase of an arbitrary single-qubit quantum gate. It then became possible to address the optimization of the input probe state ρ_0, so as to maximize the quantum Fisher information upon estimation from the noisy qubit. In such circumstance, the optimal probe is proved to always be a pure state, yet specifically matched to the noise. The optimal input probes have been determined for important quantum noises relevant to the qubit, including Pauli noises and nonunital noises as GAD or SGAD noises, with any other noise model which can be equally handled by our approach. In highly symmetric configurations, for instance with the isotropic depolarizing noise or with privileged orientations of the qubit rotation axis $\vec{n}$ relative to the eigenaxes of a Pauli noise, an optimal input probe ρ_0 independent of the unknown parameter ξ is found to exist. In other, less symmetric configurations, the optimal input probe ρ_0 comes out as a pure state with a Bloch vector orthogonal to the rotation axis $\vec{n}$ and with a ξ-dependent azimuth. For even more sophisticated configurations, for instance with the SGAD noise, the optimal input probe ρ_0 can be found in arbitrary positions relative to the rotation axis $\vec{n}$, which can all be determined from the present analysis.

The present analysis identifies the conditions with an optimal input probe ρ_0 independent of the unknown parameter ξ, defining in this way a useful setting which can be selected for efficient estimation. In addition, the situations that are identified with a ξ-dependent optimal input probe ρ_0, point in such cases to the expedient of an adaptive scheme with feedback, over a sequence of successive measurements, as explained in Sec. III F. This will, in principle, give access to the optimal probe ρ_0 maximizing the quantum Fisher information $F_q(\xi)$. Alternatively, we have presented a Bayesian approach which, based on a prior probability distribution $p_0(\xi)$ for the unknown parameter ξ, leads to a ξ-independent optimal input probe. The optimization of the input probe can also be coupled to the adaptive scheme constructing a POVM capable of reaching the maximum $F_c(\xi) = F_q(\xi)$, as explained at the end of Sec. II A. This adaptive scheme for the POVM relies on the eigen decomposition of the quantum score L_ξ, and will therefore also benefit from the present analysis for the noisy qubit. The eigen decomposition of L_ξ follows directly from the expressions for L_ξ given in Eq. (30) or (32), since any operator on $\mathcal{H}_2$ written as $L = a_0\mathbb{1} + \vec{a}\vec{\sigma}$ has the two eigenvalues $a_0 \pm \sqrt{\vec{a}^{\,2}}$ with the two projectors in $\mathcal{H}_2$ on the two associated eigenvectors reading $(\mathbb{1} \pm \vec{a}\vec{\sigma}/\sqrt{\vec{a}^{\,2}})/2$. In this way, the present analysis of quantum state estimation from a noisy qubit, offers several useful possibilities for optimizing the estimation task.

The present approach can be extended to multiparametric estimation [1] on a noisy qubit, with a qubit state $\rho_{\vec{\xi}} = \rho(\vec{\xi}\,)$ dependent on a vector of unknown parameters $\vec{\xi} = (\xi_1,\xi_2,\dots\xi_K)^\top$. The estimation performance will be controlled by the K partial derivatives $\partial_k\rho(\vec{\xi}\,) \equiv \partial\rho(\vec{\xi}\,)/\partial\xi_k$ and a score function with K components $L_k(\vec{\xi}\,)$ each defined via an analog of Eq. (7). For the noisy qubit, the Bloch representation with Bloch vector $\vec{r}(\vec{\xi}\,)$ in $\mathbb{R}^3$ and its K partial derivatives $\partial_k\vec{r}(\vec{\xi}\,) \equiv \partial\vec{r}(\vec{\xi}\,)/\partial\xi_k$ will remain central for the analysis, with more involved, multidimensional expressions, yet with the action of the noise still governed by Eq. (36).

The present approach where one seeks to optimize the input probe state prior to the action of noise, in order to maximize the performance in estimation from a noisy quantum state, can also be extended to quantum systems of dimension higher than the dimension $N = 2$ of the qubit. However, to extend the present approach in this direction, one would have to resort to some generalization of the Bloch representation decomposing a qubit state essentially on a basis of traceless Hermitian operators (plus the identity), such as, for instance, Gell-Mann matrices for the qutrit with dimension $N = 3$ or their generalization in higher dimension [28]. Here also the theoretical analysis is more involved, and moreover the characterization of quantum noises in higher dimension is another separate difficulty adding to the complication of the task.

Returning to the more tractable level of the qubit, which is a fundamental system for quantum information, informational processes other than estimation could also be envisaged along the same line, looking for optimized conditions of operation in the presence of a definite quantum noise separately characterized. In this respect, one could propose, for instance, to include the noise in specifically quantum processes involving entanglement and nonlocal quantum correlations violating Bell-type inequalities [29–31], for better exploitation of entanglement in the presence of noise.

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