Noise-enhanced Fisher information in parallel arrays of sensors with saturation

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This paper investigates stochastic resonance in parallel arrays of uncoupled saturating devices. The Fisher information is used to demonstrate the possibility of noise improved parameter estimation for arbitrary parametric signals. Especially, it is shown that improvement by noise always occurs in these arrays, for any configuration of the input signal, even in optimal configuration. The results contribute to establish stochastic resonance in parallel uncoupled arrays as a general mechanism of enhancement by noise, which can occur in wide classes of nonlinearities and for various information processing tasks. It can supplement other mechanisms of stochastic resonance that take place in isolated nonlinearities but generally in restricted configurations of the input signal.

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I. INTRODUCTION

An important index to assess the measurement of a signal in noise is provided by the Fisher information [1]. This index offers a quantification of the amount of information contained in a noisy measurement about the value of a parameter attached to a given signal. The Fisher information can be related in various ways to the fundamental process of physical measurement [2]. In particular, the reciprocal Fisher information acts in a lower bound limiting the efficacy of any conceivable estimator of the parameter from the measurement. In general, the larger the Fisher information the more accurately can the parameter be estimated. In the limit of a large number of independent measurements, the reciprocal Fisher information constitutes the mean squared estimation error achieved by a very common and convenient estimator: the maximum likelihood estimator. This relation to the maximum likelihood estimator provides a direct practical significance to the Fisher information.

For these reasons, the Fisher information has been exploited as a fundamental measure suitable to investigate the phenomenon of stochastic resonance [3–5]. Stochastic resonance designates situations where a noise and a signal interact with a system to produce a nonlinear effect of improvement by noise measured by a some meaningful index of performance (for instance see [6,7] for overviews and [8–10] for very recent specific studies). Many studies on stochastic resonance relied on nonlinear systems with thresholds or potential barriers, in which various forms of noise-improved signal transmission were demonstrated. This was done for a periodic signal assessed by a noise-enhanced signal-to-noise ratio in the frequency domain, for aperiodic and for random signals assessed by cross-correlation measures or Shannon mutual information [6,7]. Noise-enhanced Fisher information was also demonstrated for these nonlinear systems, with significance for signal estimation [3–5]. In these forms of stochastic resonance, the beneficial action of the noise essentially is to assist a small input signal in overcoming a threshold or potential barrier in the response, whence the improved transmission. Less frequently, stochastic resonance has been reported in threshold-free or barrier-free nonlinearities [11–13]. Also, [14] for isolated saturating nonlinearities, shows stochastic resonance with improvement by noise of a signal-to-noise ratio, or a cross-correlation measure, or a Shannon mutual information. More recently, [15] establishes the possibility of a noise-enhanced Fisher information in soft nonlinearities with saturations. In such systems, the mechanism of improvement is that, in the presence of a nonoptimally positioned input (for instance an input evolving in the saturation of a nonlinearity), the noise has the ability to displace the operating zone of the nonlinearity into a region more favorable to the signal. Such studies are useful as they extend the applicability of stochastic resonance to wider classes of nonlinearities, not restricted to the threshold of barrier nonlinearities.

Also recently, another distinct mechanism of stochastic resonance has been shown possible when nonlinearities are replicated into an uncoupled parallel array. This form of stochastic resonance in arrays was introduced under the name of suprathreshold stochastic resonance in [16,17] for two-state threshold comparators. Improvement by noise was registered in Refs. [16–19] for the transmission by threshold comparators of a noise-free random input with arbitrary (not necessarily subthreshold) amplitude. This form of stochastic resonance was later applied to process a signal-noise mixture as the input to the array of threshold comparators [20]. Gradually, it is becoming apparent that this form of "suprathreshold" stochastic resonance in arrays, is in fact not restricted to the threshold nonlinearities. Enhancement by noise of the transmission of a periodic signal assessed by a signal-to-noise ratio, has been shown possible in arrays of soft threshold-free power-law nonlinearities [21], and also recently in arrays of saturating sensors [22]. In the present paper, we will consider the same type of arrays of saturating sensors as in [22]. We will demonstrate that improvement by noise of the Fisher information is possible in this type of array. Also, we will show that the two mechanisms of improvement by noise evoked above can operate in conjunction, one in isolated nonlinearities but requiring ill-positioned inputs, and one as a nonlinear-array effect taking place for any inputs. The present study contributes to establish stochastic resonance in arrays as a general mechanism of enhancement by noise, which can occur in wide classes of (threshold-free) nonlinearities, and which can be assessed by different measures of efficacy, with significance for various

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signal-processing tasks, for instance signal estimation as considered here.

II. FISHER INFORMATION FOR NONLINEAR ARRAYS

A random signal $x(t)$ is dependent upon an unknown parameter a, the value of which we seek to estimate. This input signal $x(t)$ is applied onto a parallel array of N identical uncoupled sensors, conforming to the architecture also considered in [16,23,24]. Each sensor of the array is endowed with the same input-output characteristic, modeled by the static or memoryless function $g(\cdot)$. A noise $\eta_i(t)$, independent of $x(t)$, can be added to $x(t)$ at each sensor i, so as to produce the output

$$y_i(t) = g[x(t) + \eta_i(t)], \quad i = 1,2, \ldots N. \tag{1}$$

The N noises $\eta_i(t)$ are white, mutually independent, and identically distributed (i.i.d.) with cumulative distribution function $F_\eta(u)$, probability density $f_\eta(u) = dF_\eta(u)/du$, and standard deviation σ_η. The response $y(t)$ of the array is obtained by averaging the outputs of all the sensors, as follows:

$$y(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t). \tag{2}$$

Observations are performed on the array output $y(t)$ in order to estimate the parameter a. The Fisher information J_y contained in $y(t)$ about a is expressible as follows [1]:

$$J_y = \int_{-\infty}^{+\infty} \frac{1}{p_y(y)} \left[\frac{\partial}{\partial a} p_y(y) \right]^2 dy, \tag{3}$$

where $p_y(y)$ is the (a-dependent) probability density function of $y(t)$.

For a fixed given value x of the input, the resulting probability density for each $y_i(t)$ in Eq. (1) is denoted $p_{yi|x}(y,x)$. This density $p_{yi|x}(y,x)$ is usually accessible, at fixed x, from the density $f_\eta(u)$ of the noise $\eta_i(t)$ as transformed by $g(\cdot)$. For instance, when the characteristic $g(\cdot)$ is invertible, one obtains

$$p_{yi|x}(y,x) = \frac{f_\eta[g^{-1}(y) - x]}{g'[g^{-1}(y)]}. \tag{4}$$

When $g(\cdot)$ is noninvertible and maps continuous domains of finite probabilities into discrete points, then in the density $p_{yi|x}(y,x)$ these probabilities will weight Dirac delta functions located at these discrete points.

It is also possible to express separately various statistical moments of the density $p_{yi|x}(y,x)$. From Eq. (1), one for instance has the expectations

$$E[y_i(t)|x] = \int_{-\infty}^{+\infty} g(x+u) f_\eta(u) du, \tag{5}$$

and

$$E[y_i^2(t)|x] = \int_{-\infty}^{+\infty} g^2(x+u) f_\eta(u) du. \tag{6}$$

Next, we denote as $p_{y|x}(y,x)$ the probability density of $y(t)$ in Eq. (2) at fixed x. Since the noises $\eta_i(t)$ are i.i.d., so are the outputs $y_i(t)$ from Eq. (1), and $p_{y|x}(y,x)$ is therefore obtainable through an N-fold convolution of $p_{yi|x}(y,x)$. The density $p_{y|x}(y,x)$ is then averaged over the probability density $f_x(x)$ for x, so as to yield the unconditional density $p_y(y)$ needed in Eq. (3) to express Fisher information J_y.

For the sake of definiteness, concerning the parametric dependence of $x(t)$ on a, we shall consider in the sequel the broad class of processes where $x(t)$ is formed by the additive mixture $x(t) = \xi(t) + s_a(t)$. The signal $\xi(t)$ is a random noise, white, independent of the η_i's and of a, with cumulative distribution function $F_\xi(u)$, probability density $f_\xi(u) = dF_\xi(u)/du$, and standard deviation σ_ξ. The signal $s_a(t)$ is deterministic and contains the parameter a. For instance, a can be the value of a constant $s_a(t) \equiv a$, or the amplitude or frequency of a periodic $s_a(t)$, or any other parameter entering the specification of the deterministic $s_a(t)$. We then have the density $f_x(x) = f_\xi[x - s_a(t)]$, and

$$p_y(y) = \int_{-\infty}^{+\infty} p_{y|x}(y,x) f_\xi[x - s_a(t)] dx, \tag{7}$$

and for the derivative with respect to a,

$$\frac{\partial}{\partial a} p_y(y) = -\frac{\partial s_a(t)}{\partial a} \int_{-\infty}^{+\infty} p_{y|x}(y,x) f'_\xi[x - s_a(t)] dx. \tag{8}$$

The above equations expose the formal relation between the unknown parameter a to be estimated and the Fisher information J_y at the output of the array. This allows, in principle, the study of the evolution of J_y (reflecting the estimation efficacy) in various configurations of the array, for instance concerning the choice for the sensor characteristic $g(\cdot)$ and the impact of the added noises $\eta_i(t)$. The study of [20] considers the special case where the sensors are two-state threshold comparators, and it shows possibilities of improving J_y thanks to the action of the added array noises $\eta_i(t)$. In the present paper, instead of threshold comparators as in [20], we will consider a characteristic $g(\cdot)$ modeling sensors which are linear for small inputs and saturate for large inputs. This is a common behavior for sensors, and we will investigate here, through the study of J_y, the impact of the added array noises $\eta_i(t)$ on the estimation performance from these arrays.

III. FISHER INFORMATION FOR SATURATING ARRAYS

We now consider the characteristic $g(\cdot)$ with saturation, under the form

$$g(u) = \begin{cases} -\lambda & \text{for } u \leqslant -\lambda, \\ u & \text{for } -\lambda < u < \lambda, \\ \lambda & \text{for } u \geqslant \lambda. \end{cases} \tag{9}$$

The simple form of $g(\cdot)$ in Eq. (9) in particular authorizes the explicit evaluation of the integrals (5) and (6) as follows:

$$E[y_i(t)|x] = \lambda + (-\lambda - x)F_\eta(-\lambda - x) - (\lambda - x)F_\eta(\lambda - x) - G_\eta(-\lambda - x) + G_\eta(\lambda - x), \quad (10)$$

and

$$E[y_i^2(t)|x] = \lambda^2 + (\lambda^2 - x^2)[F_\eta(-\lambda - x) - F_\eta(\lambda - x)] - 2x[G_\eta(-\lambda - x) - G_\eta(\lambda - x)] - H_\eta(-\lambda - x) + H_\eta(\lambda - x), \quad (11)$$

with the functions $G_\eta(u)=\int_{-\infty}^{u} v f_\eta(v)dv$ and $H_\eta(u)=\int_{-\infty}^{u} v^2 f_\eta(v)dv$.

Beyond the conditional moments $E[y_i(t)|x]$ and $E[y_i^2(t)|x]$ of Eqs. (10) and (11), it is the complete conditional probability density $p_{yi|x}(y,x)$ which is needed in the first place, so as to deduce the density $p_{y|x}(y,x)$, and then through Eq. (7) the density $p_y(y)$ giving way to J_y of Eq. (3). The whole process of calculating the density $p_y(y)$ and then J_y, can rarely be worked out completely analytically. A difficulty in the process lies in the completion of the N-fold convolution of $p_{yi|x}(y,x)$ to obtain $p_{y|x}(y,x)$. In the sequel, we will examine conditions that make this convolution tractable, and that subsequently provide insight into the behavior of J_y. Namely, we will consider the case $N=1$ where the need for this convolution vanishes, the case $N=2$ where the convolution can be performed explicitly analytically, and the case of large N where $p_{y|x}(y,x)$ is directly accessible through the central limit theorem without the need to explicitly perform the convolution.

A. Case $N=1$

In the simple case where $N=1$, the array degenerates into a single isolated sensor, with output $y(t)=y_1(t)=g[s_a(t)+\xi(t)+\eta_1(t)]$. This case is much similar to the one treated in [15]. We introduce $p_1(u)$ the probability density of $\xi(t)+\eta_1(t)$, which is the convolution $p_1(u)=f_\xi(u)*f_\eta(u)$, associated to the cumulative distribution $F_1(u)=\int_{-\infty}^{u} p_1(v)dv$. The output density $p_y(y)$ then follows:

$$p_y(y) = F_1[-\lambda - s_a(t)]\delta(y+\lambda) + \bar{p}_1[y - s_a(t)] + [1 - F_1[\lambda - s_a(t)]]\delta(y-\lambda), \quad (12)$$

where $\bar{p}_1[y-s_a(t)]$ coincides with $p_1[y-s_a(t)]$ for $y \in (-\lambda,\lambda)$ and is zero for y elsewhere. The resulting output Fisher information J_y defined in Eq. (3), and as also found in [15], is

$$J_y = \left[\frac{\partial s_a(t)}{\partial a}\right]^2 \times \left[\frac{p_1^2[-\lambda - s_a(t)]}{F_1[-\lambda - s_a(t)]} + \int_{-\lambda}^{\lambda} \frac{p'^2_1[y - s_a(t)]}{p_1[y - s_a(t)]}dy + \frac{p_1^2[\lambda - s_a(t)]}{1 - F_1[\lambda - s_a(t)]}\right]. \quad (13)$$

For instance, in the important case where $\xi(t)$ is zero-mean Gaussian with variance σ_ξ^2, if $\eta_1(t)$ is also zero-mean Gaussian with variance σ_η^2, then the density $p_1(u)$ is zero-mean Gaussian with variance $\sigma_\xi^2+\sigma_\eta^2$; for arbitrary $f_\xi(u)$, if $\eta_1(t)$ is uniform over $[-b,b]$, then $p_1(u)=[F_\xi(u+b)-F_\xi(u-b)]/(2b)$.

B. Case $N=2$

For the array size $N=2$, at fixed x, one has $y_1(t)=g[x+\eta_1(t)]$ and $y_2(t)=g[x+\eta_2(t)]$ which both distribute according to the conditional density

$$p_{yi|x}(y,x) = F_\eta(-\lambda - x)\delta(y+\lambda) + \bar{f}_\eta(y-x) + [1 - F_\eta(\lambda - x)]\delta(y-\lambda), \quad (14)$$

where $\bar{f}_\eta(y-x)$ coincides with $f_\eta(y-x)$ for $y \in (-\lambda,\lambda)$ and is zero for y elsewhere. The conditional density $p_{y|x}(y,x)$ of the output $y(t)=[y_1(t)+y_2(t)]/2$ will result through a y-wise convolution $p_{yi|x}(y,x)*p_{yi|x}(y,x)$ of the density of Eq. (14) followed by the rescaling corresponding to the factor $1/N=1/2$. This density $p_{y|x}(y,x)$ will be zero for y outside $[-\lambda,\lambda]$; it will have a probability mass with weight $F_\eta^2(-\lambda-x)$ at $y=-\lambda$, a probability mass with weight $2F_\eta(-\lambda-x)[1-F_\eta(\lambda-x)]$ at $y=0$, and a probability mass with weight $[1-F_\eta(\lambda-x)]^2$ at $y=\lambda$. In addition, for $y \in (-\lambda,0)$ one has

$$p_{y|x}(y,x) = 4F_\eta(-\lambda - x)f_\eta(2y+\lambda - x) + 2\int_{-\lambda}^{\lambda+2y} f_\eta(u-x)f_\eta(2y-u-x)du, \quad (15)$$

and for $y \in (0,\lambda)$,

$$p_{y|x}(y,x) = 4[1 - F_\eta(\lambda - x)]f_\eta(2y-\lambda - x) + 2\int_{-\lambda+2y}^{\lambda} f_\eta(u-x)f_\eta(2y-u-x)du. \quad (16)$$

The integrals in the right-hand side of Eqs. (15) and (16) are computable analytically in standard cases where $f_\eta(u)$ is Gaussian, or uniform as we will consider later.

Once $p_{y|x}(y,x)$ is known, the integrals on x of Eqs. (7) and (8) are perfomed (usually numerically), and then the output Fisher information J_y results from Eq. (3).

C. Case of N large

When N is large, thanks to the central limit theorem, the conditional density $p_{y|x}(y,x)$ can be approximated as follows. At this place we will assume that the density $f_\eta(u)$ is uniform over $[-b,b]$, with standard deviation $\sigma_\eta=b/\sqrt{3}$. This is a convenient choice that simplifies the calculations; it also bears on the status of the noises $\eta_i(t)$ which can be viewed as noises "freely" chosen and added for the operation of the array, meanwhile the input noise $\xi(t)$ should rather be considered as imposed by the physical world.

At fixed x verifying $x \leqslant -\lambda-b$, for any i one has $x+\eta_i(t) \leqslant -\lambda$ since $\eta_i(t)$ fluctuates in $[-b,b]$. In this case, $y_i(t)=g[x+\eta_i(t)]$ saturates at $-\lambda$, for any i, and the average of Eq. (2) yields $y=-\lambda$. Especially, in such x one has $E(y|x)=-\lambda$ and $\mathrm{var}(y|x)=0$ since all the $y_i(t)$ stick at $-\lambda$.

In a similar way, at fixed x verifying $x \geqslant \lambda+b$, for any i one has $x+\eta_i(t) \geqslant \lambda$ since $\eta_i(t)$ fluctuates in $[-b,b]$. In this case, $y_i(t)=g[x+\eta_i(t)]$ saturates at λ, for any i, and the average of Eq. (2) yields $y=\lambda$. Especially, in such x one has $E(y|x)=\lambda$ and $\mathrm{var}(y|x)=0$ since all the $y_i(t)$ stick at λ.

In between, when $-\lambda-b<x<\lambda+b$, the y_i's distribute in $[-\lambda,\lambda]$ with for each y_i a nonzero probability of avoiding both $\pm\lambda$. Therefore, their average y of Eq. (2), thanks to the central limit theorem, gets normally distributed with mean $E(y|x)\in(-\lambda,\lambda)$ and variance $\text{var}(y|x)>0$, i.e., according to the Gaussian

$$\varphi(y,x)=\frac{1}{\sqrt{2\pi\,\text{var}(y|x)}}\exp\left(-\frac{[y-E(y|x)]^2}{2\,\text{var}(y|x)}\right). \tag{17}$$

From Eq. (2) one also has $E(y|x)=E(y_i|x)$, and $\text{var}(y|x)=\text{var}(y_i|x)/N$ with $\text{var}(y_i|x)=E(y_i^2|x)-E^2(y_i|x)$, which are both known through Eqs. (10) and (11).

The conditional density $p_{y|x}(y,x)$ can therefore be expressed

$$p_{y|x}(y,x)=\begin{cases}\delta(y+\lambda) & \text{for } x\leqslant-\lambda-b,\\ \varphi(y,x) & \text{for } -\lambda-b<x<\lambda+b,\\ \delta(y-\lambda) & \text{for } x\geqslant\lambda+b.\end{cases} \tag{18}$$

Performing on Eq. (18) the integration of Eq. (7) leads to

$$p_y(y)=F_\xi[-\lambda-b-s_a(t)]\delta(y+\lambda)+\Phi(y)+[1-F_\xi[\lambda+b-s_a(t)]]\delta(y-\lambda), \tag{19}$$

with the function $\Phi(y)$ defined

$$\Phi(y)=\int_{-\lambda-b}^{\lambda+b}\varphi(y,x)f_\xi[x-s_a(t)]dx. \tag{20}$$

This function $\Phi(y)$ is nonzero essentially over a support limited to $y\in[-\lambda,\lambda]$, and $\Phi(y)$ rapidly vanishes when y departs outside $[-\lambda,\lambda]$. This is due to the Gaussian integrand $\varphi(y,x)$ in Eq. (20), which, for any $x\in(-\lambda-b,\lambda+b)$, concentrates over a domain in y centered at $E(y|x)\in(-\lambda,\lambda)$ with an extension measured by $\text{var}(y|x)$ which is small at large N.

The Fisher information J_y defined in Eq. (3) then follows as such:

$$J_y=\left[\frac{\partial s_a(t)}{\partial a}\right]^2\times\left[\frac{f_\xi^2[-\lambda-b-s_a(t)]}{F_\xi[-\lambda-b-s_a(t)]}+\int_{-\lambda}^{\lambda}\frac{[\partial\Phi(y)/\partial s_a]^2}{\Phi(y)}dy+\frac{f_\xi^2[\lambda+b-s_a(t)]}{1-F_\xi[\lambda+b-s_a(t)]}\right]. \tag{21}$$

Before explicitly studying the evolutions of the Fisher information J_y in various conditions of operation of the array, it is interesting to visualize specific behaviors of J_y in some special limit conditions.

D. Limit behaviors

At the limit where the array noises $\eta_i(t)$ vanish, then at given x the dispersion $\text{var}(y|x)$ goes to zero and y tends to exactly match $E(y|x)$ to give

$$y=E(y|x)=\begin{cases}-\lambda & \text{for } x\leqslant-\lambda,\\ x & \text{for } -\lambda<x<\lambda,\\ \lambda & \text{for } x\geqslant\lambda,\end{cases} \tag{22}$$

and a conditional density $p_{y|x}(y,x)=\delta[y-E(y|x)]$. The density $p_y(y)$ resulting from Eq. (7) is then

$$p_y(y)=F_\xi[-\lambda-s_a(t)]\delta(y+\lambda)+\bar{f}_\xi[y-s_a(t)]+[1-F_\xi[\lambda-s_a(t)]]\delta(y-\lambda), \tag{23}$$

where $\bar{f}_\xi[y-s_a(t)]$ coincides with $f_\xi[y-s_a(t)]$ for $y\in(-\lambda,\lambda)$ and is zero for y elsewhere. The resulting output Fisher information J_y defined in Eq. (3) is

$$J_y=\left[\frac{\partial s_a(t)}{\partial a}\right]^2\times\left[\frac{f_\xi^2[-\lambda-s_a(t)]}{F_\xi[-\lambda-s_a(t)]}+\int_{-\lambda}^{\lambda}\frac{f_\xi'^2[y-s_a(t)]}{f_\xi[y-s_a(t)]}dy+\frac{f_\xi^2[\lambda-s_a(t)]}{1-F_\xi[\lambda-s_a(t)]}\right]. \tag{24}$$

Equation (24) can be found both as the limit of Eq. (21) when $b\to0$ or as the limit of Eq. (13) as $p_1(u)\to f_\xi(u)$ when $\sigma_\eta\to0$. Equation (24) is independent of the type of the array noises $\eta_i(t)$ which vanish, and also independent of the size N of the array, since with no array noises $\eta_i(t)$ all the sensors in the array respond in unison as a single one. The limit of the array behavior expressed by Eq. (24) in fact recaps the Fisher information at the output of a single sensor when input with $s_a(t)+\xi(t)$ and no added noise.

Another interesting limit is at nonzero σ_η when $N\to\infty$. In this condition, the width $\text{var}(y|x)=\text{var}(y_i|x)/N$ goes to zero and $\varphi(y,x)$ of Eq. (17) tends to the Dirac delta $\delta[y-E(y|x)]$. The resulting change in $\Phi(y)$ of Eq. (20) then impacts Fisher information J_y of Eq. (21) in which the central integral in the right-hand side becomes

$$\int_{-\lambda}^{\lambda}\frac{[\partial\Phi(y)/\partial s_a]^2}{\Phi(y)}dy=\int_{-\lambda-b}^{\lambda+b}\frac{f_\xi'^2[u-s_a(t)]}{f_\xi[u-s_a(t)]}du, \tag{25}$$

all the rest in J_y of Eq. (21) being unchanged.

The three special configurations (i) $N=1$ at any σ_η where J_y is given by Eq. (13), (ii) $\sigma_\eta=0$ at any N where J_y is given by Eq. (24), (iii) $N=\infty$ at any σ_η where J_y is given by Eqs. (25) and (21), are three configurations where the dispersion $\text{var}(y|x)$ in the array vanishes identically. As a result, for given x, the output y becomes the deterministic function $y=E(y|x)=E(y_i|x)$ as given by Eq. (5), returning a deterministic value $y\in[-\lambda,\lambda]$ for any x. Moreover, for any $g(u)$ in Eq. (5) that saturates at $\pm\lambda$ when $u\to\pm\infty$, this deterministic function $E(y|x)=E(y_i|x)$ also saturates at $\pm\lambda$ when $x\to\pm\infty$. Because of this deterministic dependence $y=E(y|x)$ at the array output with zero dispersion $\text{var}(y|x)$, the added array noises $\eta_i(t)$ no longer play a role in dispersing the response y. The array as a whole behaves as a deterministic device, with a deterministic transfer characteristic given by $y=E(y|x)$, with its saturations at $\pm\lambda$. We are back to the case of a deterministic sensor characteristic with saturation, as considered in [15]. In this case, for the same reasons as given

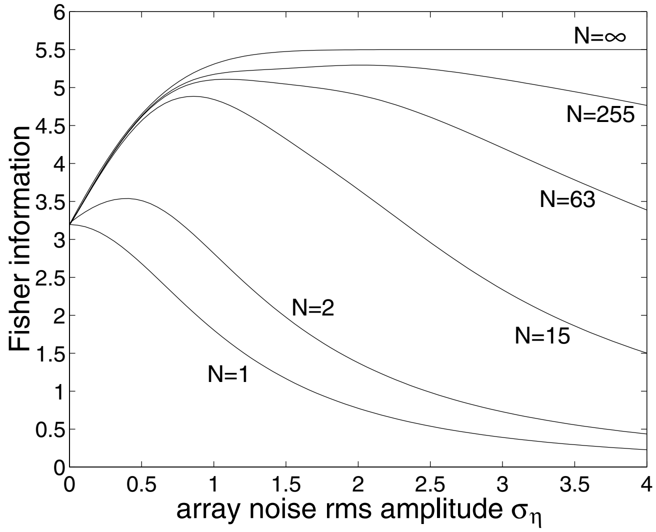

FIG. 1. Fisher information J_y at the output of the array of size N and $\lambda=1$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$. The input signal is $s_a(t)=a\sin(2\pi t/T_s)$ with $a=2$, embedded in zero-mean Gaussian input noise $\xi(t)$ with rms amplitude $\sigma_\xi=1$. A number $M=10$ observations $y(t_j)$ are made at times $t_1=\Delta t$ to $t_{10}=10\Delta t$ with $\Delta t=T_s/(M+1)$.

in [15], by inspection of Eqs. (13), (24), (25), and (21), what matters for the Fisher information J_y is the position in the sensor characteristic of the saturations at $\pm\lambda$, but not the smooth evolution of the characteristic between the saturations. In other words, the value of J_y will be the same for any saturating characteristic $g(\cdot)$ provided the saturations of $g(u)$ occur at the same levels $\pm\lambda$ for the same values of the input argument u. The smooth part of $g(\cdot)$ between the saturations, be it linear as in Eq. (9) or arbitrary curvilinear as in [15], will not affect J_y. On the contrary, in configurations differing from the three above, with nonvanishing dispersion $\mathrm{var}(y|x)$, it is expected that J_y, for instance as it will result from Eq. (21), will not be independent of the shape of the smooth part of the sensor characteristic between its saturations.

IV. NOISE-ENHANCED FISHER INFORMATION

We are now in a position to study the evolutions of the Fisher information J_y at the output of the array. We will consider the task of estimating the amplitude a of a sinusoidal input of known period T_s, i.e., $s_a(t)=a\sin(2\pi t/T_s)$. This input signal $s_a(t)$ is embedded in zero-mean Gaussian input noise $\xi(t)$. The resulting array output $y(t)$ is observed at M distinct times t_j, for $j=1$ to M. Since all the noises $\xi(t)$ and $\eta_i(t)$ are white, it results that the M data points $y(t_j)=y_j$ are independent. In this case the Fisher information is additive [1], and the total Fisher information about a contained in the complete data set $\mathbf{y}=(y_1,y_2,\dots y_M)$, call it $J_{\mathbf{y}}$, is the sum of the Fisher information J_{y_j} contained in every individual observation y_j, i.e., $J_{\mathbf{y}}=\sum_{j=1}^{M}J_{y_j}$. Figure 1 presents evolutions of the Fisher information $J_{\mathbf{y}}$ for arrays of various sizes N.

In the conditions of Fig. 1, the input sinusoidal signal $s_a(t)=a\sin(2\pi t/T_s)$ is centered in the linear part of the characteristic $g(\cdot)$ of Eq. (9); also $s_a(t)$ has a large amplitude $a>\lambda$, and it operates (on occasion) the characteristic $g(\cdot)$ in its saturating part. As a result of the centering of $s_a(t)$ in the linear part of $g(\cdot)$, with a single sensor (at $N=1$ in Fig. 1) there is no benefit gained by adding noise. This is shown in Fig. 1 by the monotonic decay of the Fisher information $J_{\mathbf{y}}$ as the level σ_η of the array noise is raised, at $N=1$. This behavior of the Fisher information was previously reported in [15] for isolated saturating sensors. It is in fact characteristic of a standard form of stochastic resonance, where improvement by noise occurs for a signal which is ill-positioned in relation to an isolated transmitting nonlinearity. The added noise, somehow, serves to displace the input signal into a more favorable operating zone of the nonlinearity. In Fig. 1 at $N=1$, the input signal $s_a(t)$ centered in the linear part of $g(\cdot)$ is not in an ill position that would allow improvement by the addition of noise, and stochastic resonance does not arise. However, the picture is quite different in a genuine array at $N>1$. In this case in Fig. 1, the same input signal $s_a(t)$ optimally centered in the linear part of $g(\cdot)$ can benefit from the addition of noise in the array. This is expressed in Fig. 1 at $N>1$, by the nonmonotonic evolutions of the Fisher information $J_{\mathbf{y}}$ as the level σ_η of the array noises $\eta_i(t)$ is raised, with $J_{\mathbf{y}}$ culminating at a maximum for a nonzero optimal level of the added array noises. As soon as an array size $N\geqslant 2$, the Fisher information $J_{\mathbf{y}}$ is always increased by addition of the array noises $\eta_i(t)$ above its value with no array noises at $\sigma_\eta=0$ in Fig. 1. The improvement by the added array noises gets more efficient as the array size N increases, as visible in Fig. 1. This is here a distinct mechanism of stochastic resonance or improvement by noise. It does not require an ill-positioned input signal in relation to a transmitting nonlinearity. It operates with optimally positioned signals at the input. The benefit brought in by the added noises is specifically an array effect, which does not take place under this form in isolated nonlinearities, and whose efficacy increases with the array size. This array effect, qualitatively, can be related to increased variability producing enhanced representation capability, when the sensors are replicated into an array with added noises. With no added noises $\eta_i(t)$ in the array, all the sensors deliver an identical output $y_i(t)$ in response to a common input $x(t)$, and the array is therefore equivalent to a single isolated sensor. The added array noises $\eta_i(t)$ force each sensor to deliver a distinct output $y_i(t)$ in response to a common input $x(t)$, and this increased variability translates into a possibility of enhancement of the Fisher information at the array output, as demonstrated here.

Figure 2 presents a similar situation with the sinusoidal input $s_a(t)=a\sin(2\pi t/T_s)$ centered in the linear part of $g(\cdot)$ but with an amplitude a which is only slightly above the saturation level λ that saturates $g(\cdot)$ of Eq. (9). In Fig. 2 the important behaviors observed in Fig. 1 are preserved: improvement by addition of the array noises $\eta_i(t)$ does not occur in an isolated sensor ($N=1$), but always takes place in arrays at $N>1$.

Further, Fig. 3 shows a situation qualitatively distinct in that the input sinusoid $s_a(t)=a\sin(2\pi t/T_s)$ permanently evolves in the linear part of the sensor $g(\cdot)$ thanks to its amplitude $a<\lambda$. Figure 3 shows that in this case also, im-

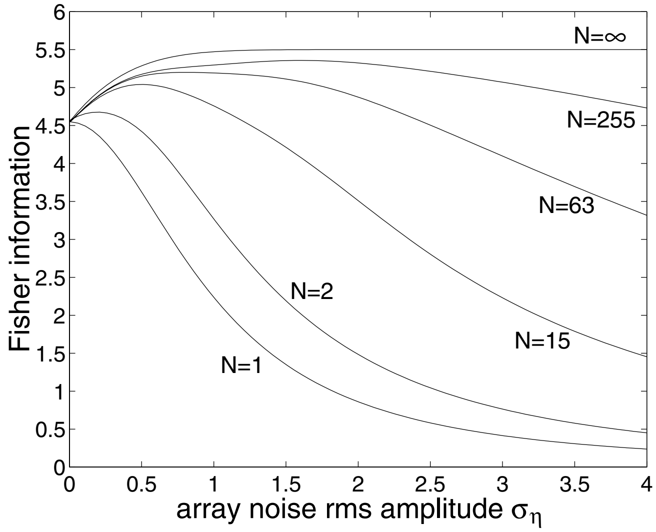

FIG. 2. Same as in Fig. 1 except for $a=1.1$.

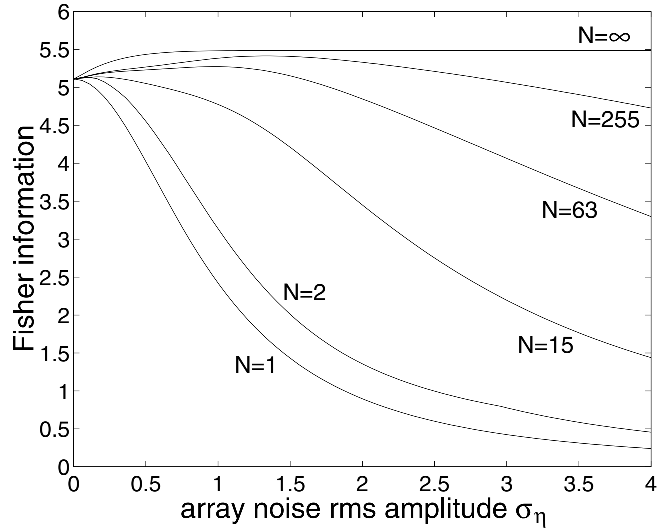

FIG. 4. Same as in Fig. 1 except for $a=0.3$.

provement by noise of the Fisher information J_y is possible in the array at $N>1$, and not possible in the isolated sensor at $N=1$. Although here $a<\lambda$, the input signal-noise mixture $a\sin(2\pi t/T_s)+\xi(t)$ still operates on occasion, because of $\xi(t)$, the nonlinearity $g(\cdot)$ in its saturating part. This nonlinear behavior is sufficient for the stochastic resonance to take place, as revealed in Fig. 3. But still some form of nonlinearity is required, because strictly linear sensors, obtained for instance as $\lambda\to\infty$ in Eq. (9), would lead to the array output $y(t)=s_a(t)+\xi(t)+N^{-1}\Sigma_{i=1}^{N}\eta_i(t)$, a purely additive signal-noise mixture allowing no possibility of improvement by noise of $J_{\mathbf{y}}$.

As the amplitude a of the input sinusoid $s_a(t)$ is further reduced below λ, Fig. 4 shows that the same type of stochastic resonance still takes place, with $J_{\mathbf{y}}$ improved by noise in genuine arrays at $N>1$ and not in an isolated sensor at $N=1$. However, as a is being reduced in relation to λ, the nonlinear saturating effect of the sensors gets less and less pronounced. As a consequence, the nonlinear effect of stochastic resonance, giving way to improvement by noise of $J_{\mathbf{y}}$, is also less and less pronounced, as visible in the sequence of Figs. 1–4.

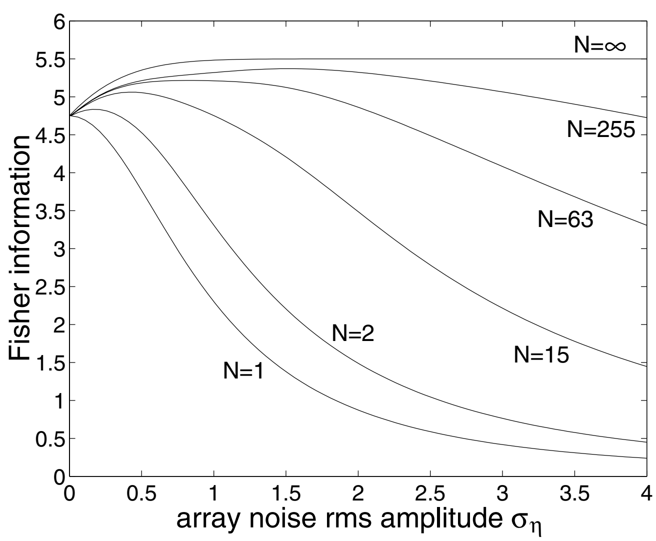

FIG. 3. Same as in Fig. 1 except for $a=0.9$.

For the input signal-noise mixture $x(t)=s_a(t)+\xi(t)$, it is possible to compute the input Fisher information J_x contained in $x(t)$ about a. This is done through a definition similar to Eq. (3) with the probability density $p_y(y)$ replaced by the density $p_x(x)=f_\xi[x-s_a(t)]$, giving

$$J_x=\left[\frac{\partial s_a(t)}{\partial a}\right]^2\int_{-\infty}^{+\infty}\frac{f'^2_\xi[u-s_a(t)]}{f_\xi[u-s_a(t)]}du. \qquad (26)$$

For the conditions of Figs. 1–4, with Gaussian input noise $\xi(t)$ and $s_a(t)=a\sin(2\pi t/T_s)$, Eq. (26) yields $J_x=\sin^2(2\pi t/T_s)/\sigma_\xi^2$ for a single measurement at time t on $x(t)$. For the M independent measurements $x(t_j)=x_j$, the total Fisher information in the data $\mathbf{x}=(x_1,x_2,\ldots x_M)$ is the sum $J_{\mathbf{x}}=\Sigma_{j=1}^{M}J_{x_j}=\Sigma_{j=1}^{M}\sin^2(2\pi t_j/T_s)/\sigma_\xi^2$. This gives $J_{\mathbf{x}}=5.5$ in the conditions of Figs. 1–4. This $J_{\mathbf{x}}$ is the Fisher information that would be accessible if the input $x(t)=s_a(t)+\xi(t)$ were directly (linearly) observable, or observable through an invertible (possibly nonlinear) sensor characteristic, because as shown for instance in [15] an invertible characteristic conserves the Fisher information. However, here we assume that $x(t)$ has to be observed through sensors with a noninvertible saturation modeled by $g(\cdot)$ of Eq. (9). In this case, a single sensor with no added array noise, provides an output Fisher information $J_{\mathbf{y}}$ which is always below the input Fisher information $J_{\mathbf{x}}$, as visible in Figs. 1–4 at $\sigma_\eta=0$. This is due to the noninvertible nature of the sensor characteristic $g(\cdot)$ of Eq. (9) which inherently entails a loss of Fisher information. This loss of information, seen in Figs. 1–4 at $\sigma_\eta=0$, gets more and more important as the input signal $s_a(t)=a\sin(2\pi t/T_s)$, with increasing a, operates more and more the nonlinearity $g(\cdot)$ in its saturating part. The interesting property is that the array with added noises $\eta_i(t)$ allows one to recover this loss of Fisher information. In large arrays, as visible at large N in

Figs. 1–4, it is even possible to recover completely the whole input Fisher information $J_{\mathbf{x}}$. At $N\to\infty$, the output Fisher information J_y behaves according to Eq. (25), and in this condition, for a sufficient level of the added noises $\eta_i(t)$ [for b large in Eq. (25)], the output Fisher information J_y from Eqs. (21) and (25) tends to the input Fisher information J_x of Eq. (26). So when saturating sensors like Eq. (9) have to be used to observe a signal, when used in isolation such sensors always entail a loss of Fisher information, but replication of them into a parallel array allows one to recover, possibly completely, this loss of Fisher information, thanks to the action of added noises.

V. TWO MECHANISMS OF ENHANCEMENT BY NOISE

The previous section addressed the case of a periodic input signal $s_a(t)$ which is by itself exactly centered in the linear part of the saturating characteristic $g(\cdot)$ of Eq. (9), sometimes with excursion at some t into the saturations of $g(\cdot)$ at $\pm\lambda$ as in Figs. 1 and 2, or sometimes completely maintained for all t inside the linear part of $g(\cdot)$ as in Figs. 3 and 4. In this configuration of $s_a(t)$, it was shown that improvement by noise of the Fisher information always takes place in genuine arrays at $N>1$ and does not occur in isolated sensors at $N=1$.

We now consider the case of a periodic input signal $s_a(t)$ which by itself permanently evolves completely in the saturation region of the sensor $g(\cdot)$ of Eq. (9). This is for instance achieved by $s_a(t)=S_0+a\sin(2\pi t/T_s)$ with an offset S_0 which is high enough to realize $S_0-a>\lambda$. In such a configuration, the periodic modulation of $s_a(t)$ around S_0 (the amplitude a of which modulation we seek to estimate) is completely invisible at the output of the sensor. In this case, the Fisher information J_y contained in the sensor output about a is strictly zero. Then, with an isolated sensor, noise which may add to $s_a(t)$ will have the ability to provoke excursions of the signal-plus-noise mixture back into the linear part of the sensor. This will make the periodic modulation visible at the sensor output, and will translate into an output Fisher information J_y raising above zero, thanks to the presence of the noise. This form of stochastic resonance in isolated saturating sensors has been reported and analyzed in [15]. It is also visible here in Fig. 5 at $N=1$. With no added array noise $\eta_1(t)$, at $\sigma_\eta=0$ in Fig. 5 at $N=1$, it is the presence of the input noise $\xi(t)$ which renders the input signal $s_a(t)$ visible at the sensor output. This leads to a nonzero output Fisher information J_y at $\sigma_\eta=0$ in Fig. 5 at $N=1$. When the input noise $\xi(t)$ is not at its optimal level of efficacy, the added array noise $\eta_1(t)$ cooperates with $\xi(t)$ to further enhance the stochastic resonance effect. This leads, in Fig. 5 at $N=1$, to an output Fisher information J_y which increases as σ_η is raised above zero, up to a maximum of J_y when the optimal amount of the noise $\xi(t)+\eta_1(t)$ is reached.

The type of enhancement by noise observed in Fig. 5 at $N=1$ resorts to a classic form of stochastic resonance, where an input signal, ill-positioned in relation to a transmitting nonlinearity, receives assistance from noise for a more efficient transmission by the isolated sensor. Yet, this form of

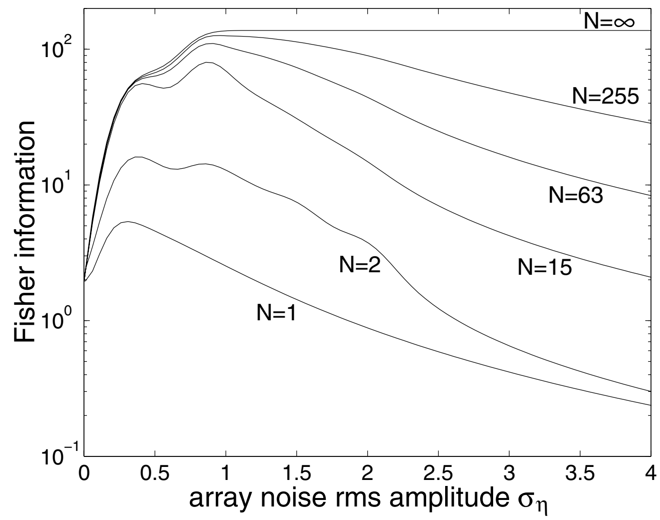

FIG. 5. Fisher information J_y at the output of the array of size N and $\lambda=1$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$. The input signal is $s_a(t)=2+a\sin(2\pi t/T_s)$ with $a=0.5$, embedded in zero-mean Gaussian input noise $\xi(t)$ with rms amplitude $\sigma_\xi=0.2$. A number $M=10$ observations $y(t_j)$ are made at times $t_1=\Delta t$ to $t_{10}=10\Delta t$ with $\Delta t=T_s/(M+1)$.

stochastic resonance in isolated sensors, can be supplemented by another form, when the sensors are replicated into parallel arrays. This is what is shown in Fig. 5 at $N>1$. The output Fisher information J_y can be further enhanced in arrays by addition of the array noises $\eta_i(t)$. This is the distinct form of stochastic resonance at $N>1$, also reported in Sec. IV, but which here coexists with the classic form at $N=1$. Two distinct mechanisms of improvement by noise are thus at work in Fig. 5, characterizing two possible forms of stochastic resonance.

VI. ESTIMATION ON NONSINUSOIDAL SIGNAL

Many reported examples of stochastic resonance involve a sinusoid in noise, which motivated here the analysis of the behavior of the Fisher information in this circumstance. However, the theoretical derivation of Secs. II and III is not restricted to a sinusoidal $s_a(t)$, and it applies equally to any type of input signal $s_a(t)$, with any parametric dependence on a.

For illustration, we consider the array of saturating sensors for estimation of a parameter of an exponential pulse. Figure 6 shows results for estimation of the amplitude a of the exponential pulse $s_a(t)=a\exp(-t/T_s)$ for $t\geqslant 0$ with known time constant T_s, while Fig. 7 shows results for estimation of the damping exponent a of the exponential pulse $s_a(t)=A\exp(-at)$ for $t\geqslant 0$ with known amplitude A. The results of both Figs. 6 and 7 confirm that the performance in estimation measured by the Fisher information J_y can always be enhanced by addition of noise in arrays of size $N>1$. In Figs. 6 and 7 noise enhancement of J_y takes place for a pulse amplitude both above or below λ, i.e., for an exponential pulse falling in the linear or in the saturating part of the sensors. Moreover in Figs. 6 and 7 at $N=63$ at the optimal

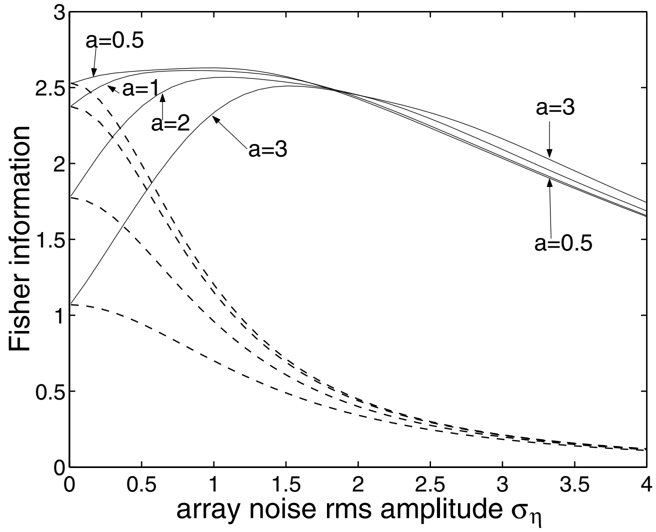

FIG. 6. Fisher information J_y at the output of the array of size N and $\lambda=1$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$. The input signal is $s_a(t)=a\exp(-t/T_s)$ embedded in zero-mean Gaussian input noise $\xi(t)$ with rms amplitude $\sigma_\xi=1$. A number $M=10$ observations $y(t_j)$ are made at times $t_j=2T_s(j-1)/(M-1)$ for $j=1$ to M. The dashed lines are for $N=1$ and the solid lines for $N=63$. For each given a the two associated curves at $N=1$ and $N=63$ connect at $\sigma_\eta=0$. The input Fisher information is $J_{\mathbf{x}}=\Sigma_{j=1}^{M}\exp(-2t_j/T_s)/\sigma_\xi^2\approx 2.75$.

level of the array noises $\eta_i(t)$, the maximum of J_y is close to the input Fisher information $J_{\mathbf{x}}$, and it can in principle be made arbitrarily close to $J_{\mathbf{x}}$ by increasing the array size N as in Figs. 1–5.

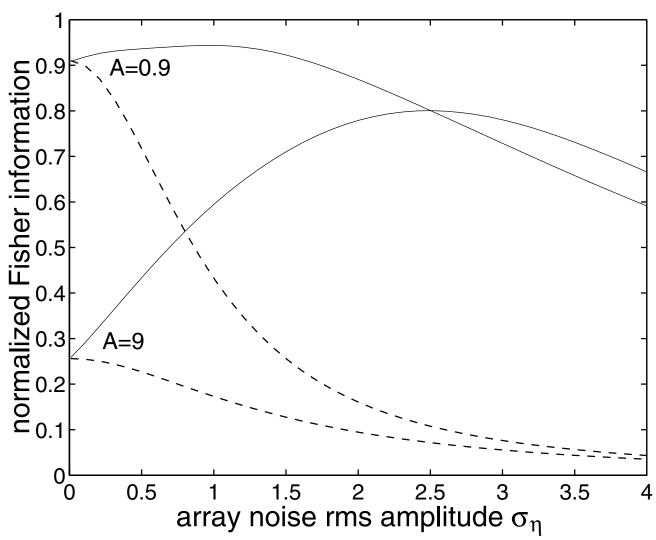

FIG. 7. Normalized Fisher information J_y/A^2 at the output of the array of size N and $\lambda=1$, as a function of the rms amplitude σ_η of the array noises $\eta_i(t)$. The input signal is $s_a(t)=A\exp(-at)$ with $a=1$, embedded in zero-mean Gaussian input noise $\xi(t)$ with rms amplitude $\sigma_\xi=1$. A number $M=10$ observations $y(t_j)$ are made at times $t_j=2(j-1)/(M-1)$ for $j=1$ to M. The dashed lines are for $N=1$ and the solid lines for $N=63$. For each given A the two associated curves at $N=1$ and $N=63$ connect at $\sigma_\eta=0$. The input Fisher information is $J_{\mathbf{x}}=A^2\Sigma_{j=1}^{M}t_j^2\exp(-2at_j)/\sigma_\xi^2\approx 0.99A^2$.

Application of the present theory shows that the stochastic resonance in the saturating arrays is largely preserved for any type of parameter estimation on any $s_a(t)$. This outcome can be attributed to the form of the Fisher information J_y exhibited for instance in Eq. (21). In J_y, from Eq. (21), the influence of the specific type of $s_a(t)$ is essentially conveyed by the prefactor $[\partial s_a(t)/\partial a]^2$. Meanwhile, the influence of the added noises $\eta_i(t)$ in the nonlinear array (those are responsible for the stochastic resonance) is conveyed by the second factor of the right-hand side of Eq. (21), in which factor $s_a(t)$ has a role reduced to a fixed constant. For this reason, improvement of the Fisher information by the noises $\eta_i(t)$, as governed by the second factor of Eq. (21), can be expected to take place in a comparable way, irrespective of the type of $s_a(t)$.

In view of the results of Figs. 6 and 7, it is plausible that multiple parameter estimation from the output of the array of saturating sensors could also be enhanced by addition of the array noises $\eta_i(t)$. This could be addressed through the computation of the Fisher information matrix for a measure of estimation efficacy [1,2], in a multidimensional extension of the derivation of Secs. II and III.

VII. DISCUSSION

We have applied the theory derived in Secs. II and III to demonstrate the enhancement by noise of the Fisher information in arrays of saturating nonlinearities. For illustration in Secs. IV and V, this was done for the Fisher information related to the amplitude of a sinusoidal input signal. This condition authorizes a comparison of the present results with those of [22] which deal with the behavior of the signal-to-noise ratio of a sinusoidal signal transmitted by similar arrays of saturating sensors with added noises. For a sinusoidal signal, the signal-to-noise ratio of [22] characterizes the efficacy in the detection of the sinusoid that could be made in the frequency domain by a narrow bandpass filter for instance; the Fisher information here, fundamentally, characterizes the efficacy in the estimation of the amplitude of the sinusoid. These are two distinct signal processing tasks, assessed by two specific measures, but when signals are processed through saturating sensors like $g(\cdot)$ of Eq. (9), both measures can always be enhanced by the replication of the sensors into parallel arrays with added noises. This confirms the general character of stochastic resonance in arrays.

Yet, a significant difference exists with the results of [22]. The signal-to-noise ratio in [22] of a sinusoid in Gaussian noise, can be increased by the parallel array of saturations, above the signal-to-noise ratio that would be afforded by the direct linear measurement of the input signal-noise mixture, in place of its measurement constrained to occur through the saturating sensors. Achievement in [22] of the maximum amplification of the signal-to-noise ratio however, requires a tunable array, with a specific λ matched to the amplitude of the sinusoidal input. By contrast here, the Fisher information J_y at the output of the array of saturations can always be increased above the Fisher information at the output of a single saturating sensor, but J_y cannot be increased above the input Fisher information J_x that would be afforded by the direct linear measurement of the input signal-noise mixture $x(t)=s_a(t)+\xi(t)$. At best, as seen in Figs. 1–5, J_y reaches J_x in large arrays, but never exceeds J_x. Achievement here of this best performance however, does not require any specific tuning of the array and is reachable at arbitrary λ by using sufficiently large arrays at sufficiently large levels of the added noises. For the Fisher information, the action of the arrays with added noises can in fact be viewed as to transform sensors with inherent unavoidable saturations into an equivalent purely linear sensing device with infinite input range.

Another important point to discuss is the noninvertible nature of the sensor characteristic $g(\cdot)$ considered here. Noninvertibility is required for the stochastic resonance measured by the Fisher information to arise; and it does not arise with invertible devices. This is the same with isolated sensors, as already pointed out in [15], and with arrays as considered here. The reason is that the integral, of the form of Eq. (3), defining the Fisher information, is invariant under any invertible change of variable. However, for sensors with saturation, which is a common behavior in practice, the modeling with an invertible saturation may be a somewhat formal hypothesis not always reflecting correctly the physical behavior of the device. Let us consider an invertible saturation model under the form $g(u)=\lambda \tanh(u/\lambda)$ in place of the noninvertible model of Eq. (9), fed at its input by an offset sinusoid $s_a(t)=S_0+a\sin(2\pi t/T_s)$ as in Fig. 5. With $S_0 \gg \lambda > a$, the input sinusoid is at the output completely "squashed" in the vicinity of λ. The input sinusoidal modulation is then completely squeezed on the output about λ, over an arbitrarily small interval as the offset S_0 grows. At some point, in practice, the sinusoidal modulation will become invisible on the output. However, when the saturation is modeled as invertible, the Fisher information for the modulation amplitude a is unaffected in the process, and it keeps the same value at the input and at the output, whatever the position of the offset S_0. In practice, there will always be a physical limit of resolution on the output, enabling one to distinguish or not, a small sinusoidal modulation about the saturation level λ. The invertible model of saturation is unable to represent this inherent resolution limit, and the unavoidable loss of information associated with it. Meanwhile, the noninvertible model of Eq. (9) offers a natural representation of this resolution limit, and for this it may constitute a more appropriate physical model for sensors with saturation. With this choice, the reduction of the Fisher information is captured when the sensors operate in the region of the saturation, and our present results show that this reduction of information can always be compensated by replicating the sensors into arrays with added noises.

The present results show that the improvement by noise of the Fisher information, and hence of the estimation efficacy, can occur in the array for any position of the input signal in relation to the sensor characteristic. In particular, the input signal can be optimally centered in relation to the nonlinear sensor characteristic, and even in this configuration, replication of the sensors into an array with added noise always leads to an improvement. This is in marked contrast with stochastic resonance in isolated nonlinearities, where improvement by noise generally occurs only for ill-positioned inputs, for instance an input lying below a threshold as in [25] or masked by a saturation as in [15]. In fact, one can view at the root two distinct mechanisms of improvement by noise, one mechanism specifically applies to assist ill-positioned inputs, the other is a nonlinear-array effect based qualitatively on enriched representation and that can occur for any input signal. In appropriate configurations, both mechanisms can take place in conjunction, as illustrated by Fig. 5. Also, one can note that the present arrays, like those of [16,17,24], are uncoupled arrays, where no coupling is necessary between the nonlinear devices in order to yield improvement of the collective response from the added array noises. This is in contrast with other forms of stochastic resonance in arrays, with other types of nonlinear devices, where coupling between devices, under various forms, is an essential ingredient of the process [8–10,26–31].

Beyond the context of physical sensors, the present results can also bear significance for signal processing by neurons. Neurons inherently display saturation in their output activity, parallel arrays are a common organization in neural structures, and they are commonly exposed to various sources of noise or external or internal origins. The present form of stochastic resonance in the saturating arrays we studied here, could constitute an additional form of stochastic resonance available in neuronal processes for noise-assisted information processing.

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