accounts for the conditional probabilities of going from X_i to X_j in one time step so that its powers can be used to compute the auto-correlation function [3]. For an (n, t)-tailed shift we have

$$\mathbf{K} = \left[\begin{array}{c|c} 0 & t^{-1} \\ \hline (n-t)^{-1} & 0 \end{array}\right] \quad \mathbf{K}^k = \frac{1}{n}\left[\begin{array}{c|c} 1-h^{k-1} & 1-h^{k-2} \\ \hline 1-h^k & 1-h^{k-1} \end{array}\right]$$

where the upper left blocks are $t \times (n - t)$ and $h = -t/(n - t)$. As the probability of being in a given X_i is $1/n$, indicating with $Q(X_j)$ the value of Q for all the points in X_j (i.e. -1 for $j \le n/2$ and $+1$ for $j > n/2$) we have

$$A_k = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} Q(X_i)Q(X_j)\mathbf{K}_{ij}^k = h^k$$

For any given n, this trend approximates the optimal choosing t as the integer closest to $nr/(1 + r)$. The accuracy of this approximation obviously increases as $n \to \infty$. Since $r \simeq 0.2679$, for $n = 10$ we have $t = 2$ and the approximation is with $h = -0.25$. Simulations of the resulting sequences for $U = 10$ users are reported in Fig. 2, in which the theoretical optimum, the theoretical performance of the exponential approximation and the theoretical performance of purely random sequences are reported for $2 \le N \le 128$.

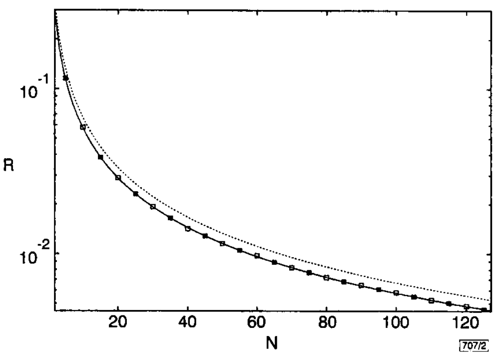

Fig. 2 *Performance of chaos-based spreading compared with optimal and random spreading*

——— optimal
■ exponential approximation
·········· random
□ simulation

Note how the exponential auto-correlation of chaos-based spreading results in a performance improvement with respect to purely random spreading. Such an improvement leads to an extremely good approximation of the maximum achievable performance of any system which employs second-order stationary sequences and for which the SGA is valid.

20 April 1999
Electronics Letters Online No. 19990754
DOI: 10.1049/el:19990754

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Noise-assisted propagation over a nonlinear line of threshold elements

F. Chapeau-Blondeau

The propagation of a periodic wave over a nonlinear line of two-state threshold elements is considered. A theoretical model and an experimental realisation confirm that the propagation of a low-amplitude wave can be improved by the addition of noise. This represents a new instance of the nonlinear phenomenon of stochastic resonance for signal enhancement by noise.

Stochastic resonance is a nonlinear phenomenon in noise-enhanced signal transmission that has been reported, in different forms, in a growing variety of systems, including electronic circuits, optical devices, and neurons [1]. Essentially, stochastic resonance has been reported to act as a noise enhancement factor in the input-output transmission of a signal applied to individual systems. Only recently has stochastic resonance been extended to coupled systems where the signal is applied only to one of them, and where the effect is interpreted as a noise-enhanced propagation of the signal among the coupled systems [2 – 4]. Such conditions may have relevance for wave propagation among nonlinear cells such as neurons or over excitable or nonlinear media such as those supporting solitons. In the few studies that have appeared [2 – 4], numerical simulation or experiments are usually resorted to, since the nonlinear systems used to exhibit a noise-enhanced propagation are sufficiently complicated to hinder an exact theoretical analysis. Here we introduce a nonlinear line of threshold elements that we demonstrate is amenable to a complete theoretical analysis and which lends itself to a direct electronic implementation, establishing one of the simplest conceivable settings for noise-enhanced propagation.

A line is formed by cascading nonlinear cells consisting of two-state comparators with threshold θ. The input end of the line is fed by a coherent T_s-periodic signal $s(t)$. The cell $n \ge 1$ receives at its input the sum $y_{n-1}(t) + \eta_n(t) = x_n(t)$ where $\eta_n(t)$ is the local noise (independent from cell to cell) on cell n, and $y_{n-1}(t)$ is the output signal from cell $n - 1$ except for the first cell $n = 1$ for which the input $y_0(t) \equiv s(t)$. Each cell $n \ge 1$ produces a binary output $y_n(t) = 1$ if $x_n(t) > \theta$, and $y_n(t) = -1$ if $x_n(t) \le \theta$.

The output $y_n(t)$ results as a random signal which bears some correlation with the T_s-periodic input $s(t)$ propagating down the noisy line. We apply and extend for this nonlinear propagation the theory of [5, 6] for stochastic resonance in a single static nonlinearity. Strictly speaking, the output $y_n(t)$ is a cyclostationary random signal with period T_s. As a result, the power spectral density of $y_n(t)$ is formed [1, 5] by spectral lines at integer multiples of $1/T_s$ emerging out of a broadband continuous-noise background. At the output of cell n, a standard signal-to-noise ratio (SNR) R_n is defined [1, 5] as the power contained in the coherent spectral line at $1/T_s$ divided by the power contained in the noise background in a small frequency band ΔB around $1/T_s$; it takes the form [5]

$$R_n = \frac{|\langle E[y_n(t)]\exp(-i2\pi t/T_s)\rangle|^2}{\langle \mathrm{var}[y_n(t)]\rangle \Delta t \Delta B} \quad (1)$$

with the time average $\langle ... \rangle = T_s^{-1}\int_0^{T_s} ... \, dt$. Also Δt is the time resolution of the measurement (i.e. the signal sampling period in a discrete-time implementation). For the two-state threshold comparator here, we have, for the output expectation

$$E[y_n(t)] = 1 - 2\Pr\{y_n(t) = -1\} \quad (2)$$

and for the output variance

$$\mathrm{var}[y_n(t)] = 4\Pr\{y_n(t) = -1\}[1 - \Pr\{y_n(t) = -1\}] \quad (3)$$

To compute the SNR at cell R_n n with eqns. 1 – 3, we need to relate both $E[y_n(t)]$ and $\mathrm{var}[y_n(t)]$ to the coherent input $s(t)$ to the line and to the properties of the noise sources down the line.

The input $x_n(t)$ to cell n is a random signal for which the cumulative distribution function $F_{x_n} = \Pr\{x_n(t) \le u\}$ verifies

$$F_{x_n}(u)=\Pr\{y_{n-1}=-1\}F_{\eta_n}(u+1)+\Pr\{y_{n-1}=1\}F_{\eta_n}(u-1) \quad (4)$$

where $F_{\eta_n}(u)$ is the cumulative distribution of the noise $\eta_n(t)$. The binary ouput $y_n(t)$ thus occurs with probabilities $\Pr\{y_n(t) = -1\} = \Pr\{x_n(t) \le \theta\} = F_{x_n}(\theta)$ and $\Pr\{y_n(t) = 1\} = 1 - F_{x_n}(\theta)$. Owing to eqn. 4, we can write

$$\Pr\{y_n=-1\}=F_{x_n}(\theta)=F_{x_{n-1}}(\theta)Q_n(\theta)+F_{\eta_n}(\theta-1) \quad (5)$$

with $Q_n(\theta) = F_{\eta_n}(q+1) - F_{\eta_n}(\theta-1)$. Applying a chain rule, we can obtain $\Pr\{y_n(t) = -1\}$ as a function of $F_{x_1}(\theta) = F_{\eta_1}[\theta-s(t)]$ and of the $F_{\eta_j}(\theta+1)$ and $F_{\eta_j}(\theta-1)$ values for j =2 to n. For simplicity, we choose to write this expression in the case where the noise sources $\eta_j(t)$, although independent, share the same cumulative distribution $F\eta(u)$, yielding

$$\Pr\{y_n=-1\}=F_\eta[\theta-s(t)]Q^n(\theta)+F_\eta(\theta-1)\frac{1-Q^{n-1}(\theta)}{1-Q(\theta)} \quad (6)$$

with $Q(\theta) = F_\eta(\theta + 1) - F_\eta(\theta - 1)$.

Eqn. 6, together with eqns. 1 – 3, provides an explicit expression for the SNR R_n at cell n, for a T_s-periodic input $s(t)$ with arbitrary waveform feeding the line corrupted by noise with arbitrary distribution $F_\eta(u)$.

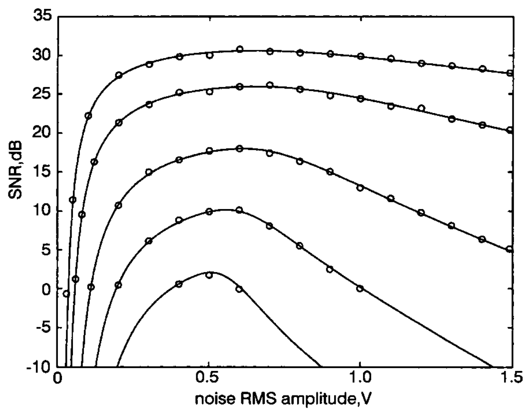

Fig. 1 *Output SNR R_n at cell n against (zero-mean Gaussian) noise RMS amplitude*

——— theoretical SNR R_n from eqns. 1 – 3
○ experimental results

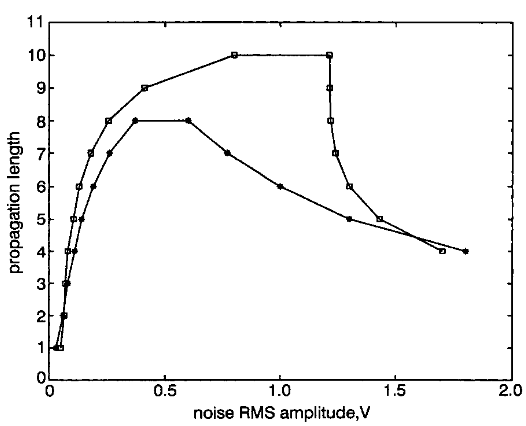

Fig. 2 *Propagation length against noise RMS amplitude over line with threshold θ = 1.1 V*

∗ $s(t) = A\cos(2\pi t/T_s)$ with A = 1V, and zero-mean Gaussian noise
□ $s(t)$ square wave half period at 1V and half period at –1V, and zero-mean uniform noise

A line has been realised with each nonlinear comparator implemented by an operational amplifier. The details of the experiment for the line reproduce the conditions of [7] for a single comparator. By adjusting the operational amplifiers and their coupling resistances, we have set the two levels of the signals y_n at ±1V. We had $\Delta t \Delta B = 10^{-4}$ for eqn. 1. The ouput SNR can be experimentally evaluated at each cell down the line, and compared to the theoretical predicition from eqns. 1 – 3. Fig. 1 shows this comparison when the coherent input is $s(t) = A\cos(2\pi t/T_s)$ with A = 1V and the threshold θ = 1.1V. In this case, the coherent signal $s(t)$ and every ouput y_n = ±1V are below the threshold θ, and no propagation can take place in the absence of noise, this translating into a zero SNR R_n at every cell n. When noise is added over the line, a co-operative effect takes place by which the noise assists the signal in overcoming the threshold, thus allowing the propagation of the coherent signal down the line. This is a stochastic resonance effect under the form of a noise-assisted propagation. This translates, as is visible in Fig. 1, into a SNR R_n which can be maximised at every cell n by a optimal nonzero noise level. If ever the noise is set to zero at any cell n then the propagation is blocked at this cell.

Another characterisation of the noise-assisted propagation that can be deduced from SNR curves such as those of Fig. 1 is the evaluation of a propagation length defined as the index n of the remotest cell that can be reached before the SNR R_n drops below a given reference level, say 0dB. Such a propagation length is shown in Fig. 2, under various conditions, as a function of the level of the noise over the line. Again, we observe in Fig. 2 that the propagation length goes to zero at zero noise, and that there exists an optimal nonzero noise level at which the propagation length is maximised.

The model reveals how stochastic resonance in propagation is preserved over many conditions for the periodic waveform $s(t)$ and the noise distribution. This work is the first to establish a noise-enhanced propagation based on an exact theoretical model and its experimental realisation. It constitutes a unique framework for further studies on stochastic resonance and its applications in nonlinear processing.

5 May 1999
Electronics Letters Online No: 19990713
DOI: 10.1049/el:19990713

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