Rényi entropy measure of noise-aided information transmission in a binary channel

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This paper analyzes a binary channel by means of information measures based on the Rényi entropy. The analysis extends, and contains as a special case, the classic reference model of binary information transmission based on the Shannon entropy measure. The extended model is used to investigate further possibilities and properties of stochastic resonance or noise-aided information transmission. The results demonstrate that stochastic resonance occurs in the information channel and is registered by the Rényi entropy measures at any finite order, including the Shannon order. Furthermore, in definite conditions, when seeking the Rényi information measures that best exploit stochastic resonance, then nontrivial orders differing from the Shannon case usually emerge. In this way, through binary information transmission, stochastic resonance identifies optimal Rényi measures of information differing from the classic Shannon measure. A confrontation of the quantitative information measures with visual perception is also proposed in an experiment of noise-aided binary image transmission.

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I. INTRODUCTION

Information measures are important both for the foundation of information sciences and for practical applications of information processing. For measuring information, a general approach is provided in a statistical framework based on the information entropy introduced by Shannon [1]. As a measure of information, the Shannon entropy satisfies some desirable axiomatic requirements and also it can be assigned operational significance in important practical problems, for instance, in coding and telecommunication [2]. For information measure, an extension to the Shannon entropy has been proposed with the Rényi entropy [3]. As an extension, the Rényi entropy satisfies a set of axiomatic requirements [3], slightly less restrictive than the Shannon case, but still preserving suitability as a natural measure of information. The Rényi entropy offers a parametric family of measures, from which the Shannon entropy is accessible as a special case. On the operational side, it is appropriate to test the Rényi entropy as a general parametric approach for the measure of information in concrete practical processes. In this respect, it is the special Shannon case of the measure that has been most largely involved in practice. Beyond, it remains interesting and useful to identify and investigate the possibilities of assigning some specific role, significance, or properties to the Rényi entropy outside the special Shannon case. In this direction, the present paper proposes an analysis, with Rényi-entropy-based measure of information, of a basic process of information transmission with a binary channel. This analysis is a natural extension of, and contains as a special case, the classic reference model of information transmission formed by the binary channel quantified with Shannon-entropy-based measure [2]. In addition, in the analysis of the binary channel based on the Rényi entropy, we will specifically examine the possibility of a stochastic resonance phenomenon or a regime of information transmission aided by noise.

Stochastic resonance, in a broad acceptation, designates situations where the noise can play a constructive role in transmission or processing of information [4–6]. Stochastic resonance has been reported to operate in many different areas, including electronic circuits [7–9], neuronal processes [10–13], nonlinear sensors [14–16], and optical devices [17–19]. For example, the currently vibrant state of the topic of stochastic resonance in relation to neuroscience can be illustrated in recent works [20–24]. Stochastic resonance is usually identified by a relevant measure of performance which culminates (resonates) at a maximum for a nonzero optimal amount of noise. Stochastic resonance was first demonstrated in the transmission of a periodic signal aided by noise. In such situation, the standard measure of performance is a signal-to-noise ratio defined in the frequency domain [4,25,26]. Later, stochastic resonance was extended to noise-aided transmission of aperiodic signals, assessed by cross-correlation measures [27–29]. Situations of noise-aided signal detection or estimation were also reported and characterized by appropriate performance measures, such as a probability of detection [5,30,31], an estimation error [32–34], or a Fisher information [35–37] improvable by the noise. Owing to their status of general information measures, information-theoretic quantities based on the Shannon entropy were also employed to characterize stochastic resonance. Situations were reported of mutual information [38–41] or information capacity [42–44] improved by noise. A natural extension is to test the ability of general information measures based on the Rényi entropy for the characterization of stochastic resonance. A study is proposed on this issue in the present paper. We will demonstrate that Rényi-entropy-based measures are capable of registering noise-aided transmission in an information channel. Moreover, when seeking the information measures benefiting from the largest amount of noise, we will show that nontrivial optimal orders emerge for the Rényi entropy, which differ from the special order defining the Shannon entropy.

A previous study appeared in [45] associating the Rényi entropy and stochastic resonance. The problem addressed in [45] is, for a deterministic signal in additive noise, to estimate the signal-to-noise ratio classically defined as the ratio of the signal power to the noise power. For this purpose, Ref.

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[45] proposed a scheme exploiting trajectories in symbolic dynamics and measure of their complexity by running cylinder or word entropies. From time averages on such entropies, an estimator is suggested in [45] for the signal-to-noise ratio. The estimator based on the Shannon entropy is proved in [45] to converge, for a sinusoidal signal in noise, asymptotically at large signal-to-noise ratio. An improvement factor is defined by the derivative of the estimator with respect to the signal-to-noise ratio and [45] finds that there is a maximum of the improvement factor at intermediate noise levels, which is interpreted as a form of stochastic resonance. When the scheme is used with the Rényi entropy, Ref. [45] reports a shift of the maximum of the improvement factor toward lower noise amplitudes. By contrast, our present study does not deal with the problem of estimating the signal-to-noise ratio of a sinusoid in noise, but it realizes the analysis of an information channel.

In the present paper, we will first review definitions and basic properties of Rényi entropy based information measures. We will especially present a practical problem, in source coding, where a specific operational role is assigned to the Rényi entropy in a configuration differing from the traditional Shannon entropy. Next, we will present a binary information channel and its analysis based on the Rényi entropy. We will demonstrate stochastic resonance or noise-aided transmission in this information channel and optimality conditions that single out nontrivial orders for the Rényi entropy also differing from the traditional Shannon entropy. A confrontation of the quantitative information measures with visual perception will also be proposed in an experiment of noise-aided binary image transmission. The results of the paper seek to contribute in two directions: to enlarge the vision of the capabilities of Rényi-entropy-based information measures and to consolidate stochastic resonance as a universal phenomenon characterizable with general information measures.

II. RÉNYI ENTROPY MEASURES AND PROPERTIES

A. Rényi entropy

From an alphabet of N symbols, an information source emits symbols independently with probabilities P_i, for $i=1$ to N. The Rényi entropy of the source is defined as [3]

$$H_\alpha(P_i) = \frac{1}{1-\alpha}\log\left(\sum_{i=1}^{N} P_i^\alpha\right), \tag{1}$$

for an order $\alpha \geq 0$. At the limit $\alpha=1$, the L'Hospital rule yields $H_1(P_i)=-\sum_{i=1}^N P_i \log(P_i)$, i.e., the Shannon entropy. For any order $\alpha \geq 0$, the Rényi entropy $H_\alpha(P_i)$ of Eq. (1) is nonnegative and it reaches its maximum $H_{\max}=\log(N)$ at equiprobability $P_i=1/N$ for all $i=1$ to N [46]. $H_\alpha(P_i)$ is concave $(\cap)$ for $0\leq\alpha\leq 1$ and pseudoconcave (a single maximum) for $1<\alpha$ [47]. For a given probability distribution, $H_\alpha(P_i)$ is a decreasing function of α [46], decaying from $H_0(P_i)=\log(N)=H_{\max}$ down to $H_\infty(P_i)=-\log[\max_i(P_i)]$. The Rényi entropy is additive for independent random variables.

For illustration, Fig. 1 shows the Rényi entropy $H_\alpha(P_i)$ of Eq. (1) for a binary source (of interest to us in the sequel).

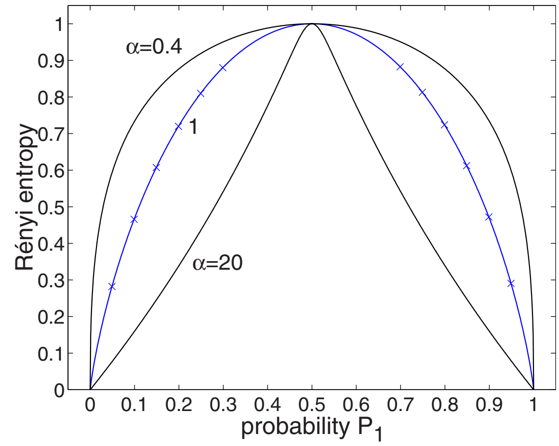

FIG. 1. (Color online) Rényi entropy $H_\alpha(P_i)$ of Eq. (1), as a function of the probability P_1 of a binary source $\{P_1, 1-P_1\}$, for three values of the order $\alpha=0.4$, $\alpha=20$, and $\alpha=1$ identified by crosses $(\times)$ corresponding to the Shannon entropy.

Especially, Fig. 1 depicts how the entropy $H_\alpha(P_i)$ loses its concavity $(\cap)$ at $\alpha>1$ while keeping a single maximum (pseudoconcave).

B. Rényi relative entropy

Associated with the Rényi entropy is the Rényi relative entropy or divergence, which refers to two probability distributions $\{P_i\}$ and $\{Q_i\}$, $i=1$ to N, over the same alphabet, and is defined as [3,46]

$$D_\alpha(P_i \,\|\, Q_i) = \frac{1}{\alpha-1}\log\left(\sum_{i=1}^{N} P_i^\alpha Q_i^{1-\alpha}\right). \tag{2}$$

At the limit $\alpha=1$, the L'Hospital rule yields $D_1(P_i\|Q_i)=\sum_{i=1}^N P_i \log(P_i/Q_i)$, i.e., the Kullback-Leibler relative entropy [2]. For any order $\alpha\geq 0$, the Rényi relative entropy $D_\alpha(P_i\|Q_i)$ of Eq. (2) is nonnegative and vanishes if and only if $P_i=Q_i$ for all $i=1$ to N [46].

For two given probability distributions $\{P_i\}$ and $\{Q_i\}$, $i=1$ to N, the Rényi relative entropy $D_\alpha(P_i\|Q_i)$ is an increasing function of α. At the limit $0\leftarrow\alpha$, one has $D_{0\leftarrow\alpha}(P_i\|Q_i)=\alpha D_1(Q_i\|P_i)$, meaning that the Rényi relative entropy $D_\alpha(P_i\|Q_i)$ reaches zero as a lower bound when $0\leftarrow\alpha$, with the limiting behavior controlled by the Kullback-Leibler relative entropy $D_1(Q_i\|P_i)$. This is in accordance with a general property of $D_\alpha(P_i\|Q_i)$ of Eq. (2) which for any $0\leq\alpha\leq 1$ verifies $(1-\alpha)D_\alpha(P_i\|Q_i)=\alpha D_{1-\alpha}(Q_i\|P_i)$. At the limit $\alpha\to\infty$, one has $D_{\alpha\to\infty}(P_i\|Q_i)=\log[\max_i(P_i/Q_i)]$ as the upper bound reached by $D_\alpha(P_i\|Q_i)$. By choosing the reference probabilities $\{Q_i\}$ as the uniform distribution $\{Q_i=1/N\}$ for all $i=1$ to N, one obtains

$$D_\alpha(P_i \,\|\, Q_i=1/N) = H_{\max} - H_\alpha(P_i), \tag{3}$$

expressing a connection between entropy and relative entropy at any Rényi order α.

C. Rényi transinformation

One now considers an input alphabet with N symbols, an output alphabet with M symbols, and over those two a joint probability distribution $\{P_{ij}\}$, for $(i,j)\in[1,N]\times[1,M]$, as would occur between the emitting and receiving ends of a communication channel. The N input symbols, indexed by i, have marginal probabilities $P_i=\Sigma_{j=1}^{M}P_{ij}$. The M output symbols, indexed by j, have marginal probabilities $Q_j=\Sigma_{i=1}^{N}P_{ij}$. A Rényi transinformation or mutual information follows as

$$I_\alpha = D_\alpha(P_{ij}\,\|\,P_iQ_j) = \frac{1}{\alpha-1}\log\left(\sum_{i=1}^{N}\sum_{j=1}^{M}P_{ij}^{\alpha}P_i^{1-\alpha}Q_j^{1-\alpha}\right). \tag{4}$$

In the case $\alpha=1$ the Rényi transinformation $I_{\alpha=1}$ from Eq. (4) is the Shannon transinformation. The limit behaviors of the Rényi relative entropy D_α indicated in Sec. II B give for the Rényi transinformation $I_{0\leftarrow\alpha}=\alpha D_1(P_iQ_j\|P_{ij})$ and $I_{\alpha\to\infty}=\log[\max_{i,j}(P_{ij}/P_iQ_j)]$.

D. Source coding with the Rényi entropy

We now describe a practical problem of source coding introduced in [48] and in the resolution of which the Rényi entropy emerges at an order α differing from the traditional Shannon entropy. The N symbols of a source alphabet are coded with an encoding alphabet of D characters. Symbol i, having probability P_i, is coded by a word with a length ℓ_i of D-ary characters, for $i=1$ to N. For a uniquely decipherable code, the lengths ℓ_i must satisfy [2,49] the Kraft inequality

$$\sum_{i=1}^{N}D^{-\ell_i}\le 1. \tag{5}$$

The traditional approach to optimal source coding [2,49] is to measure the elementary cost c_i of encoding symbol i directly by its code length $\ell_i=c_i$ and then to seek those lengths ℓ_i that minimize the average coding length $\Sigma_{i=1}^{N}P_i\ell_i$ while satisfying the constraint (5). The optimal lengths come out as $\ell_i=-\log_D(P_i)$, for all $i=1$ to N, and these achieve the minimum average coding length $-\Sigma_{i=1}^{N}P_i\log_D(P_i)$ which is the Shannon entropy of the source. For all other code lengths ℓ_i, the Shannon entropy forms a lower bound to the average coding length $\Sigma_{i=1}^{N}P_i\ell_i$.

A less traditional approach to optimal source coding [48] is to measure the elementary cost of encoding symbol i as $c_i=D^{\beta\ell_i}$, introducing a cost c_i which is an exponential function of the code length ℓ_i, with a parameter $\beta>0$ to have the cost c_i an increasing function of the length ℓ_i. The global cost of encoding the source is expressed by the average

$$C_\beta=\sum_{i=1}^{N}P_ic_i=\sum_{i=1}^{N}P_iD^{\beta\ell_i}. \tag{6}$$

Minimizing the average cost C_β of Eq. (6) is equivalent to minimizing the monotonic increasing function of C_β as

$$L_\beta=\frac{1}{\beta}\log_D\left(\sum_{i=1}^{N}P_iD^{\beta\ell_i}\right). \tag{7}$$

By measuring the coding performance with L_β of Eq. (7), the traditional approach can be recovered as a special case. At the limit $0\leftarrow\beta$ in Eq. (7), the L'Hospital rule yields $L_0=\Sigma_{i=1}^{N}P_i\ell_i$ which is the traditional average coding length. Also, at the limit $\beta\to\infty$, Eq. (7) yields $L_\infty=\max_i(\ell_i)$ putting all the weight on the longest code word. For intermediate values of β, the quantity L_β of Eq. (7) is interpretable as a generalized average coding length[1] of order β and is a nondecreasing function of β verifying $L_0\le L_\beta\le L_\infty$. As $0\leftarrow\beta$, the average coding length L_β tends to distribute the weights among the code words in proportion of their lengths, while as $\beta\to\infty$ more weight is put on the long code words.

Now this extended approach to optimal source coding [48] seeks those lengths ℓ_i that minimize the generalized average coding length L_β of Eq. (7) while satisfying the constraint (5). The optimal lengths come out as

$$\ell_i=-\log_D\frac{P_i^{\alpha}}{\displaystyle\sum_{j=1}^{N}P_j^{\alpha}}=-\alpha\log_D(P_i)+(1-\alpha)H_\alpha(P_i), \tag{8}$$

for all $i=1$ to N, with $\alpha=1/(\beta+1)$. And the optimal lengths ℓ_i of Eq. (8) achieve in Eq. (7) the minimum average coding length $H_\alpha(P_i)$ which is the Rényi entropy of order α of the source. For all other code lengths ℓ_i, the Rényi entropy $H_\alpha(P_i)$ at $\alpha=1/(\beta+1)$ forms a lower bound to the generalized average coding length L_β of Eq. (7). These results contain, as a special case, the traditional approach to optimal source coding, in the limit $0\leftarrow\beta$. This generalized approach to optimal coding finds application to minimize the probability of buffer overflow [51] or other design optimization in communication systems [52,53]. These results are further developed in [46,50,54]. They are especially interesting since arguably they represent, among the rare instances of this kind, the most simple and concrete, yet fundamental, situation where a special and operational role is assigned to the Rényi entropy at an order α differing from the traditional Shannon case $\alpha=1$. This is realized at the occasion of a source coding problem where the generalized (nonlinear) average coding length being optimized determines the nontrivial order $\alpha\ne 1$ of the Rényi entropy. In the sequel, we will present another process, under the form of a noise-aided information transmission over a binary channel, which also points to a nontrivial order $\alpha\ne 1$ for the Rényi entropy.

III. BINARY INFORMATION CHANNEL

An information channel emits input symbols X from the binary alphabet $\{0,1\}$. The successive input symbols are independent and identically distributed with the probabilities $P_1=\text{Prob}\{X=1\}$ and $P_0=1-P_1=\text{Prob}\{X=0\}$. At the receiving end of the channel, the output symbols Y are in the binary alphabet $\{0,1\}$. Transmission over the channel is character-

[1]Two interpretations are possible for Eqs. (6) and (7): an exponential elementary cost $c_i=D^{\beta\ell_i}$ associated with a conventional linear average in Eq. (6) [followed by the increasing transformation of Eq. (7) with no change to the minimizer] or a conventional linear elementary cost $c_i=\ell_i$ associated with a generalized [50] exponential average $\varphi^{-1}[\Sigma_iP_i\varphi(\ell_i)]$ realized by Eq. (7) with the strictly increasing function $\varphi(u)=D^{\beta u}$.

ized by the four conditional probabilities $P_{j|i}=\mathrm{Prob}\{Y=j|X=i\}$, for $(i,j)\in\{0,1\}^2$. The joint input-output probabilities result as $P_{ij}=P_{j|i}P_i$ and the output probabilities $Q_j=\mathrm{Prob}\{Y=j\}=\Sigma_{i=0}^{1}P_{j|i}P_i$. Thus in Eq. (4), one has $P_{ij}^{\alpha}P_i^{1-\alpha}=P_{j|i}^{\alpha}P_i$ and the input-output Rényi transinformation follows as

$$I_\alpha(X;Y) = \frac{1}{\alpha-1}\log\left(\sum_{j=0}^{1}(P_{j|0}^{\alpha}P_0 + P_{j|1}^{\alpha}P_1)Q_j^{1-\alpha}\right). \quad (9)$$

Equation (9) is the input-output Rényi transinformation for any (memoryless) binary channel characterized by the four transmission probabilities $P_{j|i}$. We now specify concrete physical conditions that determine a definite channel and its probabilities $P_{j|i}$. We consider the binary input X in the transmission corrupted by a white noise W to yield $X+W$ and then at the receiver $X+W$ is compared to a fixed decoding threshold θ to determine the binary output Y of the channel according to

$$\text{If } X+W>\theta \text{ then } Y=1, \quad \text{else } Y=0. \quad (10)$$

The noise W has the cumulative distribution function $F(w)=\mathrm{Prob}\{W\le w\}$. The input X and the noise W are statistically independent.

The input-output transmission probabilities of this binary channel are readily derived. For instance, the probability $P_{0|1}=\mathrm{Prob}\{Y=0|X=1\}$ is also $\mathrm{Prob}\{X+W\le\theta|X=1\}$ which amounts to $\mathrm{Prob}\{W\le\theta-1\}=F(\theta-1)$. With similar rules, one arrives at

$$P_{0|1} = \mathrm{Prob}\{Y=0|X=1\} = F(\theta-1), \quad (11)$$

$$P_{1|1} = \mathrm{Prob}\{Y=1|X=1\} = 1-F(\theta-1), \quad (12)$$

$$P_{0|0} = \mathrm{Prob}\{Y=0|X=0\} = F(\theta), \quad (13)$$

$$P_{1|0} = \mathrm{Prob}\{Y=1|X=0\} = 1-F(\theta). \quad (14)$$

These transmission probabilities $P_{j|i}$, $(i,j)\in\{0,1\}^2$, define an asymmetric binary channel. For this channel, a typical evolution of the Rényi transinformation $I_\alpha(X;Y)$ of Eq. (9) is shown in Fig. 2, with the binary input $X=0$ or 1 evolving on both sides of the decoding threshold $\theta=0.8$. In such condition, the presence of the channel noise W in Eq. (10) hinders the recovery of the information signal at the receiving end. It results that the performance of the transmission, as measured by the input-output Rényi transinformation $I_\alpha(X;Y)$, decreases as the level of noise increases, as visible in Fig. 2.

In Fig. 2, a similar decreasing evolution of the Rényi transinformation $I_\alpha(X;Y)$ as the level of noise increases is observed for any order α, especially, but not only, in the Shannon case $\alpha=1$. In this respect, this shows that the Rényi transinformation $I_\alpha(X;Y)$ at any order α is capable of manifesting the detrimental action of the noise in the transmission of information through the channel. We will now consider another regime of operation of the channel of Eq. (10) and show the possibility of a constructive action of the noise in the transmission of information assessed with the Rényi transinformation $I_\alpha(X;Y)$.

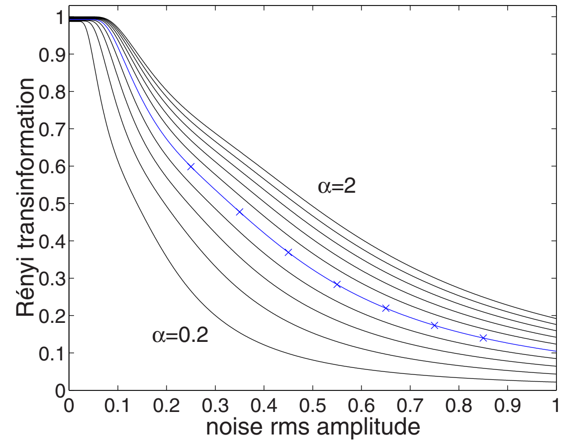

FIG. 2. (Color online) Input-output Rényi transinformation $I_\alpha(X;Y)$ from Eq. (9), as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with input probability $P_1=0.45$ and threshold $\theta=0.8$. The order α goes from 0.2 to 2 with step 0.2. The crosses ($\times$) identify $\alpha=1$ when $I_{\alpha=1}(X;Y)$ is the Shannon transinformation.

IV. NOISE-IMPROVED INFORMATION TRANSMISSION

A. Noise-improved Rényi transinformation

For an information channel with a decoding threshold $\theta=1.2$ in Eq. (10), the evolution of the input-output Rényi transinformation $I_\alpha(X;Y)$ from Eq. (9) is presented in Fig. 3. The results of Fig. 3 clearly demonstrate a nonmonotonic action of the noise. In the conditions of Fig. 3, the binary input X by itself is always below the response threshold $\theta=1.2$ on the output. As a consequence, in the absence of noise at $\sigma=0$ in Fig. 3, the channel output permanently remains at $Y=0$. No information is transmitted through the channel, as expressed by the Rényi transinformation $I_\alpha(X;Y)$ which stays at zero when $\sigma=0$ for any finite order α. However, as the noise level σ is progressively raised above zero in Fig. 3, a cooperative effect can take place, with the noise W assisting the subthreshold input X to overcome the response threshold θ. This elicits transitions in the output Y bearing statistical dependence with the input X. As a consequence, a nonzero input-output transmission of information occurs, as registered by the Rényi transinformation $I_\alpha(X;Y)$ which starts to increase in Fig. 3 as the noise level σ rises above zero. There exists a nonzero amount of noise for which the information transfer measured by $I_\alpha(X;Y)$ is maximized and such a maximum of $I_\alpha(X;Y)$ occurs for any finite order α as shown in Fig. 3. This is the effect of stochastic resonance or noise-aided information transmission, registered by the Rényi transinformation $I_\alpha(X;Y)$ in Fig. 3 at any finite order α.

In Fig. 3, the order $\alpha=1$ corresponds to the situation where $I_{\alpha=1}(X;Y)$ is the Shannon transinformation and $I_{\alpha=1}(X;Y)$ in Fig. 3 culminates at a maximum for a nonzero level of noise σ, as also reported in [42]. For the limit order $0\leftarrow\alpha$, the Rényi transinformation goes to zero as

$$I_{0\leftarrow\alpha}(X;Y) = \alpha D_1(P_iQ_j \,\|\, P_{ij}), \quad (15)$$

and meanwhile, $I_{0\leftarrow\alpha}(X;Y)$ of Eq. (15) keeps a maximum as a function of σ which is a maximum inherited from

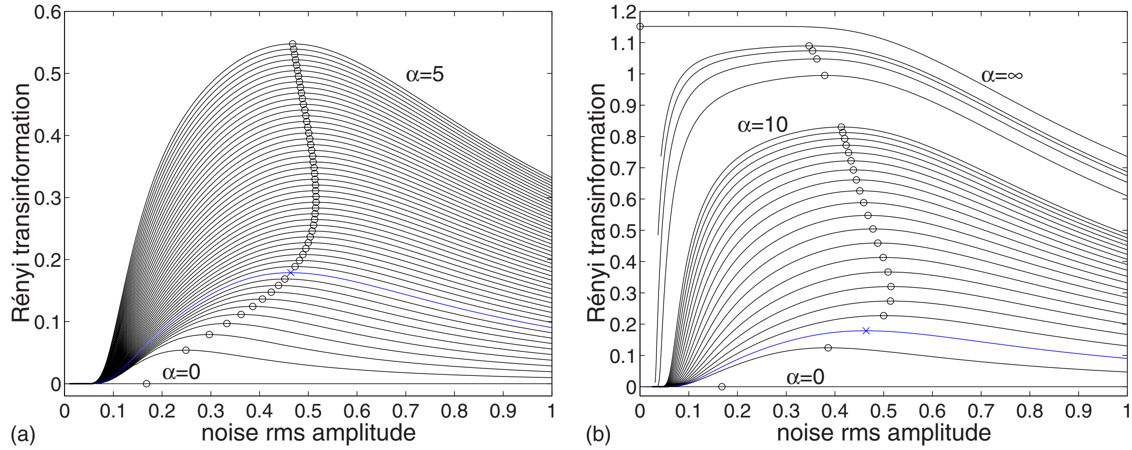

FIG. 3. (Color online) Input-output Rényi transinformation $I_\alpha(X;Y)$ from Eq. (9), as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with input probability $P_1=0.45$ and threshold $\theta=1.2$. On each curve, the maximum is indicated by a circle (○), except for $\alpha=1$ identified by a cross (×) when $I_{\alpha=1}(X;Y)$ is the Shannon transinformation. The order α goes from 0 to 5 with step 0.1 (panel A) and from 0 to 10 with step 0.5, then $\alpha=20$, 30, 40, 50, ∞ (panel B).

$D_1(P_iQ_j \| P_{ij})$. It is the location of this maximum, occurring also at a nonzero level of noise σ, which is indicated by a circle on the curve of $I_{\alpha=0}(X;Y)$ in Fig. 3.

For the limit order $\alpha \to \infty$, the Rényi transinformation behaves as $I_{\alpha\to\infty}(X;Y)=\log[\max_{i,j}(P_{j|i}/Q_j)]$. For the binary channel resulting from Eq. (10), we have $P_{0|0} \geq P_{0|1}$ and $P_{1|1} \geq P_{1|0}$, so that

$$\max_{i,j}(P_{j|i}/Q_j) = \max\left(\frac{P_{0|0}}{Q_0}, \frac{P_{1|1}}{Q_1}\right). \tag{16}$$

And from Eqs. (11)–(14), it comes

$$\max_{i,j}(P_{j|i}/Q_j) = \max\left(\frac{F(\theta)}{P_0F(\theta)+P_1F(\theta-1)}, \frac{1-F(\theta-1)}{P_0[1-F(\theta)]+P_1[1-F(\theta-1)]}\right). \tag{17}$$

In this way, Eq. (17) provides access to the Rényi transinformation $I_{\alpha\to\infty}(X;Y)=\log[\max_{i,j}(P_{j|i}/Q_j)]$ especially as a function of the noise level σ intervening through the cumulative distribution function $F(\cdot)$ of the channel noise W. In Eq. (17), the maximum in the right-hand side switches from one term to the other depending on σ (at given P_1 and θ). An evolution resulting for $I_{\alpha\to\infty}(X;Y)$ is depicted in Fig. 3(b). In the conditions of Fig. 3, at $P_1=0.45$ and $\theta>1$, the maximum in Eq. (17) is realized by the second term in the right-hand side to yield

$$I_{\alpha\to\infty}(X;Y) = \log\left(\frac{1-F(\theta-1)}{P_0[1-F(\theta)]+P_1[1-F(\theta-1)]}\right), \tag{18}$$

which forms a decreasing function of σ, depicted in Fig. 3(b). The maximum of $I_{\alpha\to\infty}(X;Y)$ occurs at $\sigma=0$ and comes out as $-\log(P_1)$ indicated in Fig. 3(b) as $-\log_2(0.45)=1.152$. The order $\alpha\to\infty$ is the only configuration where the maximum of $I_\alpha(X;Y)$ occurs at $\sigma=0$, while for any finite order α, the maximum of $I_\alpha(X;Y)$ occurs at a nonzero level σ of the channel noise. This is the manifestation of the stochastic resonance effect, taking place with any finite order α of the Rényi transinformation $I_\alpha(X;Y)$.

It is visible in Fig. 3 that the maximum of the Rényi transinformation $I_\alpha(X;Y)$ occurs at an optimal level σ_{opt} of the noise which varies with the Rényi order α. This variation of σ_{opt} with α is further analyzed in Fig. 4 for different values of the decoding threshold θ.

A remarkable property observed in Fig. 4 is that the optimal noise level σ_{opt} maximizing $I_\alpha(X;Y)$ experiences a nonmonotonic evolution with the order α. There exists in Fig. 4 an optimal value $\alpha_{\mathrm{opt}}=2.20$ of the Rényi order where σ_{opt} is maximized. At $\alpha_{\mathrm{opt}}=2.20$ in Fig. 4, the maximum reached by σ_{opt} is usually dependent on the decoding threshold θ and σ_{opt} increases as θ increases since more noise is required to assist the subthreshold input at higher threshold. Nevertheless, the nonmonotonic evolution of σ_{opt} maximized at α_{opt} in Fig. 4 demonstrates that the stochastic resonance effect selects a specific order α_{opt} of the Rényi transinformation $I_\alpha(X;Y)$. This optimal order α_{opt} identifies the Rényi transinformation $I_{\alpha_{\mathrm{opt}}}(X;Y)$ that is capable of drawing the most pronounced benefit of the added noise in stochastic resonance, since $I_{\alpha_{\mathrm{opt}}}(X;Y)$ stands as the measure of input-output infor-

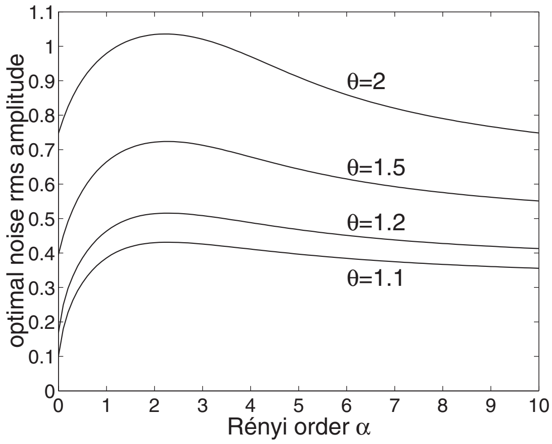

FIG. 4. Optimal noise rms amplitude σ_{opt} maximizing the input-output Rényi transinformation $I_\alpha(X;Y)$ from Eq. (9), as a function of the order α. The information channel is with input probability $P_1=0.45$, zero-mean Gaussian noise W, and threshold θ.

mation transfer that gets maximized at the largest optimal noise level σ_{opt}.

Additionally, another property also observed in Fig. 4 is a quasi-invariance of the optimal Rényi order α_{opt} with the decoding threshold θ. Due to the intricacy of the (nonlinear) dependencies involved in the Rényi transinformation $I_\alpha(X;Y)$, we were not able to obtain an analytical characterization of the theoretical relation implied between α_{opt} and θ. Alternatively, we have performed a numerical characterization which presents in Fig. 5 the optimal Rényi order α_{opt}, as a function of the input probability P_1, at different values of the decoding threshold θ. The results of Fig. 5 show that in the region around $P_1 \approx 0.5$, which corresponds to an approximately balanced binary input, almost no influence of the decoding threshold is observed on the optimal order α_{opt}. In such conditions, the optimal order α_{opt} appears intrinsic to the channel structure and independent of the decoding threshold θ, at least in the range tested in Fig. 5 for θ. These are the conditions that prevail in Fig. 4, at $P_1=0.45$, with no detected dependence of α_{opt} on θ. For small or large values

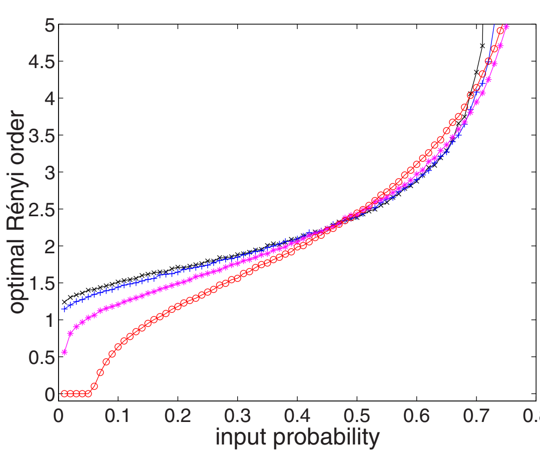

FIG. 5. (Color online) Optimal Rényi order α_{opt}, as a function of the input probability P_1, for the binary channel of Eq. (10) with threshold $\theta=1.1$ (×), 1.2(+), 1.5(∗), and 2 (○).

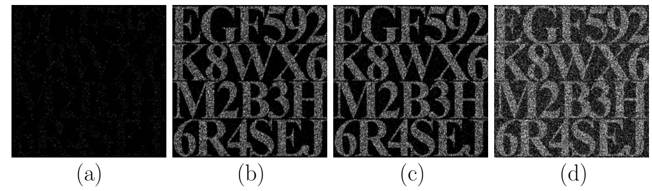

FIG. 6. Binary image Y at the output of an information channel according to Eq. (10). Black and white input image X has size 610×555 pixels, with the fraction $P_1=0.45$ of white pixels. The decoding threshold $\theta=1.2$. The channel noise W is zero-mean Gaussian with rms amplitude (a) $\sigma=0.1$, (b) $\sigma=0.46$ maximizing the Shannon transinformation $I_1(X;Y)$ in Fig. 3, (c) $\sigma=0.52$ maximizing the Rényi transinformation $I_{\alpha_{\mathrm{opt}}}(X;Y)$ at the optimal order $\alpha_{\mathrm{opt}}=2.20$ in Fig. 3, and (d) $\sigma=1$.

of P_1, far from the balanced case $P_1\approx 0.5$, a dependence of α_{opt} on θ gradually appears, as visible in Fig. 5. Nevertheless, this dependence, whenever it applies, does not critically impact the essential property of the existence of an optimal order α_{opt} selected by stochastic resonance in the operation of the binary channel.

It is also interesting to note that this optimal Rényi order α_{opt} selected by stochastic resonance at its maximum usually differs from the Shannon order $\alpha=1$. The Rényi transinformation $I_\alpha(X;Y)$ that best exploits the stochastic resonance is usually not the Shannon transinformation $I_1(X;Y)$. It is neither a degenerate configuration $I_{\alpha=0}(X;Y)$ nor $I_{\alpha=\infty}(X;Y)$ occurring as the limit of a monotonic evolution. It is a truly nontrivial instance $I_{\alpha_{\mathrm{opt}}}(X;Y)$ that usually emerges for the Rényi transinformation in the presence of stochastic resonance.

B. Image transmission

As another point of view complementing the quantification of stochastic resonance performed in Sec. IV A with the Rényi transinformation, a visual illustration of the noise-enhanced information transmission is proposed in an experiment with images. The early study of [55] reported stochastic resonance or improvement by noise in an experiment of visual perception of images, with an evaluation based on the psychovisual assessment performed by human subjects. We extend this approach with a similar experiment of image transmission, associated with an evaluation by quantitative information measures. We consider an experiment of binary image transmission. The binary input X is a black and white image where the probability $P_1=\mathrm{Prob}\{X=1\}$ is the fraction of white pixels. A noise W is added, followed by the threshold decoding according to Eq. (10) to reconstruct a black and white output image Y. Various output images Y are shown in Fig. 6 at different levels of the noise W on the transmission channel, in conditions matching those quantified in Figs. 3 and 4.

The nonmonotonic action of the noise is visually perceivable on the images of Fig. 6. For a subthreshold input image X in Fig. 6, when the level σ of the noise is too small as in Fig. 6(a), a poor reconstructed image Y is obtained at the output. For intermediate noise levels as in Figs. 6(b) and

6(c), there is a favorable action of the noise which assists the subthreshold input image to overcome the decoding threshold. This results in a much better quality of image reconstruction in Figs. 6(b) and 6(c) at the output, thanks to the action of the noise. Further, at still higher noise level as in Fig. 6(d), the noise gradually recovers its detrimental impact resulting in poorer quality of the reconstructed image Y at the output.

When improvement by noise of image transmission takes place in Figs. 6(b) and 6(c), the noise level is set successively at the maximum of the Shannon transinformation $I_1(X;Y)$ and of the Rényi transinformation $I_{\alpha_{\text{opt}}}(X;Y)$ at the optimal order $\alpha_{\text{opt}}=2.20$, as deduced from Figs. 3 and 4. The visual perception in Figs. 6(b) and 6(c) also records improved image transmission in these two configurations. This expresses that $I_1(X;Y)$ or $I_{\alpha_{\text{opt}}}(X;Y)$ are both acceptable measures to quantify the benefit from noise in image transmission as perceived visually and that also no one measure emerges against the other as being more suitable for this purpose. For noise-improved image transmission, other performance measures more specific to images have also been shown appropriate for quantifying stochastic resonance [56]. In this respect, the main point of the present study is to demonstrate the ability of a general information measure such as the Rényi transinformation to quantify stochastic resonance. And the present experiment with images shows that the optimal Rényi measure $I_{\alpha_{\text{opt}}}(X;Y)$ emerging with stochastic resonance is consistent with noise-improved image transmission as registered by visual perception.

C. Noise-improved Rényi information capacity

The optimal Rényi order α_{opt} selected by the stochastic resonance as discussed in Sec. IV A is usually related to a given information source characterized by the input probability P_1. This is for instance illustrated by the evolutions of α_{opt} with P_1 in Fig. 5. A point of view not impacted by this dependence with P_1 is accessible by considering the Rényi information capacity of the channel. The Rényi transinformation $I_\alpha(X;Y)$ observed in Figs. 3 and 4 assesses the transmission through the noisy channel of a binary input X with given probability $P_1=\text{Prob}\{X=1\}=1-\text{Prob}\{X=0\}$. The Rényi information capacity C_α is defined by seeking the value P_1^* of P_1 achieving the maximum C_α of $I_\alpha(X;Y)$ at given order α. For the optimal input probability P_1^*, an explicit analytical expression exists in the special case $\alpha=1$ of the Shannon information capacity $C_{\alpha=1}$ of the asymmetric binary channel, taking the form [42]

$$P_1^* = \frac{AP_{0|0}-1}{A(P_{0|0}-P_{0|1})}, \tag{19}$$

with

$$A = 1+\exp\left[\frac{h(P_{0|0})+h(P_{1|0})-h(P_{1|1})-h(P_{0|1})}{P_{0|0}-P_{0|1}}\right], \tag{20}$$

with the function $h(u)=-u\ln(u)$. Then, the Shannon capacity $C_{\alpha=1}$ follows from Eq. (19) in $I_\alpha(X;Y)$ at $\alpha=1$. For an arbitrary order α other than $\alpha=1$, the optimal probability P_1^* usually needs to be computed numerically by maximizing $I_\alpha(X;Y)$ from Eq. (9). In this way, we have realized an evaluation of the optimal probability P_1^* in Fig. 7, and of the corresponding Rényi information capacity C_α in Fig. 8, for the binary channel according to Eq. (10).

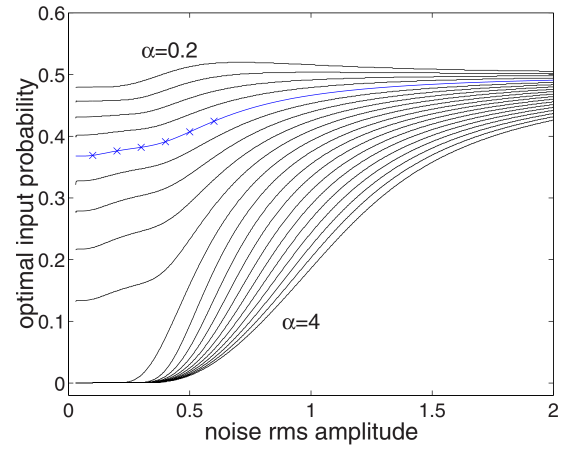

FIG. 7. (Color online) Optimal input probability P_1^* achieving the Rényi information capacity C_α, as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with decoding threshold $\theta=1.2$. The order α goes from 0.2 to 4 with step 0.2. Crosses ($\times$) identify $\alpha=1$ when P_1^* achieves the Shannon information capacity $C_{\alpha=1}$ according to Eq. (19).

In addition to the case $\alpha=1$ ruled by Eq. (19), analytical insight on the capacity C_α can also be obtained in two other limit cases for the Rényi order α. In the limit $0\leftarrow\alpha$, the Rényi transinformation $I_{0\leftarrow\alpha}(X;Y)$ of Eq. (15) is ruled by $D_1(P_iQ_j\|P_{ij})$ which takes its maximum at $P_1^*\approx 1/2$, especially $P_1^*\to 1/2$ when the noise level σ goes to zero or to infinity. In these conditions, the capacity $C_{0\leftarrow\alpha}$ as well as the Rényi transinformation $I_{0\leftarrow\alpha}(X;Y)$ of Eq. (15) go to zero. The evolutions to these limit behaviors for P_1^* and $C_{0\leftarrow\alpha}$ are

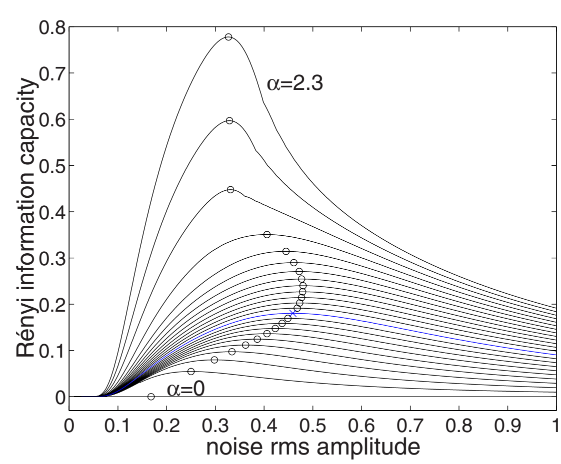

FIG. 8. (Color online) Input-output Rényi information capacity C_α, as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with decoding threshold $\theta=1.2$. On each curve, the maximum is indicated by a circle ($\bigcirc$), except for $\alpha=1$ identified by a cross ($\times$) when $C_{\alpha=1}$ is the Shannon information capacity. The order α goes from 0 to 2.3 with step 0.1.

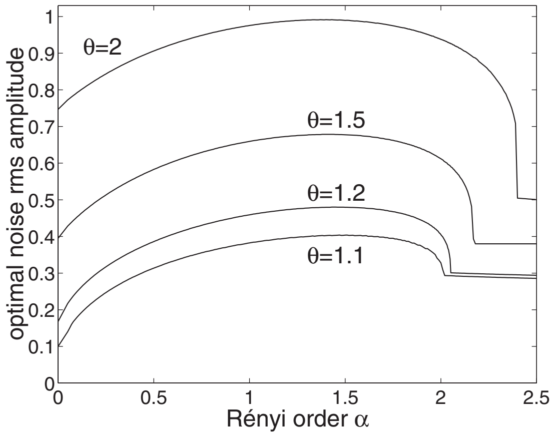

FIG. 9. Optimal noise rms amplitude σ_{opt} maximizing the Rényi information capacity C_α as a function of the order α. The information channel is with zero-mean Gaussian noise W and threshold θ.

discernable in Figs. 7 and 8, respectively. At the limit $\alpha \to \infty$, the Rényi transinformation $I_{\alpha\to\infty}(X;Y)$ from Eq. (17) is maximized by $P_1^* \to 0$ to yield the capacity $C_{\alpha\to\infty} = \log\{[1-F(\theta-1)]/[1-F(\theta)]\}$ which is the limit of Eq. (18) as $P_1^* \to 0$.

Compared to the Rényi transinformation $I_\alpha(X;Y)$ at fixed P_1 as in Fig. 3, the same remarkable properties related to stochastic resonance are observed for the Rényi information capacity C_α at the optimal P_1^* in Fig. 8. At any order α, the Rényi information capacity C_α undergoes a nonmonotonic evolution as the noise level σ increases. In the regime $\theta > 1$ of a subthreshold binary input X, when no noise is present, no information is transmitted, as marked by a vanishing Rényi capacity C_α at any order α when $\sigma = 0$ in Fig. 8. Adding noise then modifies the channel, in a way where the subthreshold input $X=1$ has more chance to get across and be correctly decoded as $Y=1$ at the output. This constructive action of the noise induces a capacity C_α rising above zero in Fig. 8. Moreover, a nonzero level of noise exists where the capacity C_α is maximized in Fig. 8. This is again a manifestation of the stochastic resonance with a Rényi information capacity C_α maximized at a nonzero optimal level of noise on the channel. Also, as in Fig. 3, the optimal noise level σ_{opt} maximizing C_α in Fig. 8 is found dependent on the Rényi order α, yet with a nonmonotonic dependence. This dependence of σ_{opt} with α is represented in Fig. 9.

The nonmonotonic dependence in Fig. 9 identifies an optimal Rényi order $\alpha_{\mathrm{opt}} = 1.44$ at which the optimal noise level σ_{opt} of stochastic resonance assumes its largest value. And this optimal order $\alpha_{\mathrm{opt}} = 1.44$ is found in Fig. 9 invariant with the decoding threshold θ of the channel. This invariance is at least observed, as in Fig. 4, in the absence of a theoretical proof, at the precision of our numerical analysis and for the "reasonable" range of θ tested in Fig. 9. However, independently of this invariance with the channel threshold θ, we have now a characterization of an optimal Rényi order α_{opt} which is not attached to a given information source via its P_1. The stochastic resonance selects a nontrivial Rényi order $\alpha_{\mathrm{opt}} = 1.44$ through the Rényi information capacity $C_{\alpha_{\mathrm{opt}}}$ that exploits stochastic resonance in the most pronounced way since $C_{\alpha_{\mathrm{opt}}}$ gets maximized by the largest possible optimal noise level σ_{opt}. The optimal Rényi order $\alpha_{\mathrm{opt}} = 1.44$ selected by stochastic resonance in the capacity is now intrinsic to the binary channel and insensitive to the input probability (and to the decoding threshold). And the optimal order α_{opt}, selected by stochastic resonance in the Rényi information capacity, differs from the Shannon order $\alpha = 1$.

V. DISCUSSION AND CONCLUSION

After reviewing basic properties of generalized information measures based on the Rényi entropy, we have applied these generalized measures for the analysis of a binary information channel. This analysis contains as a special case: the classic reference model of information transmission over a binary channel quantified with Shannon entropy based measure. The analysis with the Rényi entropy therefore offers an extended reference model useful to describe information transmission in broader conditions. We have exploited this extended model to investigate further possibilities and properties of stochastic resonance or noise-aided information transmission. The results demonstrate that stochastic resonance occurs and is registered by the Rényi entropy measures at any finite order α, including the Shannon order $\alpha = 1$. Moreover, in definite conditions when one seeks the Rényi information measure that best exploits stochastic resonance, i.e., the information measure that is maximized by the largest optimal amount of noise, then a nontrivial order α_{opt} differing from the Shannon case $\alpha = 1$ usually emerges. In this way, in binary information transmission over the channel, stochastic resonance selects a specific nontrivial Rényi measure of information differing from the classic Shannon measure.

The optimal Rényi information measure can be selected as the Rényi transinformation $I_{\alpha_{\mathrm{opt}}}(X;Y)$ associated with a given information source to be transmitted. In this case, the optimal order α_{opt}, such as the Rényi transinformation $I_\alpha(X;Y)$ itself, is usually dependent on the input probability P_1 of the binary information source applied to the channel. The optimal Rényi transinformation $I_{\alpha_{\mathrm{opt}}}(X;Y)$ is in this way optimal for transmission of a given information source. Such a dependence of the optimal Rényi order with the binary information source disappears when one turns to the Rényi capacity C_α of the channel. We have shown that the capacity C_α, as well as $I_\alpha(X;Y)$, is capable of registering a noise-aided information transmission at any finite order α. And in general, the optimal order selected for $I_{\alpha_{\mathrm{opt}}}(X;Y)$ or C_α by stochastic resonance differs from the Shannon order $\alpha = 1$.

In addition to the demonstration of feasibility of stochastic resonance in the binary channel, it therefore appears as a remarkable and robust observation that stochastic resonance acts to select an optimal Rényi information measure, under the form of an optimal Rényi transinformation $I_{\alpha_{\mathrm{opt}}}(X;Y)$ for a given information source, or of an optimal Rényi capacity $C_{\alpha_{\mathrm{opt}}}$. Beyond its nominal defining condition of optimality, it remains difficult for the moment to further specify a more concrete interpretation to the optimal Rényi order α_{opt} emerging in different situations of stochastic resonance. On this issue, we have conducted an experiment of noise-aided binary image transmission. It showed that the assessment of the improvement by the optimal Rényi measure is consistent with the visual perception, as the Shannon transinformation also is. In any case, we find it helpful to have access to definite quantitative processes capable of assigning special roles or properties to Rényi information measures at nontrivial orders differing from the Shannon order. This may serve for getting more insight on the various possible ways of quantitatively measuring information, in different contexts and for different prospects where information contents are relevant and need be formalized. In this way, the results of the paper want to contribute in two directions: to consolidate stochastic resonance as a universal phenomenon characterizable with general information measures and to enlarge the appreciation of quantitative measures of information.

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