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CONSTRUCTIVE ACTION OF ADDITIVE NOISE IN OPTIMAL DETECTION

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The optimal detection of a signal of known form hidden in additive white noise is examined in the framework of stochastic resonance and noise-aided information processing. Conditions are exhibited where the performance in the optimal detection increases when the level of the additive (non-Gaussian bimodal) noise is raised. On the additive signal–noise mixture, when a threshold quantization is performed prior to the optimal detection, another form of improvement by noise can be obtained, with subthreshold signals and Gaussian noise. Optimization of the quantization threshold shows that even in symmetric detection settings, the optimal threshold can be away from the center of symmetry and in subthreshold configuration of the signals. These properties concerning non-Gaussian noise and nonlinear preprocessing in optimal detection, are meaningful to the current exploration of the various modalities and potentialities of stochastic resonance.

Keywords: Stochastic resonance; signal detection; quantizer; noise.

1. Introduction

More and more studies have shown that noise is not necessarily always a nuisance, but can sometimes have a beneficial constructive action. This possibility has now been concretized in many different settings and conditions. Stochastic resonance is a generic denomination that can be used to designate such constructive manifestations of the noise [Moss *et al.*, 1994; Chapeau-Blondeau & Godivier, 1996; Gammaitoni *et al.*, 1998; Andò & Graziani, 2000]. Instances of stochastic resonance have been registered in electronic circuits [Anishchenko *et al.*, 1992, 1994; Godivier *et al.*, 1997; Harmer & Abbott, 2000; Morfu *et al.*, 2003], optical devices [McNamara *et al.*, 1988; Dykman *et al.*, 1995; Jost & Saleh, 1996; Vaudelle *et al.*, 1998], neural processes [Bulsara *et al.*, 1991; Douglass *et al.*, 1993; Pantazelou *et al.*, 1995; Chapeau-Blondeau & Godivier, 1996], nanotechnologies [Lee *et al.*, 2003]. Many possible distinct forms have appeared for stochastic resonance, depending on the types of processes coupling signal and noise, and the various measures of performance receiving improvement from the noise. Inventory and analysis of these various forms and modalities of stochastic resonance are still ongoing endeavors. The developments are driven both by the important conceptual significance of stochastic resonance concerning the status of noise, and by its potentialities for applications, especially for information processing. In particular, stochastic resonance has been investigated within standard signal processing problems, like detection [Zozor & Amblard, 2002; Saha & Anand, 2003] or estimation [Chapeau-Blondeau & Rojas Varela, 2001; Rousseau *et al.*, 2003] of signals in noise.

Most forms of stochastic resonance observed so far concern suboptimal processes, in which a processing system is not tuned at its best, and where the noise is used to alter the operating conditions of the system so as to bring them closer to the best performance. Very recently, the possibility of some form of stochastic resonance has been extended to optimal processes. Constructive action of the noise was reported in optimal detection of signals corrupted by non-additive phase noise in [Rousseau & Chapeau-Blondeau, 2002; Chapeau-Blondeau, 2003]. In the present paper, we shall show that a similar property can be obtained in the more common case of an additive signal–noise mixture. We shall also study the possibility of another type of improvement by noise when nonlinear preprocessing under the form of threshold quantization is performed prior to the optimal detection.

2. Optimal Detection

We consider a standard detection situation, where a deterministic signal $s(t)$ can assume one among two known expressions $s_0(t)$ (with prior probability P_0) or $s_1(t)$ (with prior probability $P_1 = 1 - P_0$). This signal $s(t)$ is mixed to a noise $\eta(t)$, the resulting mixture forming the observable signal $x(t)$. This signal $x(t)$ is measured at N distinct times t_k, for $k = 1$ to N, so as to provide N data points $x_k = x(t_k)$. We wish to use the data $\mathbf{x} = (x_1, \ldots, x_N)$ to decide whether the signal $s(t)$ is $s_0(t)$ (hypothesis H_0) or is $s_1(t)$ (hypothesis H_1).

According to classical detection theory [Van Trees, 2001; Kay, 1998], a given detector will decide hypothesis H_0 whenever the data $\mathbf{x} = (x_1, \ldots, x_N)$ falls in the region $\mathcal{R}_0$ of $\mathbb{R}^N$, and it will decide H_1 when $\mathbf{x}$ falls in the complementary region $\mathcal{R}_1$ of $\mathbb{R}^N$. In doing so, the detector achieves an overall probability of detection error P_{er} expressable as

$$P_{\mathrm{er}} = P_1 \int_{\mathcal{R}_0} p(\mathbf{x}|\mathrm{H}_1) d\mathbf{x} + P_0 \int_{\mathcal{R}_1} p(\mathbf{x}|\mathrm{H}_0) d\mathbf{x}, \quad (1)$$

where $p(\mathbf{x}|\mathrm{H}_j)$ is the probability density for observing $\mathbf{x}$ when H_j holds, with $j \in \{0, 1\}$, and the notation $\int \cdot \, d\mathbf{x}$ stands for the N-dimensional integral $\int \cdots \int \cdot \, dx_1 \cdots dx_N$.

Since $\mathcal{R}_0$ and $\mathcal{R}_1$ are complementary in $\mathbb{R}^N$, one has

$$\int_{\mathcal{R}_0} p(\mathbf{x}|\mathrm{H}_1) d\mathbf{x} = 1 - \int_{\mathcal{R}_1} p(\mathbf{x}|\mathrm{H}_1) d\mathbf{x}, \quad (2)$$

which substituted in Eq. (1) yields

$$P_{\mathrm{er}} = P_1 + \int_{\mathcal{R}_1} [P_0 p(\mathbf{x}|\mathrm{H}_0) - P_1 p(\mathbf{x}|\mathrm{H}_1)] d\mathbf{x}. \quad (3)$$

The detector that minimizes P_{er} can be obtained by making as negative as possible the integral over $\mathcal{R}_1$ on the right-hand side of Eq. (3). This is realized by including into $\mathcal{R}_1$ all and only those points $\mathbf{x}$ for which the integrand $P_0 p(\mathbf{x}|\mathrm{H}_0) - P_1 p(\mathbf{x}|\mathrm{H}_1)$ is negative. This yields the optimal detector, that uses the likelihood ratio

$$L(\mathbf{x}) = \frac{p(\mathbf{x}|\mathrm{H}_1)}{p(\mathbf{x}|\mathrm{H}_0)}, \quad (4)$$

to implement the test

$$L(\mathbf{x}) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \frac{P_0}{P_1}. \quad (5)$$

The minimal P_{er} reached by the optimal detector of Eq. (5) is expressable as

$$P_{\mathrm{er}} = \int_{\mathbb{R}^N} \min[P_0 p(\mathbf{x}|\mathrm{H}_0), P_1 p(\mathbf{x}|\mathrm{H}_1)] d\mathbf{x}. \quad (6)$$

Since $\min(a, b) = (a + b - |a - b|)/2$, the minimal probability of error of Eq. (6) reduces to

$$P_{\mathrm{er}} = \frac{1}{2} - \frac{1}{2} \int_{\mathbb{R}^N} |P_1 p(\mathbf{x}|\mathrm{H}_1) - P_0 p(\mathbf{x}|\mathrm{H}_0)| d\mathbf{x}. \quad (7)$$

We consider here that the signal–noise mixture $x(t)$ is the additive mixture

$$x(t) = s(t) + \eta(t), \quad (8)$$

with $\eta(t)$ a stationary white noise of cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u) = dF_\eta/du$. Additive signal–noise mixture is a case very often met in practice. A more complicated nonlinear mixture is considered in [Rousseau & Chapeau-Blondeau, 2002; Chapeau-Blondeau, 2003]. It follows then, that the conditional densities factorize as $p(\mathbf{x}|\mathrm{H}_j) = \prod_{k=1}^N p(x_k|\mathrm{H}_j)$, with

$$p(x_k|\mathrm{H}_j) = f_\eta[u - s_j(t_k)], \quad (9)$$

for $j \in \{0, 1\}$. Now this last Eq. (9) makes possible the explicit evaluation of the optimal detector of Eqs. (4) and (5), and of its performance [Eq. (7)].

3. Constructive Role of Noise

The level of noise $\eta(t)$ is quantified by its rms amplitude σ_η. A situation often met in practice is the case where $\eta(t)$ in the mixture of Eq. (8) is a Gaussian noise. In this case, it is well-known that the performance P_{er} in Eq. (7) of the optimal detector experiences a monotonic degradation as the noise level σ_η increases. However, it is important to realize that the expectation of a monotonic degradation of the performance $P_{\rm er}$ of an optimal detector when the noise level is raised, is not true in generality. This was illustrated in [Rousseau & Chapeau-Blondeau, 2002; Chapeau-Blondeau, 2003] with a nonlinear signal–noise mixture. We shall show here that the same can occur with the more common linear signal–noise mixture of Eq. (8) when it operates with certain non-Gaussian noises.

For our demonstration, we consider in the sequel the basic situation where the signals to be detected are the constant signals $s_0(t) = s_0$ and $s_1(t) = s_1$, for all t, with two constants $s_0 < s_1$. In the standard case where the white noise $\eta(t)$ in Eq. (8) is zero-mean Gaussian, it is well-known that the optimal detector of Eqs. (4) and (5) reduces to

$$\overline{x} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \frac{s_0+s_1}{2} + \frac{\frac{\sigma_\eta^2}{N}}{s_1-s_0}\ln\left(\frac{P_0}{P_1}\right) = x_T, \qquad (10)$$

with $\overline{x} = N^{-1}\sum_{k=1}^{N} x_k$. This optimal test of Eq. (10) achieves the probability of error in Eq. (7) which reads

$$P_{\rm er} = \frac{1}{2}\left[1 + P_1\,\mathrm{erf}\left(\sqrt{N}\frac{x_T-s_1}{\sqrt{2}\sigma_\eta}\right) - P_0\,\mathrm{erf}\left(\sqrt{N}\frac{x_T-s_0}{\sqrt{2}\sigma_\eta}\right)\right]. \qquad (11)$$

It is easy to verify that the performance $P_{\rm er}$ of Eq. (11) experiences a monotonic degradation as the noise level σ_η increases.

For the white noise $\eta(t)$ in Eq. (8) we now turn to a non-Gaussian case, by way of the class of zero-mean Gaussian mixture with standardized probability density $(0 < m < 1)$

$$f_{\rm gm}(u) = \frac{1}{2\sqrt{2\pi}\sqrt{1-m^2}}\left\{\exp\left[-\frac{(u+m)^2}{2(1-m^2)}\right] + \exp\left[-\frac{(u-m)^2}{2(1-m^2)}\right]\right\}, \qquad (12)$$

and cumulative distribution function

$$F_{\rm gm}(u) = \frac{1}{2} + \frac{1}{4}\left[\mathrm{erf}\left(\frac{u+m}{\sqrt{2}\sqrt{1-m^2}}\right) + \mathrm{erf}\left(\frac{u-m}{\sqrt{2}\sqrt{1-m^2}}\right)\right]. \qquad (13)$$

As $m \to 0$, Eq. (12) approaches the zero-mean unit-variance Gaussian density; as $m \to 1$, Eq. (12) approaches the zero-mean unit-variance dichotomic density at ± 1. With $f_\eta(u) = f_{\rm gm}(u/\sigma_\eta)/\sigma_\eta$, Fig. 1 shows different evolutions of the performance $P_{\rm er}$ in Eq. (7) of the optimal detector, as the noise rms amplitude σ_η increases.

Figure 1 reveals the possibility of nonmonotonic evolutions of performance measure $P_{\rm er}$ of the optimal detector, as the level σ_η of the Gaussian-mixture noise is raised. With no noise, at $\sigma_\eta = 0$,

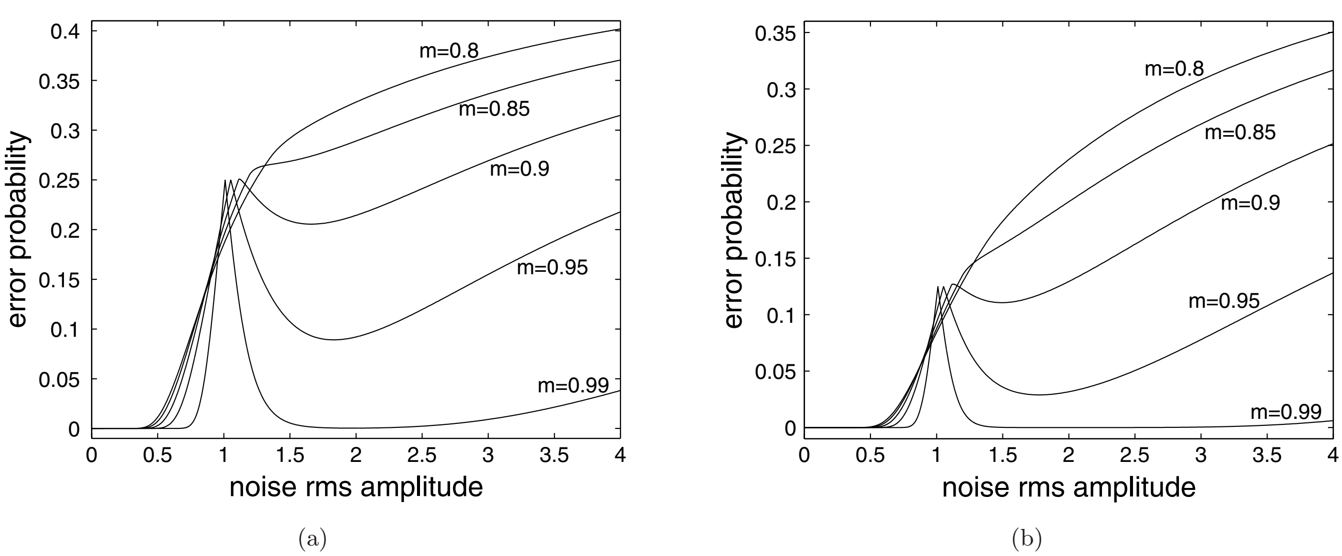


Fig. 1. Probability of error $P_{\rm er}$ of Eq. (7) for the optimal detector of Eq. (5), as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ from Eq. (12) at different m. Also, $s_0(t) \equiv s_0 = -1$, $s_1(t) \equiv s_1 = 1$ and $P_0 = 1/2$; (a) $N = 1$ or (b) $N = 2$.

the probability of detection error $P_{\rm er}$ is always at its best value $P_{\rm er} = 0$ in Fig. 1, as expected for an optimal detector operating in noise-free condition. When the noise level σ_η rises above zero in Fig. 1, the probability of error $P_{\rm er}$ starts gradually to degrade (to increase). However, this degradation of $P_{\rm er}$ does not always proceed monotonically as σ_η is further increased. Conditions exist in Fig. 1, where the performance $P_{\rm er}$ can improve (decrease) when the noise level σ_η is further raised, over some ranges. This constructive action of the additive noise $\eta(t)$ on the performance $P_{\rm er}$ of the optimal detector, can be interpreted as a novel aspect of stochastic resonance. At even larger levels, the detrimental action of the noise resumes, and $P_{\rm er}$ degrades again by increasing towards the least favorable value of 1/2. As seen in Fig. 1, the constructive action occurs when the noise $\eta(t)$ departs sufficiently from a Gaussian noise, i.e. when m in Eq. (12) is sufficiently close to 1. On the contrary, Gaussian noise or values of m approaching zero in Fig. 1, lead to a monotonic increase of $P_{\rm er}$ as σ_η is raised.

A similar behavior with a constructive action of the noise, can be obtained with other non-Gaussian densities for $\eta(t)$. Let us consider passing a noise uniform over $[-1, 1]$ through the nonlinearity

$$g(u) = \frac{1}{a}\frac{\beta u}{\sqrt{1+(\beta u)^2}} \tag{14}$$

parameterized by $\beta > 0$, with $a = \sqrt{1-\arctan(\beta)/\beta}$. This produces a standardized noise whose probability density $f_{\rm sq}(u)$ is zero for u outside $[-g(1), g(1)]$, and otherwise

$$f_{\rm sq}(u) = \frac{1}{2\beta}\frac{a}{[1-(au)^2]^{3/2}}, \tag{15}$$

and its cumulative distribution function is

$$F_{\rm sq}(u) = \frac{1}{2} + \frac{1}{2\beta}\frac{au}{\sqrt{1-(au)^2}} \tag{16}$$

over the support $u \in [-g(1), g(1)]$, and $F_{\rm sq}(u) = 0$ for $u < -g(1)$ and $F_{\rm sq}(u) = 1$ for $u > g(1)$. As $\beta \to 0$, one recovers the uniform noise over $[-\sqrt{3}, \sqrt{3}]$. For increasing β, the density $f_{\rm sq}(u)$ develops peaks at its two modes in $-g(1)$ and $g(1)$, up to $\beta \to +\infty$ which yields a dichotomic noise at ± 1. With $f_\eta(u) = f_{\rm sq}(u/\sigma_\eta)/\sigma_\eta$, Fig. 2 shows different evolutions of the performance $P_{\rm er}$ in Eq. (7) of the optimal detector, as the noise rms amplitude σ_η increases.

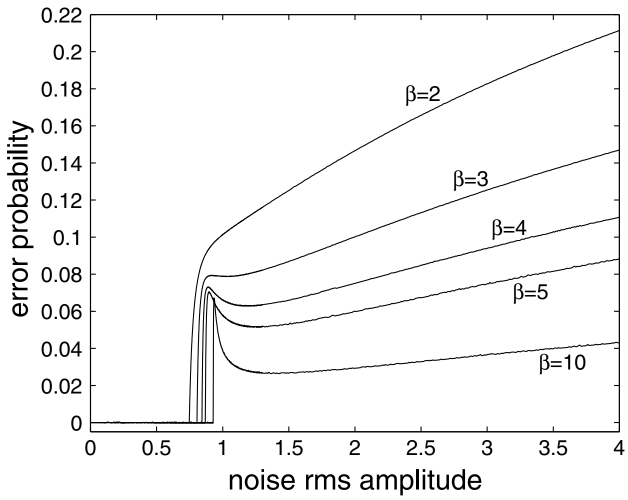


Fig. 2. Probability of error $P_{\rm er}$ of Eq. (7) for the optimal detector of Eq. (5), as a function of the rms amplitude σ_η of the noise $\eta(t)$ from Eq. (15) at different β. Also, $s_0(t) \equiv s_0 = -1$, $s_1(t) \equiv s_1 = 1$, $P_0 = 1/2$ and $N = 1$.

Figure 2 reveals that, as the noise level σ_η increases, the possibility of a nonmonotonic evolution of the performance $P_{\rm er}$, rather than a monotonic degradation, is preserved with the noise density of Eq. (15). This occurs in Fig. 2 for sufficiently large values of β, associated to the noise $\eta(t)$ with a density from Eq. (15) with a sufficiently pronounced bimodal structure.

The bimodal structure of the noise $\eta(t)$ seems here, both in Figs. 1 and 2, to be an essential ingredient for observing the improvement by noise of $P_{\rm er}$. This can be understood qualitatively, because a zero-mean bimodal noise with rms amplitude σ_η, as used in Figs. 1 and 2, tends to concentrate its amplitudes around $\pm\sigma_\eta$, as opposed to a unimodal noise which concentrates its amplitudes around zero. In a binary detection task, noise fluctuations around $\pm\sigma_\eta$, especially for well-chosen ranges of σ_η in relation to the levels s_0 and s_1 to be detected, can be less damageable than noise fluctuations around zero to the capacity of distinguishing between s_0 and s_1. At the extreme, a purely dichotomic noise at $\pm\sigma_\eta$ would let intact the capacity of distinguishing between s_0 and s_1, as long as $s_0 \pm \sigma_\eta$ cannot be confused with s_1, which is the rule, except in the very special configuration where $\sigma_\eta = s_1 - s_0$. Especially, when $\sigma_\eta > s_1 - s_0$, the capacity of distinguishing between s_0 and s_1 is unaffected by the dichotomic noise, however large σ_η may be. It is a reminiscence of this property of dichotomic noise, which is at work in Figs. 1 and 2, to allow the nonmonotonic evolution of the performance $P_{\rm er}$ with continuous bimodal noises interpolating between dichotomic and unimodal noises.

Beyond the necessity of bimodal noises for the above mechanism to apply, what we wish to emphasize here is the conceptual significance in principle of the results in Figs. 1 and 2. These results establish that an optimal detector operating on an addditive signal–noise mixture can experience an improvement of its performance $P_{\rm er}$, when the noise level increases, over some ranges of the noise, instead of a monotonic degradation of $P_{\rm er}$. The same property was shown possible with nonlinear signal–noise mixture in [Rousseau & Chapeau-Blondeau, 2002; Chapeau-Blondeau, 2003], and it is extended here to the more common linear (additive) signal–noise mixture. It is clear in Figs. 1 and 2 that the improvement of $P_{\rm er}$ does not appear as soon as the noise level σ_η is raised above zero. A nonzero amount of noise $\eta(t)$ has to pre-exist before improvement of $P_{\rm er}$ by a further increase in σ_η is obtained; but a pre-existing amount of noise is usually the rule in a signal-processing task. Also, for a non-Gaussian $\eta(t)$, the increase of the rms amplitude σ_η cannot be achieved by a simple addition of more noise if one wants to keep the same shape for the probability density $f_\eta(u)$, so as to match the conditions of Figs. 1 and 2 which increase σ_η at $f_\eta(u)$ constant in shape. A more internal adjustable parameter, analog to a physical temperature, has to be assumed to increase σ_η at $f_\eta(u)$ constant in shape. Alternatively, the change of the density $f_\eta(u)$ can be explicitly modeled as more noise is added (what is not done here). Improvement by noise can still be expected in these more elaborate conditions, since we show here that such improvement is robustly preserved over various shapes for $f_\eta(u)$, this point remaining to be explicitly explored. But again, beyond such issues which are oriented towards practical implementation of the proposed framework, what we want to emphasize here is its conceptual significance: the feasibility in principle of a form of improvement by noise in optimal detection with additive signal–noise mixture. This possibility is an important feature which complements all the aspects and properties known to stochastic resonance.

4. Nonlinear Transformation Before Detection

It sometimes happens that a nonlinear transformation is performed on the signal–noise mixture $x(t) = s(t) + \eta(t)$ prior to the detection process. Such a nonlinear transformation may be imposed by the physics of the sensing or measuring device. Let us consider here the nonlinear transformation, very often considered in the context of stochastic resonance, which produces the output signal $y(t)$ as

$$y(t) = \mathrm{sign}[s(t) + \eta(t) - \theta] = \pm 1. \tag{17}$$

The transformation of Eq. (17) realizes a one-bit quantization of the input signal–noise mixture $x(t) = s(t) + \eta(t)$, with quantization threshold θ. It offers a parsimonious signal representation which can be useful for fast real-time processing; it also bears some similarity with neuronal coding.

When the detection is based on $y(t)$, the same considerations as in Sec. 2 yield the optimal detector as

$$L(\mathbf{y}) = \frac{\Pr\{\mathbf{y}|\mathrm{H}_1\}}{\Pr\{\mathbf{y}|\mathrm{H}_0\}} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \frac{P_0}{P_1}, \tag{18}$$

with $\mathbf{y} = (y_1, \ldots, y_N)$ and $y_k = y(t_k)$ for $k = 1$ to N. Equation (18) is the minimal-$P_{\rm er}$ detector, achieving the minimum $P_{\rm er}$ among all detection schemes based on $\mathbf{y}$, this minimum being

$$P_{\rm er} = \frac{1}{2} - \frac{1}{2} \sum_{\mathbf{y} \in \{-1,1\}^N} |P_1 \Pr\{\mathbf{y}|\mathrm{H}_1\} - P_0 \Pr\{\mathbf{y}|\mathrm{H}_0\}| \tag{19}$$

with the sum performed over the 2^N distinct states $(y_1 = \pm 1, \ldots, y_N = \pm 1)$ accessible to the data $\mathbf{y}$. For $\eta(t)$ a white noise, we have $\Pr\{\mathbf{y}|\mathrm{H}_j\} = \prod_{k=1}^N \Pr\{y_k|\mathrm{H}_j\}$, for $j \in \{0, 1\}$. At any time t, we have according to Eq. (17), the conditional probability $\Pr\{y(t) = -1|\mathrm{H}_j\}$ which is also $\Pr\{s_j(t) + \eta(t) \le \theta\}$, this amounting to

$$\Pr\{y(t) = -1|\mathrm{H}_j\} = F_\eta[\theta - s_j(t)]. \tag{20}$$

In the same way, we have $\Pr\{y(t) = 1|\mathrm{H}_j\} = 1 - F_\eta[\theta - s_j(t)]$. This allows the explicit evaluation of the optimal detector of Eq. (18) and of its performance of Eq. (19).

We again consider in the following, the basic situation where the signals to be detected are the constant signals $s_0(t) = s_0$ and $s_1(t) = s_1 > s_0$, for all t. In this case, the optimal test of Eq. (18) can be reduced to a simpler expression taking the form

$$N_1 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln\left(\dfrac{P_0}{P_1}\right) - N \ln\left[\dfrac{F_\eta(\theta - s_1)}{F_\eta(\theta - s_0)}\right]}{\ln\left[\dfrac{1 - F_\eta(\theta - s_1)}{1 - F_\eta(\theta - s_0)}\right] - \ln\left[\dfrac{F_\eta(\theta - s_1)}{F_\eta(\theta - s_0)}\right]} = N_T, \tag{21}$$

where N_1 is the number, between 0 and N, of components y_k at +1 in the data $\mathbf{y}$. This optimal test of Eq. (21) achieves the probability of error

$$P_{\rm er} = P_1 \sum_{N_1 < N_T} C_{N_1}^N [1 - F_\eta(\theta - s_1)]^{N_1} \times F_\eta(\theta - s_1)^{N - N_1} + P_0 \sum_{N_1 \geq N_T} C_{N_1}^N [1 - F_\eta(\theta - s_0)]^{N_1} \times F_\eta(\theta - s_0)^{N - N_1}, \tag{22}$$

with $C_{N_1}^N$ as the binomial coefficients.

When the noise $\eta(t)$ has a Gaussian-mixture density as in Eq. (12), then Fig. 3 represents the evolution of the probability of error $P_{\rm er}$ of Eq. (22) for the optimal detector of Eq. (21), as the rms amplitude σ_η of the noise $\eta(t)$ is raised. It is interesting to compare Fig. 3 characterizing the detection from the quantized data $\mathbf{y}$, to Fig. 1 characterizing the detection from the analog (unquantized) data $\mathbf{x}$ in otherwise similar conditions. Two important observations can be made in this respect.

The first observation is that $P_{\rm er}$ in the detection with the quantized data $\mathbf{y}$ is always larger (in the same noise condition) than with the analog unquantized data $\mathbf{x}$. This observation has a natural explanation: one-bit quantization by $\mathbf{y}$ of the analog data $\mathbf{x}$ entails a loss of information, whence the reduced performance in detection. The reduced performance is even more pronounced at $N = 2$ in Fig. 3(b) than at $N = 1$ in Fig. 3(a), because the loss of information increases with the number N of data points. The trade-off is that $\mathbf{y}$ represents a much more parcimonious representation with only one bit per data point which can be useful for fast real-time processing, compared to $\mathbf{x}$ which in principle requires a infinite number (12 to 16 in practice) of bits per data point.

The second observation is that the constructive action of noise present in Fig. 1 is absent in Fig. 3. When the thresholding is done prior to the detection, the detector cannot benefit from a soft, smooth, analog representation of the data, which is somehow richer than the hard-thresholded representation imposed to base the decision in Fig. 3. The constructive action of the noise can operate in the soft analog representation of Fig. 1, but is suppressed in the thresholded representation of Fig. 3. This has been revealed by the present analysis. This is somehow a novel aspect which enriches the

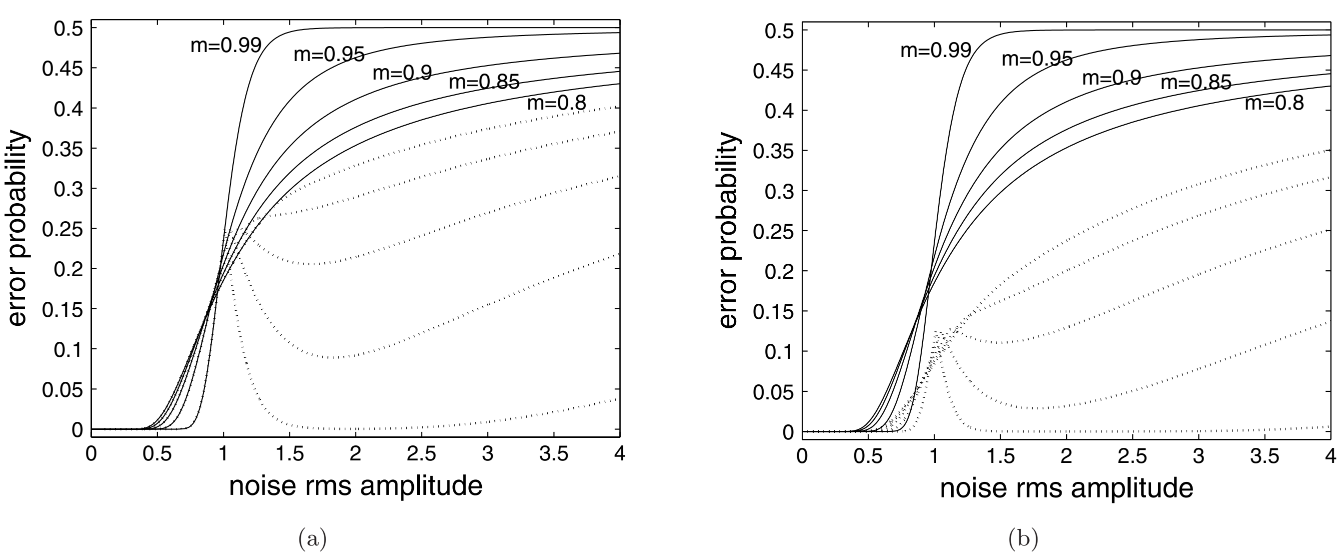


Fig. 3. Solid lines: probability of error $P_{\rm er}$ of Eq. (22) for the optimal detector of Eq. (21) from the quantized data $\mathbf{y}$, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ from Eq. (12) at different m. The quantization threshold in Eq. (17) is $\theta = 0$. Also as in Fig. 1: $s_0(t) \equiv s_0 = -1$, $s_1(t) \equiv s_1 = 1$ and $P_0 = 1/2$; (a) $N = 1$ or (b) $N = 2$. The dotted lines are $P_{\rm er}$ redrawn from Fig. 1 for the detection from the analog (unquantized) data $\mathbf{x}$.

properties known to stochastic resonance. Improvement by noise usually occurs, in known forms of stochastic resonance, for a nonlinear transformation of an input signal–noise mixture. Here, improvement by noise is possible on the linear input signal–noise mixture $x(t)$, and disappears after a nonlinear transformation on $x(t)$.

However, if the nonlinear transformation of Fig. 3 performed with the quantization threshold $\theta = 0$, is modified by varying θ, then a constructive action of the noise can be recovered. Figure 4 addresses a situation where the thresholding of Eq. (17) is performed in such a way that both signals $s_0(t)$ and $s_1(t)$ to be detected are on the same side of the threshold θ.

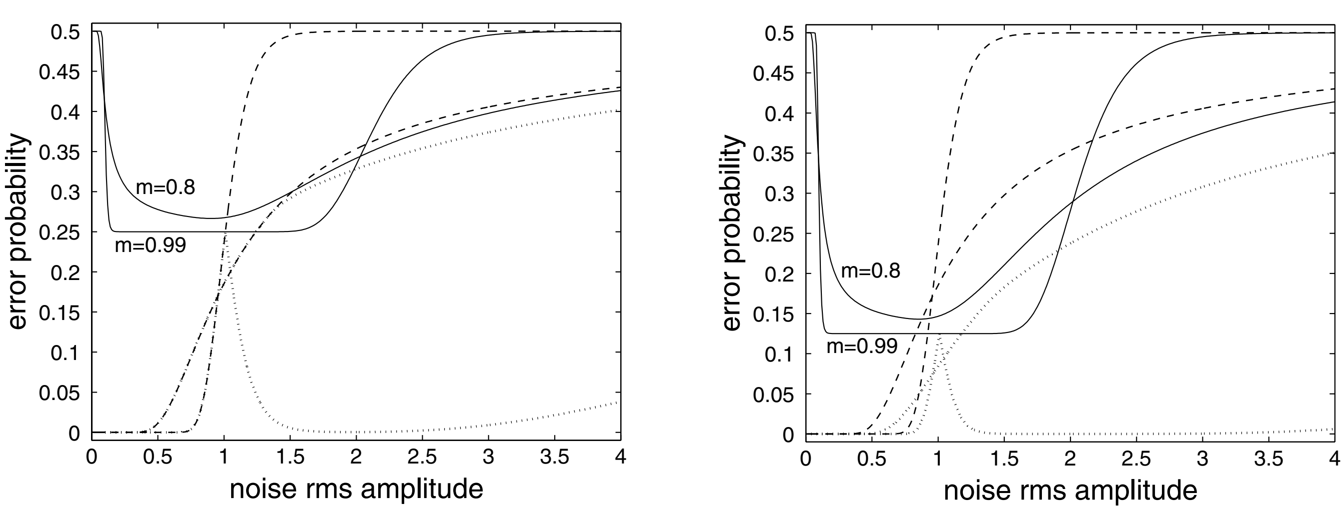
In Fig. 4, when the noise $\eta(t)$ is absent in Eq. (17), both signals $s_0(t)$ and $s_1(t)$ are always below the quantization threshold θ, and are therefore always quantized exactly in the same way. In this case, at $\eta(t) \equiv 0$, no discriminating detection is possible based on the quantized data $\mathbf{y}$ and the performance is at its worst, i.e. $P_{\rm er} = 1/2$ in Fig. 4. Next, as the noise level σ_η is raised above zero, the presence of the noise $\eta(t)$ progressively allows the signals $s_0(t)$ and $s_1(t)$ to be quantized differently by Eq. (17). This translates in Fig. 4, into an improvement of the detection performance $P_{\rm er}$ as the noise level σ_η increases, up to an optimal nonzero noise level where the probability of detection error $P_{\rm er}$ is minimized. The constructive action of the noise, or stochastic resonance, is recovered, under the form of a noise-assisted detection of subthreshold signals.

For subthreshold signals, the constructive action of the noise is possible with the Gaussian mixture noise of Fig. 4, but it is also possible with Gaussian noise, as shown in Fig. 5.

Figures 4 and 5 also confirm a remark made above for Fig. 3, that the detection from the quantized data $\mathbf{y}$ never improves over the detection from the analog (unquantized) data $\mathbf{x}$. This is observed whatever the position of the quantization threshold θ, either in a suprathreshold (Fig. 3) or a subthreshold (Fig. 4) configuration of the signals $s_0(t)$ and $s_1(t)$, as also confirmed in Fig. 5.

Another important observation in Figs. 4 and 5, is that a quantization threshold θ in a subthreshold configuration, can lead to a better detection performance $P_{\rm er}$ compared to θ in a suprathreshold configuration of the signals $s_0(t)$ and $s_1(t)$. For instance, in Figs. 4 and 5, for the detection of $s_0(t) \equiv -1$ and $s_1(t) \equiv 1$, at large noise levels σ_η, the performance $P_{\rm er}$ in the subthreshold configuration $\theta = 1.1$ is generally better than that in the suprathreshold configuration $\theta = 0$. This is true for any N in Fig. 4 with Gaussian-mixture noise, and for $N > 1$ in Fig. 5 with Gaussian noise. This is an important property for the use of quantization devices, as often considered for stochastic resonance: even in completely symmetric conditions of the signals and noise and process, the optimal configuration for the quantization threshold is not necessarily at the center of symmetry $\theta = 0$.

The above observation naturally leads to raise the issue of optimizing the quantization threshold θ



(a) (b)

Fig. 4. Same as Fig. 3, except that $\theta = 1.1$. The dashed lines are redrawn from Fig. 3.

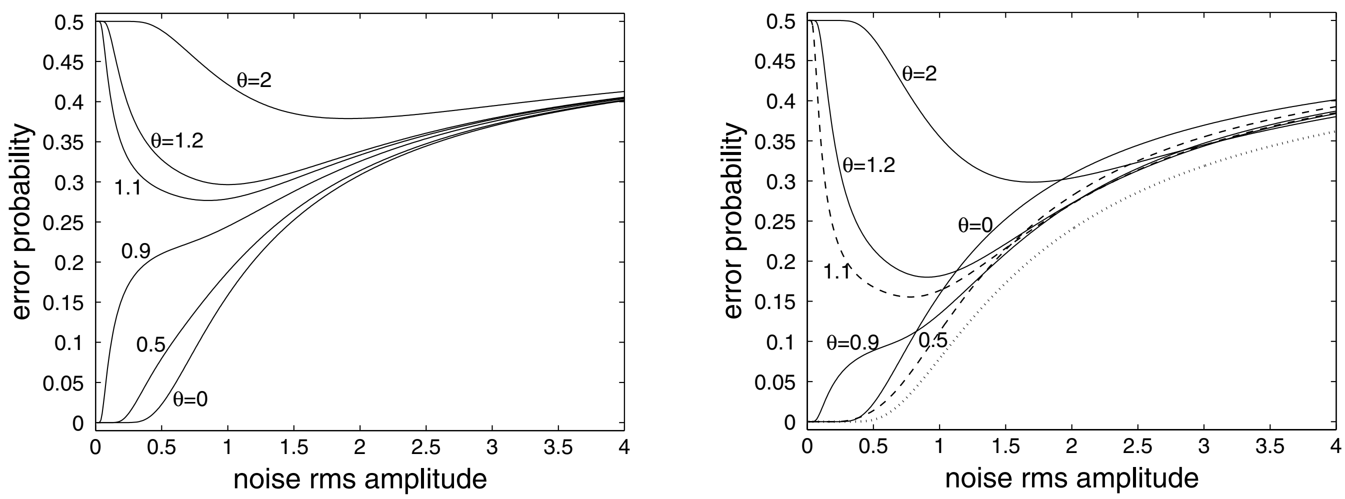


Fig. 5. Solid and dashed lines: probability of error $P_{\rm er}$ of Eq. (22) for the optimal detector of Eq. (21) from the quantized data $\mathbf{y}$, as a function of the rms amplitude σ_η of the Gaussian noise $\eta(t)$, and different quantization thresholds θ. Also as in Fig. 1: $s_0(t) \equiv s_0 = -1$, $s_1(t) \equiv s_1 = 1$ and $P_0 = 1/2$; (a) $N = 1$ or (b) $N = 2$. In (b), the dotted line is $P_{\rm er}$ of Eq. (11) for detection from the analog (unquantized) data $\mathbf{x}$; in (a) this line is superimposed to the curve at $\theta = 0$.

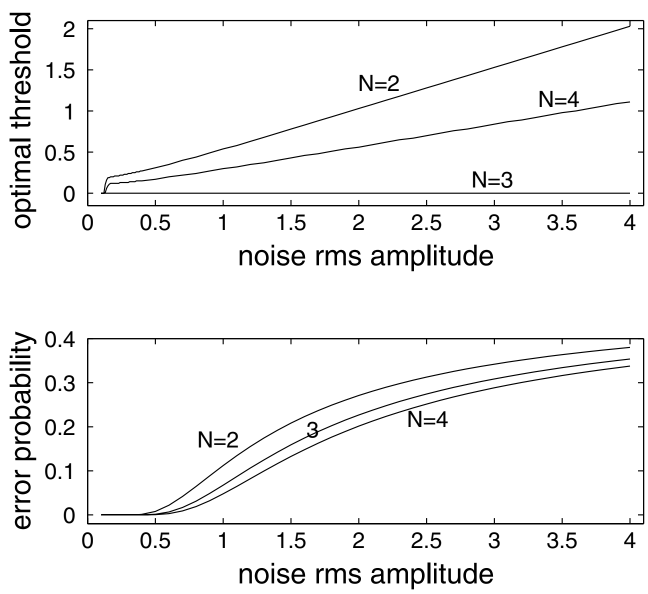


Fig. 6. Performance of the optimal detector of Eq. (21) from the quantized data $\mathbf{y}$, as a function of the rms amplitude σ_η of the Gaussian noise $\eta(t)$, and different number N of data points. Also as in Fig. 1: $s_0(t) \equiv s_0 = -1$, $s_1(t) \equiv s_1 = 1$ and $P_0 = 1/2$. Upper panel: optimal value $\theta_{\rm opt}$ of the quantization threshold θ in Eq. (17) minimizing $P_{\rm er}$ of Eq. (22). Due to the symmetry of the process, $-\theta_{\rm opt}$ is also an optimal threshold. Lower panel: Minimum $P_{\rm er}$ at $\theta_{\rm opt}$.

so as to maximize the performance (minimize $P_{\rm er}$) in given conditions of the noise $\eta(t)$ and signals to be detected. For illustration, this issue is solved in Fig. 6 for conditions with Gaussian noise $\eta(t)$.

With Gaussian noise, the results of Fig. 6 reveal that the optimal threshold $\theta_{\rm opt}$ is never zero for N even, although it is always zero for N odd. The same trend that takes $\theta_{\rm opt}$ away from zero is even more manifest with non-Gaussian noise $\eta(t)$; for instance, with the Gaussian-mixture noise of Eq. (12), $\theta_{\rm opt}$ never remains at zero, even for N odd. Moreover, Fig. 6 shows that the optimal threshold $\theta_{\rm opt}$ can lie in a subthreshold configuration of the signals $s_0(t)$ and $s_1(t)$ to be detected, i.e. $\theta_{\rm opt}$ is above 1 for detection between $s_0(t) \equiv -1$ and $s_1(t) \equiv 1$ at large values of the noise rms amplitude σ_η and N even. However, as also shown in Fig. 6, when the process is tuned at $\theta_{\rm opt}$, the resulting probability of detection error $P_{\rm er}$ is generally an increasing function of the noise level σ_η. Improvement of $P_{\rm er}$ by increasing σ_η is feasible when θ is not at its optimal position, as in Fig. 4, but disappears when θ is at its optimum $\theta_{\rm opt}$. Noise improvement occurs here in nonoptimal processes, while it occurred in optimal processes in Sec. 3.

5. Summary and Outlook

We have examined the optimal detection of a signal of known form hidden in additive white noise, in the framework of stochastic resonance or noise-aided information processing. Several conclusions, meaningful in this framework, can be emphasized as follows.

- It is in principle possible for an optimal detector operating on an addditive signal–noise mixture to experience an improvement of its performance when the noise level increases, over some ranges of the noise, instead of a monotonic degradation. This property is obtained here with non-Gaussian bimodal noise. A similar property was observed in [Rousseau & Chapeau-Blondeau, 2002; Chapeau-Blondeau, 2003], yet with a nonlinear signal–noise mixture, but unimodal noise. Other forms of optimal processing with a constructive role of noise, could also be found, with linear or nonlinear signal–noise mixtures, bimodal or unimodal or Gaussian noises. It is important, in the current endeavor of inventory and analysis of all the modalities and potentialities of stochastic resonance or improvement by noise in information processing, to have in mind this possibility of a constructive action of the noise in *optimal* processing, which although counterintuitive is authorized in principle, as confirmed here.
- When the optimal detection is performed after a common nonlinear transformation like a threshold quantization, another form of improvement by noise can be registered. When the quantization threshold is not placed in an optimal position and is associated to a subthreshold configuration of the signals to be detected, then injection of noise can improve the detection performance, and this improvement can also occur with Gaussian noise.
- Even in completely symmetric detection settings, the optimal location of the quantization threshold θ is not necessarily at the center of symmetry $\theta = 0$. The optimal quantization threshold θ_{opt} can in principle be determined by application of the present treatment. Especially, we have shown that conditions exist where the optimum θ_{opt} can lie in a subthreshold configuration of the signals to be detected.

These properties of optimal detection with additive signal–noise mixtures, possibly with nonlinear preprocessing, are meaningful to the current ongoing explorations and analyses of the various modalities of stochastic resonance and of its potentialities for noise-aided information processing. We have focused here essentially on the case of constant signals, which is a basic configuration of detection; yet the theory of Sec. 2 is general and can be applied to investigate comparable properties of the optimal detection of arbitrary signals. Also, similar properties can be investigated in other forms of optimal detection, for instance in Bayes or Neyman-Pearson sense, following an approach much alike to what is done here for the minimal-P_{er} detector. Further, other types of optimal processing with a constructive action of noise, in particular additive noise, can also be sought for investigation, since again, as verified here, this possibility although counterintuitive is not *a priori* prohibited in principle. This could form the basis for innovative information processing strategies exploiting the noise instead of combating it.

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