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Exploring weak-periodic-signal stochastic resonance in locally optimal processors with a Fisher information metric

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ABSTRACT

For processing a weak periodic signal in additive white noise, a locally optimal processor (LOP) achieves the maximal output signal-to-noise ratio (SNR). In general, such a LOP is precisely determined by the noise probability density and also by the noise level. It is shown that the output–input SNR gain of a LOP is given by the Fisher information of a standardized noise distribution. Based on this connection, we find that an arbitrarily large SNR gain, for a LOP, can be achieved ranging from the minimal value of unity upwards. For stochastic resonance, when considering adding extra noise to the original signal, we here demonstrate via the appropriate Fisher information inequality that the updated LOP fully matched to the new noise, is unable to improve the output SNR above its original value with no extra noise. This result generalizes a proof that existed previously only for Gaussian noise. Furthermore, in the situation of non-adjustable processors, for instance when the structure of the LOP as prescribed by the noise probability density is not fully adaptable to the noise level, we show general conditions where stochastic resonance can be recovered, manifested by the possibility of adding extra noise to enhance the output SNR.

1. Introduction

Stochastic resonance (SR), originally introduced in the field of climate dynamics [1], is now emerging as a nonlinear signal processing method [2–8]. This method considers the possibility of adding an appropriate amount of noise to a nonlinear system (or network) in order to improve its performance described by an appropriate quantitative measure, such as the output signal-to-noise ratio (SNR) [2,3,6–14], the mutual information [15,16], the Fisher information [17–19], the detection probability [20–39], the mean-square-error of estimator [26,40], etc. A proven SR result is that, within the regime of validity of linear response theory, the output–input SNR gain cannot exceed unity for a nonlinear system subjected to a weak sinusoidal signal plus Gaussian white noise [4,8,41]. But, beyond the conditions where linear response theory applies, the possibility of SNR gain above unity is demonstrated for certain static nonlinearities [6,7] and dynamical systems [13,14]. More recently, many significant studies on SR in the areas of statistical signal detection and estimation [21–40] show the applicability of SR in nonlinear signal processing for the improvement of system performance by noise. However, most studies of SR first establish a fixed nonlinearity, and observe the noise-enhanced phenomenon therein. When an optimal processor can be updated according to the actual noise, specific examples [18,26,30,31] showed that the updated optimal processor operating on the data with extra noise can

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outperform the original optimal processor operating on the original data without extra noise. For the generic situation of the detection of a deterministic weak signal in noise, it is shown that the asymptotic efficacy of a locally optimal detector is generally determined by the Fisher information of the noise distribution [42]. Then, based on a Fisher information inequality, we prove that the improvement by adding noise is impossible for the detection probability of a weak known signal [42].

In this paper, we focus on the possibility of the SR effect in a locally optimal processor (LOP) that processes a weak periodic signal in additive white noise [24,44,45]. In this case, Zozor and Amblard [24] demonstrated that a LOP possesses the maximal output SNR. We further demonstrate that the output SNR of a LOP is closely related to the Fisher information of the noise probability density function (PDF) [43]. It is interesting to note that the output–input SNR gain of a LOP is given by the Fisher information of a standardized noise PDF. It is well known that a standardized Gaussian PDF has a minimum Fisher information of unity [44]. As a consequence, for any non-Gaussian noise, it is always possible to achieve an output–input SNR gain of a LOP larger than unity. Some types of noise and their corresponding LOPs are discussed for obtaining an arbitrary large output–input SNR gain. When adding extra noise to the original signal, we assume that the LOP can be updated according to the composite noise, aiming to achieve the maximal output SNR. Then, we prove that the updated LOP, via the Fisher information convolution inequality, is unable to improve the output SNR. This result generalizes a proof that existed previously only for a weak periodic signal in additive Gaussian noise [4,8,41]. This result and its domain of applicability leave open the possibility of SR or improvement by noise in situations with less flexibility, for instance, when the exact optimal updated LOP is not accessible or too complex to be implemented. Then, we further prove that, if the structure of a normalized LOP is a function of the noise root-mean-square amplitude, then such a prescribed LOP can exhibit the SR effect. Moreover, utilizing dichotomous noise as the added noise to the signal, a family of LOPs is elicited with their structures as a function of the root-mean-square amplitude of dichotomous noise. The SR effect is shown to always occur in such a prescribed LOP by increasing the added noise level to the special value given by the LOP. Based on the relationship of Fisher information of noise distribution and the output SNR, we show a new example of the Fisher information equality for the uniform noise and the dichotomous noise. Finally, some open questions are discussed.

2. No SR effect in an updated LOP

2.1. SNR gain of a LOP

Consider a static (memoryless) nonlinearity g with its output

$$y(t)=g[x(t)], \tag{1}$$

where $x(t)=s(t)+z(t)$ is a signal-plus-noise mixture input. The component $s(t)$ is a periodic signal with a maximal amplitude A $(0<|s(t)|\leq A)$ and period T. The zero-mean white noise $z(t)$, independent of $s(t)$, is with the PDF f_z and a root-mean-square amplitude σ_z [7]. The input SNR for $x(t)$ can be defined as the power contained in the spectral line $1/T$ divided by the power contained in the noise background in a small frequency bin ΔB around $1/T$ [7], that is

$$R_{\mathrm{in}}=\frac{|\langle s(t)\exp[-i2\pi t/T]\rangle|^2}{\sigma_z^2\Delta t\Delta B}, \tag{2}$$

where Δt indicates the time resolution in a discrete-time implementation and the temporal average defined as $\langle\cdots\rangle=(1/T)\int_0^T\cdots dt$ [7]. Here, we assume $\Delta t\ll T$ and observe the output $y(t)$ for a sufficiently large time interval of NT $(N\gg 1)$. Then, the practical discrete-time white noise $z(j\Delta t)$ has the autocorrelation function $\mathrm{E}[z(j\Delta t)z(j\Delta t+k\Delta t)]=\sigma_z^2\Delta t\delta(k\Delta t)$ with the discrete-time version of the Dirac delta function $\delta(k\Delta t)=1/\Delta t$ for $k=0$ and zero otherwise [7]. Here, σ_z^2 is the variance of zero-mean white noise $z(t)$ [7]. Similarly, based on the cyclostationarity property of $y(t)$, the output SNR for $y(t)$ is given by

$$R_{\mathrm{out}}=\frac{|\langle \mathrm{E}[y(t)]\exp[-i2\pi t/T]\rangle|^2}{\langle\mathrm{var}[y(t)]\rangle\Delta t\Delta B}, \tag{3}$$

with nonstationary expectation $\mathrm{E}[y(t)]$ and nonstationary variance $\mathrm{var}[y(t)]$ [7].

Assume $s(t)$ is weak $(A\to 0)$, and make a Taylor expansion of g around z at a fixed time t as

$$y(t)=g[z+s(t)]\approx g(z)+s(t)g'(z), \tag{4}$$

with $g'(z)=dg(z)/dz$ existing for almost all z. Here, the Taylor expansion of g is up to first order in the small signal $s(t)$. We further assume that g has zero mean and finite variance under f_z, i.e. $\mathrm{E}[g(z)]=\int_{-\infty}^{\infty}g(z)f_z(z)\,dz=0$ and $\mathrm{E}[g^2(z)]=\int_{-\infty}^{\infty}g^2(z)f_z(z)\,dz<\infty$. For an arbitrary memoryless nonlinearity g, the zero mean of $\mathrm{E}[g(z)]$ is not restrictive since any arbitrary g can always include a constant bias to cancel this average [44,45]. Therefore, we have

$$\mathrm{E}[y(t)]=\mathrm{E}[g(z)]+s(t)\mathrm{E}[g'(z)]\approx s(t)\mathrm{E}[g'(z)]. \tag{5}$$

Using Eqs. (4) and (5), we obtain

$$\begin{aligned}\mathrm{var}[y(t)]&=\mathrm{E}[y^2(t)]-\mathrm{E}[y(t)]^2\\&\approx\mathrm{E}[y^2(t)]-s^2(t)\mathrm{E}^2[g'(z)]\\&\approx\mathrm{E}[g^2(z)]+2s(t)\mathrm{E}[g(z)g'(z)]+s^2(t)\{\mathrm{E}[g'^2(z)]-\mathrm{E}^2[g'(z)]\}\\&\approx\mathrm{E}[g^2(z)]+2s(t)\mathrm{E}[g(z)g'(z)],\end{aligned} \tag{6}$$

up to first order in the small signal $s(t)$. Here, as $A\to 0$ $(0<|s(t)|\leq A)$, the higher-order term of $s^2(t)\{\mathrm{E}[g'^2(z)]-\mathrm{E}^2[g'(z)]\}$ is neglected [24,44,45]. Substituting Eqs. (5) and (6) into Eq. (3), we have

$$\begin{aligned}R_{\mathrm{out}}&\approx\frac{|\langle s(t)\exp[-i2\pi t/T_s]\rangle|^2}{\Delta B\Delta t}\frac{\mathrm{E}^2[g'(z)]}{\langle\mathrm{E}[g^2(z)]+2s(t)\mathrm{E}[g(z)g'(z)]\rangle}\\&\approx R_{\mathrm{in}}\sigma_z^2\frac{\mathrm{E}^2[g'(z)]}{\mathrm{E}[g^2(z)]},\end{aligned} \tag{7}$$

where the first-order term $2s(t)\mathrm{E}[g(z)g'(z)]$, compared with $\mathrm{E}[g^2(z)]$ and $\mathrm{E}^2[g'(z)]$, has no contribution for the calculation of R_{out} in the weak signal condition ($A\to 0$ and $0<|s(t)|\leq A$). The above derivations of Eqs. (5)–(7) are valid in the limit of a vanishing $s(t)$ [44,45].

Then, the output–input SNR gain of the static nonlinearity is bounded by

$$G=\frac{R_{\text{out}}}{R_{\text{in}}}\approx\sigma_z^2\frac{\mathrm{E}^2[g'(z)]}{\mathrm{E}[g^2(z)]}$$
$$=\sigma_z^2\frac{\left(\int_{-\infty}^{\infty}g(z)f_z'(z)/f_z(z)f_z(z)\,dz\right)^2}{\int_{-\infty}^{\infty}g^2(z)f_z(z)\,dz}$$
$$\le\sigma_z^2\int_{-\infty}^{\infty}\frac{f_z'^2(z)}{f_z^2(z)}f_z(z)\,dz$$
$$=\sigma_z^2\mathrm{E}\left[\frac{f_z'^2(z)}{f_z^2(z)}\right]=\sigma_z^2 I(f_z)=I(f_{z_0}). \tag{8}$$

It is noted that the equality of Eq. (8) occurs as g becomes a LOP

$$g_{\text{opt}}(z)\triangleq Cf_z'(z)/f_z(z), \tag{9}$$

by the Schwarz inequality for the derivative $f_z'(z)=df_z(z)/dz$ (without loss of generality with $C=-1$) [24,44,45]. Here, the scaled noise $z(t)=\sigma_z z_0(t)$ has PDF $f_z(z)=f_{z_0}(z/\sigma_z)/\sigma_z$, and the standardized noise PDF f_{z_0} is with unity variance $\sigma_{z_0}^2=1$ [19,46]. Then, the Fisher information $I(f_z)$ of f_z can be expressed as

$$I(f_z)=\mathrm{E}\left[\frac{f_z'^2(z)}{f_z^2(z)}\right]=\sigma_z^{-2}\mathrm{E}\left[\frac{f_{z_0}'^2(z_0)}{f_{z_0}^2(z_0)}\right]=\sigma_z^{-2}I(f_{z_0}), \tag{10}$$

with the Fisher information $I(f_{z_0})$ of f_{z_0}. In Eqs. (8) and (9), note that not only the constant C but also any irrelevant multiplicative coefficient in g_{opt} can be reduced, resulting in a normalized LOP g_{opt}^n [44,45]. For instance, a normalized LOP of Eq. (26) corresponds to the generalized Gaussian noise with PDF of Eq. (25).

It is also indicated in Eq. (8) that the maximal G achieved by g_{opt} is completely determined by the Fisher information $I(f_{z_0})$. For a standardized PDF f_{z_0}, we have

$$I(f_{z_0})=\mathrm{E}\left[\frac{f_{z_0}'^2(z_0)}{f_{z_0}^2(z_0)}\right]\mathrm{E}[z_0^2]\ge\mathrm{E}\left[\frac{f_{z_0}'(z_0)}{f_{z_0}(z_0)}z_0\right]^2=1, \tag{11}$$

with $\mathrm{E}[z_0^2]=\sigma_{z_0}^2=1$ and the equality occurring if $f_{z_0}'(z_0)/f_{z_0}(z_0)=cz_0$ for a constant $c\neq 0$. Then, $f_{z_0}(z_0)=\exp[k+cz_0^2/2]$ [44]. In order to be a PDF, $c<0$ and $\exp(k)$ is the normalized constant. This is a standardized Gaussian PDF $f_{z_0}(z_0)=\exp(-z_0^2/2)/\sqrt{2\pi}$ [44]. Thus, a standardized Gaussian PDF f_{z_0} has a minimal $I(f_{z_0})=1$ and any standardized non-Gaussian PDF f_{z_0} has $I(f_{z_0})>1$ [44]. In other words, Eq. (8) shows that the output–input SNR gain achieved by a LOP in Eq. (9) certainly exceeds unity for a weak periodic signal in additive non-Gaussian white noise. An interesting question arises, this is, which type of a standardized PDF f_{z_0} has the maximal Fisher information $I(f_{z_0})$? Therefore, can the output–input SNR gain achieved by a LOP be arbitrarily large ranging from unity upwards? Here, we consider the Gaussian mixture noise that is frequently employed in previous studies [21,24,26].

Example 1. Consider the Gaussian mixture noise $z(t)$ with its PDF [26]

$$f_z(z)=\frac{1}{2\sqrt{2\pi\epsilon^2}}\left[\exp\left(\frac{-(z-\mu)^2}{2\epsilon^2}\right)+\exp\left(\frac{-(z+\mu)^2}{2\epsilon^2}\right)\right], \tag{12}$$

where the variance $\sigma_z^2=\mu^2+\epsilon^2$ and parameters $\mu,\epsilon\ge 0$. Note that Eq. (12) can be expressed as

$$f_z(z)=\exp[-y(z)]/\sqrt{2\pi\epsilon^2}, \tag{13}$$

with $y(z)=(z^2+\mu^2)/2\epsilon^2-\ln[\cosh(\mu z/\epsilon^2)]$ [21,24]. Based on Eq. (13), the corresponding normalized LOP can be expressed as

$$g_{\text{opt}}^n(x)=x-\mu\tanh\left(\frac{\mu x}{\epsilon^2}\right). \tag{14}$$

For $0\le m\le 1$, assume $\mu=m\sigma_z$ and $\epsilon^2=(1-m^2)\sigma_z^2$, Eq. (13) becomes a standardized Gaussian mixture PDF [26]

$$f_{z_0}(z_0)=\exp[-y(z_0)]/\sqrt{2\pi(1-m^2)}, \tag{15}$$

with $y(z_0)=(z_0^2+m^2)/2(1-m^2)-\ln[\cosh(mz_0/(1-m^2))]$. The Fisher information $I(f_{z_0})$ versus the parameter m of the standardized Gaussian mixture PDF is shown in Fig. 1, and $I(f_{z_0})$ can be calculated as (no explicit expression exists)

$$I(f_{z_0})=\mathrm{E}\left\{\left[\frac{z_0}{1-m^2}-\frac{m}{1-m^2}\tanh\left(\frac{mz_0}{1-m^2}\right)\right]^2\right\}. \tag{16}$$

It is interesting to note that, as $m=0$, Eq. (15) is the standardized Gaussian PDF with $I(f_{z_0})=1$, as shown in Fig. 1. While, $I(f_{z_0})\to+\infty$ as $m\to 1$. In Eq. (12), for $m\to 1$, $\mu=m\sigma_z\to\sigma_z$ and $\epsilon^2=(1-m^2)\sigma_z^2\to 0$, the noise $z(t)$ becomes the dichotomous noise with equiprobable values at $\pm\sigma_z$ (but randomly takes levels $\pm\sigma_z$). In this case, the PDF of $z(t)$ can be represented as

$$f_z(z)=\frac{1}{2}[\delta(z-\sigma_z)+\delta(z+\sigma_z)], \tag{17}$$

where $\delta(z)$ is the Dirac delta function. The corresponding normalized LOP for dichotomous noise is

$$g_{\text{opt}}^n(x)=x-\sigma_z\,\text{sign}(x), \tag{18}$$

where $\lim_{\epsilon\to 0,\mu\to\sigma_z}\tanh(\mu x/\epsilon^2)=\text{sign}(x)$ $(\mu>0)$ in Eq. (14). Here, the normalized LOP g_{opt}^n is not continuous at $x=0$, but the above analysis is valid for processing a known weak

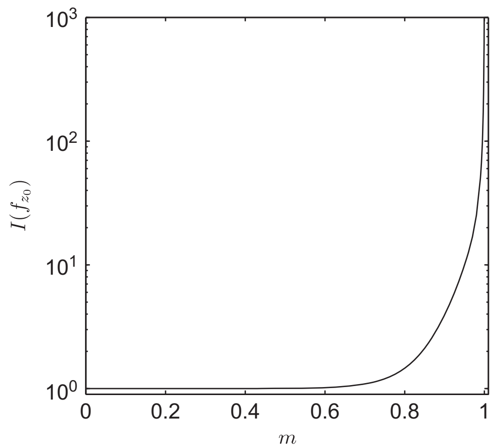

Fig. 1. Fisher information $I(f_{z_0})$ versus the parameter m of the standardized Gaussian mixture noise PDF of Eq. (15). As $m=0$ and $m=1$, Eq. (15) represents the Gaussian noise PDF with $I(f_{z_0})=1$ and the dichotomous noise PDF with $I(f_{z_0})=\infty$, respectively.

signal in dichotomous noise. This point is like the case of LOP $g^n_{\rm opt}(x)={\rm sign}(x)$ for Laplacian noise [44].

When the dichotomous noise $z(t)$ randomly takes two levels $\pm\sigma_z$ and $s(t)$ is weak compared with $z(t)$ ($\sigma_z>|s(t)|$), the signs of input $x(t)=s(t)+z(t)$ always take the sign of $z(t)$, i.e. ${\rm sign}(x)={\rm sign}(z)$ in Eq. (18). Therefore, the LOP of Eq. (18) at a fixed time t can be solved as $g^n_{\rm opt}[x(t)]=x(t)-\sigma_z\,{\rm sign}[x(t)]=s(t)+z(t)-\sigma_z\,{\rm sign}[z(t)]=s(t)$. Moreover, Refs. [7,41] have pointed out that there exists a scheme allowing a perfect recovery of $s(t)$ corrupted by dichotomous noise $z(t)$ with the PDF of Eq. (17). Thus, according to the optimal performance of the LOP of Eq. (8), $I(f_{z_0})=\infty$ contained in the type of PDF of Eq. (17), as shown in Fig. 1. Using Eq. (15), $I(f_{z_0})$ of Eq. (16) can be computed as

$$I(f_{z_0})={\rm E}\left[\frac{f'^2_{z_0}(z_0)}{f^2_{z_0}(z_0)}\right]={\rm E}\left[\left(\frac{dy(z_0)}{dz_0}\right)^2\right]$$
$$=\lim_{m\to1}\int_{-\infty}^{\infty}\frac{\left[1-m^2-m^2\,{\rm sech}^2\left(\frac{mz_0}{1-m^2}\right)\right]}{(1-m^2)^2}\frac{\exp[-y(z_0)]}{\sqrt{2\pi(1-m^2)}}\,dz_0=\infty, \tag{19}$$

where $\lim_{m\to1}[m^2\,{\rm sech}^2(mz_0/(1-m^2))]=0$, the numerator is the infinitesimal $O(1-m^2)$ and the denominator is a higher-order infinitesimal $O((1-m^2)^2)$ in the integral. Note that the noise type with infinite Fisher information is not unique, and another noise type is uniform noise indicated in Eq. (30).

2.2. No SR effect in an updated LOP by adding extra noise

We now consider the method of adding extra noise $\eta(t)$, independent of $z(t)$ and $s(t)$, to the given data $x(t)$, and investigate the possibility of improving the output SNR of $y(t)=g[x(t)+\eta(t)]$. Here, it is noted that the input SNR $R_{\rm in}$ for the given data $x(t)$ is fixed.

After adding $\eta(t)$ to $x(t)$, the resulting data $\hat{x}(t)=s(t)+z(t)+\eta(t)$. The composite noise $\hat{z}(t)=z(t)+\eta(t)$, $z(t)$ and $\eta(t)$ have PDFs of $f_{\hat{z}}(\hat{z})=\int_{-\infty}^{\infty}f_z(\hat{z}-u)f_\eta(u)\,du$, f_z and f_η, respectively. From Eq. (9), as $s(t)$ is currently buried in $\hat{z}(t)$, the corresponding LOP should be updated as

$$\hat{g}_{\rm opt}(x)=-f'_{\hat{z}}(x)/f_{\hat{z}}(x), \tag{20}$$

for achieving the maximal output SNR $\hat{R}_{\rm out}$. Substituting $\hat{g}_{\rm opt}$ into Eq. (8), the output SNR $\hat{R}_{\rm out}$ of $\hat{g}_{\rm opt}$ is

$$\hat{R}_{\rm out}=R_{\rm in}\sigma_z^2 I(f_{\hat{z}}). \tag{21}$$

Then, noting Eqs. (7) and (8), we can evaluate the ratio

$$\frac{\hat{R}_{\rm out}}{R_{\rm out}}=\frac{R_{\rm in}\sigma_z^2 I(f_{\hat{z}})}{R_{\rm in}\sigma_z^2 I(f_z)}=\frac{I(f_{\hat{z}})}{I(f_z)}, \tag{22}$$

to judge the role of the addition of $\eta(t)$ to $x(t)$.

Since $z(t)$ and $\eta(t)$ are independent, it is well known that $I(f_{\hat{z}})$, $I(f_z)$ and $I(f_\eta)$ of $\hat{z}(t)$, $z(t)$ and $\eta(t)$, satisfy the convolution inequality [46,47]

$$I^{-1}(f_{\hat{z}})\ge I^{-1}(f_z)+I^{-1}(f_\eta), \tag{23}$$

where the Fisher information $I(f)>0$ for any PDF f [46,47]. Thus, we have

$$\frac{\hat{R}_{\rm out}}{R_{\rm out}}=\frac{I(f_{\hat{z}})}{I(f_z)}\le 1-\frac{I(f_{\hat{z}})}{I(f_\eta)}\le 1, \tag{24}$$

which indicates that $\hat{R}_{\rm out}\le R_{\rm out}$ and the addition of $\eta(t)$ to $x(t)$ can not improve the output SNR.

This result of Eq. (24) generalizes a proof that existed previously only for a weak periodic signal in additive Gaussian white noise [4,8,41]. Using Eq. (24), the conclusion in [4,8,41] can be easily explained. The original Gaussian noise $z(t)$ is with PDF $f_z(z)=\exp[-z^2/(2\sigma_z^2)]/\sqrt{2\pi\sigma_z^2}$ and Fisher information $I(f_z)=1/\sigma_z^2$, and the added Gaussian noise $\eta(t)$ has its Fisher information $I(f_\eta)=1/\sigma_\eta^2$ and PDF $f_\eta(\eta)=\exp[-\eta^2/(2\sigma_\eta^2)]/\sqrt{2\pi\sigma_\eta^2}$. Then, the composite noise $\hat{z}(t)=z(t)+\eta(t)$ is also Gaussian distributed with PDF $f_{\hat{z}}(\hat{z})=\exp[-\hat{z}^2/(2\sigma_{\hat{z}}^2)]/\sqrt{2\pi\sigma_{\hat{z}}^2}$ and the Fisher information $I(f_{\hat{z}})=1/\sigma_{\hat{z}}^2$. It is noted that the equality of Eq. (23) occurs, this is $I(f_{\hat{z}})^{-1}=\sigma_{\hat{z}}^2=I(f_z)^{-1}+I(f_\eta)^{-1}=\sigma_z^2+\sigma_\eta^2$. From Eq. (24), the ratio of $\hat{R}_{\rm out}/R_{\rm out}=I(f_{\hat{z}})/I(f_z)=\sigma_z^2/(\sigma_z^2+\sigma_\eta^2)\le 1$ for $\sigma_\eta\ge 0$. We also note that the normalized LOP for a weak signal in Gaussian noise $z(t)$ is simply the linear system $g_{\rm L}(x)=x$, and the output SNR $R_{\rm out}=R_{\rm in}$ [4,18,24,44]. Therefore, the conclusion in [4] is comprehensible that, for a static nonlinearity driven by a weak periodic signal plus Gaussian noise, the SNR at the output can not exceed the input SNR by adding more Gaussian noise to the signal. Here, Eq. (24) generalizes this conclusion to arbitrary noise types of $z(t)$ and $\eta(t)$ for processing a weak periodic signal.

3. SR effect in a prescribed LOP

In Section 2, it is proven that, if a LOP can be updated according to the actual noise, the SR method cannot improve the output SNR in the weak-signal condition. However, this result also indicates that the SR effect should be observed outside the restricted conditions that a weak periodic signal in additive white noise is processed by an updated LOP [21,24–26,42]. In this section, we mainly explore the possibility of SR in a prescribed LOP that is simply locally optimum at a specific noise level for processing a weak periodic signal.

3.1. A prescribed LOP that matches the background noise

Given a LOP deduced from Eq. (9), we here give a general conclusion of the observation of SR effect in it. This is, if a normalized LOP $g^n_{\rm opt}(x,\sigma_z)$ is a function of the noise root-mean-square amplitude σ_z, then such a prescribed nonlinearity $g^n_{\rm opt}(x,\sigma_z^*)$ can exhibit the SR effect. Here, the fixed parameter σ_z^* is preestablished. This is because, when $g^n_{\rm opt}(x,\sigma_z^*)$ is a function of σ_z^*, it is only locally optimal for a specific value of σ_z^*, yielding the maximal output–input SNR gain $G=I(f_{z_0})$ of Eq. (8) as $\sigma_z=\sigma_z^*$. In other words, if σ_z is less or larger than σ_z^*, the output–input SNR gain of $g^n_{\rm opt}(x,\sigma_z^*)$ cannot reach its maximal value of Eq. (8), and the prescribed $g^n_{\rm opt}(x,\sigma_z^*)$ is not the corresponding LOP for the background noise. This is the typical characteristic of SR [1,8]. On the other hand, if a prescribed normalized LOP $g^n_{\rm opt}$ is not a function of σ_z, then no statistics of noise parameters can be tuned to improve the performance of $g^n_{\rm opt}$, and no SR effect will occur [24].

Example 2. For generalized Gaussian noise $z(t)$ with PDF [44]

$$f_z(z)=\frac{c_1}{\sigma_z}\exp\left(-c_2\left|\frac{z}{\sigma_z}\right|^{\alpha}\right), \quad (25)$$

whereby $c_1=(\alpha/2)\Gamma^{1/2}(3/\alpha)/\Gamma^{3/2}(1/\alpha)$ and $c_2=[\Gamma(3/\alpha)/\Gamma(1/\alpha)]^{\alpha/2}$ for the exponent $\alpha>0$. The corresponding normalized LOP is [44,45]

$$g_{\rm opt}^n(x)=|x|^{\alpha-1}\,{\rm sign}(x). \quad (26)$$

Based on Eq. (8) and the theoretical results of [44,45], the output–input SNR gain achieved by the LOP of Eq. (26) is

$$G=I(f_{z_0})=\alpha^2\Gamma(3\alpha^{-1})\Gamma(2-\alpha^{-1})/\Gamma^2(\alpha^{-1}). \quad (27)$$

It is noted that the normalized LOP of Eq. (26) is not a function of σ_z, and the maximal output–input SNR gain G of Eq. (27) [44] is a given quantity for a fixed exponent α that indicates the noise type. Thus, no SR will appear in a prescribed LOP of Eq. (26).

Example 3. For hyperbolic secant noise $z(t)$ with an unimodal PDF $f_z(z)=(1/2\sigma_z){\rm sech}(\pi x/2\sigma_z)$ [45], the corresponding normalized LOP $g_{\rm opt}^n(x,\sigma_z)=\tanh(\pi x/2\sigma_z)$ is a function of σ_z. Then, from Eq. (7), the output–input SNR gain G of a prescribed nonlinearity $g_{\rm opt}^n(x,\sigma_z^*)=\tanh(\pi x/2\sigma_z^*)$ can be calculated, as shown in Fig. 2. Here, σ_z^* is a prior fixed parameter. It is clearly seen that, as $\sigma_z/\sigma_z^*=1$, the output–input SNR gain can reach the maximum of $G=I(f_{z_0})=\pi^2/8$ determined by Eq. (8). This example also solves the conjecture by Zozor and Amblard [24], i.e. a prescribed LOP deduced from an unimodal noise PDF exhibits the SR effect.

3.2. A prescribed LOP that matches the composite noise

There is another realization of SR in a prescribed LOP: when a prescribed static nonlinearity is not the matched LOP for the initial noise, but it is a corresponding LOP to the composite noise. Then, if this prescribed LOP is a function of the root-mean-square amplitude of the added noise, the SR effect will certainly appear. Especially, when dichotomous noise acts as the added noise, a family of LOPs exists with their structures determined by both the noise PDF and the level of added noise. Such a prescribed LOP always exhibits the SR effect, whatever the type of initial noise is. This is because, for a weak signal $s(t)$ corrupted by initial noise $z(t)$ with PDF f_z, we add the dichotomous noise $\eta(t)$ with its PDF $f_\eta(\eta)=[\delta(\eta-\sigma_\eta)+\delta(\eta+\sigma_\eta)]/2$ to the signal. Then, the composite noise $\hat z(t)=z(t)+\eta(t)$ has PDF $f_{\hat z}(x)=[f_z(x-\sigma_\eta)+f_z(x+\sigma_\eta)]/2$, and the corresponding LOP becomes

$$g_{\rm opt}(x)=-\frac{f'_{\hat z}(x)}{f_{\hat z}(x)}=-\frac{f'_z(x-\sigma_\eta)+f'_z(x+\sigma_\eta)}{f_z(x-\sigma_\eta)+f_z(x+\sigma_\eta)}. \quad (28)$$

If $g_{\rm opt}(x)=\phi(\sigma_\eta)g_{\rm opt}^n(x)$ and the normalized LOP $g_{\rm opt}^n(x)$ is not a function of σ_η, then $f_{\hat z}(x)=C[\exp(-\int g_{\rm opt}^n(x)\,dx)]^{\phi(\sigma_\eta)}$ (C is a normalized constant). Thus, f_z does not contain $x\pm\sigma_\eta$ terms, which is contrary to the consequence of the convolved PDF $f_{\hat z}$. Therefore, $g_{\rm opt}(x)=\phi(\sigma_\eta)g_{\rm opt}^n(x,\sigma_\eta)$, and the normalized LOP $g_{\rm opt}^n(x,\sigma_\eta)$ must be a function of σ_η. Therefore, the addition of dichotomous noise $\eta(t)$ to the signal can always elicit a family of prescribed normalized LOPs $g_{\rm opt}^n(x,\sigma_\eta^*)$ that exhibit the SR effect.

Example 4. Assume $s(t)$ is initially corrupted by the generalized Gaussian noise with its PDF of Eq. (25). After adding the dichotomous noise $\eta(t)$ to $x(t)$, the generalized Gaussian mixture noise $\hat z(t)=z(t)+\eta(t)$ has its PDF $f_{\hat z}(x)=[f_z(x-\sigma_\eta)+f_z(x+\sigma_\eta)]/2$ and variance $\sigma_{\hat z}^2=\sigma_z^2+\sigma_\eta^2$. Then, the corresponding LOP can be expressed as

$$g_{\rm opt}^n(x)=\left[|x-\sigma_\eta|^{\alpha-1}\,{\rm sign}(x-\sigma_\eta)\exp\left(-c_2\left|\frac{x-\sigma_\eta}{\sigma_z}\right|^{\alpha}\right)\right.$$
$$\left.+|x+\sigma_\eta|^{\alpha-1}\,{\rm sign}(x+\sigma_\eta)\exp\left(-c_2\left|\frac{x+\sigma_\eta}{\sigma_z}\right|^{\alpha}\right)\right]$$
$$\left[\exp\left(-c_2\left|\frac{x-\sigma_\eta}{\sigma_z}\right|^{\alpha}\right)+\exp\left(-c_2\left|\frac{x+\sigma_\eta}{\sigma_z}\right|^{\alpha}\right)\right]^{-1}. \quad (29)$$

Assume a prescribed nonlinearity $g_{\rm opt}^n(x,\sigma_z^*,\sigma_\eta^*)$ in Eq. (29) is with the fixed parameters $\sigma_\eta^*=10\sigma_z^*$ and $\sigma_z^*=1$. Here, the initial generalized Gaussian noise $z(t)$ is given with its root-mean-square amplitude $\sigma_z^*=1$. Then, the input SNR $R_{\rm in}^*=|\langle s(t)\exp[-i2\pi t/T]\rangle|^2/(\Delta B\Delta t)$. Next, we add the dichotomous noise $\eta(t)$ to the signal, and compute the output SNR $R_{\rm out}$ of $g_{\rm opt}^n(x,\sigma_z^*,\sigma_\eta^*)$ by Eq. (7). For illustration, we plot the output SNR $R_{\rm out}/R_{\rm in}^*$ for Laplacian noise ($\alpha=1$) and Gaussian noise ($\alpha=2$) in Fig. 3. It is seen in Fig. 3 that, when σ_η equals to $\sigma_\eta^*=10\sigma_z^*$, $g_{\rm opt}^n(x,\sigma_z^*,\sigma_\eta^*)$ achieves its maximal output SNR, and SR phenomena appear. Since the input SNR $R_{\rm in}^*$ is a given quantity, Fig. 3 clearly shows that the output SNR $R_{\rm out}$ at the resonant point of $\sigma_\eta/\sigma_z^*=10$ is larger than the initial output SNR at $\sigma_\eta/\sigma_z^*=0$ for both Laplacian noise ($\alpha=1$) and Gaussian noise ($\alpha=2$).

Example 5. Adding noise to a nonlinear system is currently richly recognized for enhancing the system performance [21,25,26,32]. Here, we illustratively show that the method of adding noise can be used to interpret the related information inequality [47,49]. According to the

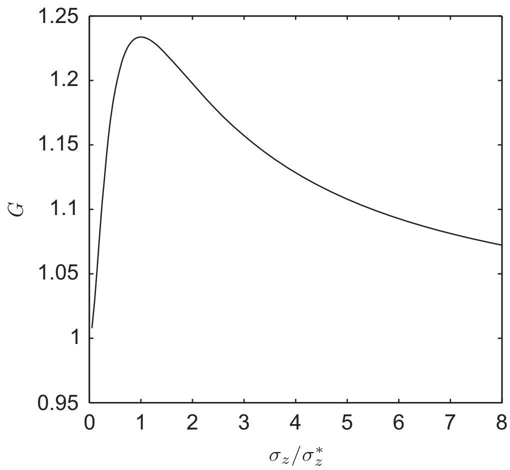

Fig. 2. The output–input SNR gain G of a prescribed normalized LOP $g_{\rm opt}^n(x,\sigma_z^*)=\tanh(\pi x/2\sigma_z^*)$ versus the noise root-mean-square amplitude σ_z/σ_z^* for the hyperbolic secant noise $z(t)$ with PDF $f_z(z)=(1/2\sigma_z)\,{\rm sech}(\pi x/2\sigma_z)$.

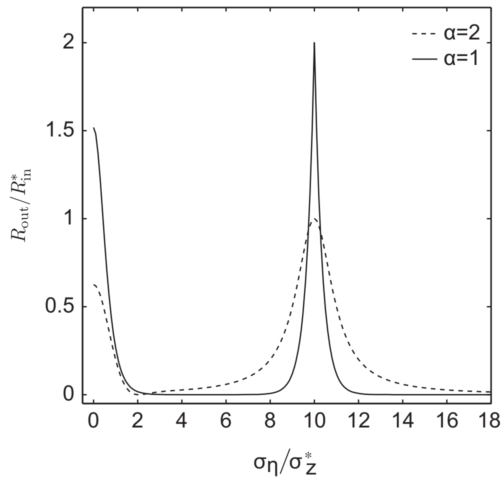

Fig. 3. The output SNR $R_{\rm out}/R^*_{\rm in}$ of a prescribed LOP $g^n_{\rm opt}(x,\sigma^*_z,\sigma^*_\eta)$ ($\sigma^*_\eta=10\sigma^*_z,\sigma^*_z=1$) versus the root-mean-square amplitude σ_η/σ^*_z of the dichotomous noise $\eta(t)$. Here, the initial noise $z(t)$ is with the root-mean-square amplitude $\sigma^*_z=1$ for $\alpha=1$ (Laplacian noise) and $\alpha=2$ (Gaussian noise), respectively. The input SNR is given by $R^*_{\rm in}=|\langle s(t)\exp[-i2\pi t/T]\rangle|^2/(\Delta t\Delta B)$.

relationship of the output SNR and the Fisher information of noise distribution, we illustrate a new example of Fisher information equality $I(f_{\hat z})^{-1}=I(f_z)^{-1}+I(f_\eta)^{-1}$ in Eq. (23) by adding the uniform noise to a weak periodic signal in the dichotomous noise, except for the Gaussian noise in Section 2.2. The Fisher information equality in Eq. (23) has been noted in [49], but not validated by a practical example. Consider the dichotomous noise $z(t)$ with its PDF of Eq. (17), the Fisher information $I(f_z)=I(f_{z_0})/\sigma_z^2=\infty$ for $0<\sigma_z^2<\infty$, as indicated in Eq. (19). From Eq. (8), this point also represents $R_{\rm out}=\infty$, since $R_{\rm in}$ of Eq. (2) is finite as the variance $0<\sigma_z^2<\infty$. Next, for a weak periodic signal $s(t)$ buried in the dichotomous noise, we add the uniform noise $\eta(t)$ to the mixture of $s(t)+z(t)$. Here, as the exponent $\alpha=\infty$, Eq. (25) represents the PDF of uniform noise as

$$f_\eta(x)=1/(2b) \tag{30}$$

for $-b\le x\le b$ ($b=\sqrt{3}\sigma_\eta>0$) and zero otherwise. Based on Eq. (27), we know that the Fisher information of uniform noise is $I(f_\eta)=I(f_{\eta_0})/\sigma_\eta^2=\infty$ for $0<\sigma_\eta^2<\infty$ [44]. Then, the weak signal $s(t)$ is currently corrupted by the composite noise $\hat z(t)=z(t)+\eta(t)$ with PDF $f_{\hat z}(x)=[f_\eta(x-\sigma_z)+f_\eta(x+\sigma_z)]/2$. This PDF is discontinuous at $x=\pm(b+\sigma_z)$, and has an infinite Fisher information $I(f_{\hat z})$ [48]. Practically, we consider the three-threshold nonlinearity [7]

$$g_{\rm th}(x)=\begin{cases}-1 & \text{for } x<-c,\\ 0 & \text{for } -c\le x\le c,\\ +1 & \text{for } x>c,\end{cases} \tag{31}$$

with response thresholds at $x=\pm c$ for processing a weak periodic signal in the composite noise $\hat z(t)$. Based on Eq. (8), the output–input SNR gain of the three-threshold nonlinearity can be computed as

$$G=\sigma_z^2\frac{{\rm E}^2[g'_{\rm th}(x)]}{{\rm E}[g^2_{\rm th}(x)]}\bigg|_{c=b+\sigma_z}=\sigma_z^2\frac{2f^2_{\hat z}(c)}{1-F_{\hat z}(c)}\bigg|_{c=b+\sigma_z}=\infty, \tag{32}$$

where $F_{\hat z}(x)=\int_{-\infty}^x f_{\hat z}(u)\,du$ represents the cumulative distribution function of $\hat z(t)$ and the response threshold $c=b+\sigma_z$. In this case, $f_{\hat z}(c)=f_\eta(c-\sigma_z)/2=f_\eta(b)/2\neq 0$ and $F_{\hat z}(c)=1$, then we have $G=\infty$. Here, the three-threshold nonlinearity in Eq. (31) corresponds to the LOP of noise $\hat z(t)$. This result of infinite G in Eq. (32) accords with the infinite Fisher information $I(f_{\hat z})$ of the composite noise $\hat z(t)$. Then, the equality of $I(f_{\hat z})^{-1}=I(f_z)^{-1}+I(f_\eta)^{-1}$ in Eq. (23) is proven for the uniform noise $\eta(t)$ and the dichotomous noise $z(t)$.

4. Conclusion

In this paper, we studied the performance of a LOP for processing a weak periodic signal in additive white noise. Under the weak-signal condition, it is known that the LOP possesses the maximal output SNR, and its structure is precisely determined by the noise probability density and also by the noise level. We further proved that the output SNR of a LOP is closely related to the Fisher information of the noise PDF. Interestingly, the output–input SNR gain of a LOP is given by the Fisher information of a standardized noise PDF. It is well known that the Gaussian noise PDF has the minimal Fisher information of unity [44]. Based on the relationship between the output–input SNR gain and the Fisher information, this result indicates that the SNR gain, for a LOP, is certainly larger than unity for a weak periodic signal in additive non-Gaussian noise. Moreover, for a LOP, an arbitrarily large output–input SNR gain can be achieved ranging from the minimal value of unity up to infinity, which is achieved via Gaussian mixture noise. Furthermore, by the Fisher information convolution inequality, we demonstrated that the updated LOP is proven unable to improve the output SNR. This result extends a proof that existed previously only for additive Gaussian white noise [4,8,41] to other noise types. Beyond these restrictive conditions, we explored the possibility of SR in a prescribed LOP. We concluded that if a normalized LOP is a function of the noise root-mean-square amplitude, such a prescribed LOP can exhibit the SR effect. Especially, using dichotomous noise as the added noise, a family of LOPs is elicited with their structures being a function of the added noise level. Then it is shown that, such a prescribed LOP can always exhibit the SR effect. In addition, using the relationship between the output SNR and the Fisher information, the Fisher information equality is valid for the uniform noise and the dichotomous noise. This new Fisher information equality provides a practical realization in the context of weak signal processing.

We now know that the output SNR of an updated LOP can not be improved by adding extra noise to a weak periodic signal in additive white noise. Beyond the restricted condition of an updated LOP, we here observed the occurrence of SR effect in some prescribed LOPs. Therefore, it is also interesting to explore the SR effect that occurs outside of other restrictive conditions given in Section 2. For instance, the LOP is unrealizable [42] or too complex to be implemented [21,45], and the input signal is non-weak [26]. In some practical signal processing tasks, the original noise distribution or the noise level is unknown [24,45] and the LOP can not be pre-established. Therefore, we can employ a suboptimal processor (compared with the corresponding LOP) to obtain a better but available system performance by exploiting the constructive role of added noise. These problems will be of interest for further studies of nonlinear signal processing.

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