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Tsallis entropy measure of noise-aided information transmission in a binary channel

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ABSTRACT

Noise-aided information transmission via stochastic resonance is shown and analyzed in a binary channel by means of information measures based on the Tsallis entropy. The analysis extends the classic reference of binary information transmission based on the Shannon entropy, and also parallels a recent study based on the Rényi entropy. The conditions for a maximally pronounced stochastic resonance identify optimal Tsallis measures. The study involves a correspondence between Tsallis and Rényi information measures, specially relevant to the characterization of stochastic resonance, and establishing that for such effects identical properties are shared in common by both Tsallis and Rényi measures.

1. Introduction

Information is an essential notion of our everyday experience. A quantitative approach to information is also fundamental to science and technology. Especially, growing impact is witnessed at the intersection with physics, where informational concepts are progressing for instance in statistical physics, thermodynamics, nonlinear physics, complexity science, neural biophysics, quantum computation, and the physics of information (see for instance [1–10] for general overviews, and [11,12] for very recent examples in this journal). A very important quantitative approach to information, expressed in a statistical framework and based on the concept of entropy, has been initiated by the work of Shannon and Weaver [13]. This statistical theory of information has shown great impact on communication technologies, and also it offered a foundation to statistical mechanics [14,15]. An important extension to the entropy underpinning the Shannon theory of information, has been provided by the Rényi entropy [16]. Applicability of Rényi-entropy-based information measures has been demonstrated for various practical problems such as source coding or classification [17–23]. Comparatively, the Rényi entropy has so far found probably less impact for its significance toward physics and the physics of information. Another, more recent, extension to the entropy of the Shannon theory of information, has been proposed with the Tsallis entropy [24,25]. The Tsallis entropy is involved into much closer connections with physics, since it has been postulated and tested to form the ground of a nonextensive generalization to statistical mechanics [24,26–28,25]. A line of thought is that, in the presence of physical constraints on known statistical averages, maximization of the Tsallis entropy leads to probability distributions for the states of a system, that generalize (and contain as a special case) the Boltzmann–Gibbs distribution of standard statistical mechanics. Furthermore, such generalized probability distributions have been found to offer efficient models to fit many experimental observations and data sets, especially associated with "complex" processes of various types, possibly with long range interactions where the specific nonextensive character of the Tsallis entropy could play a relevant role. In the present Letter, as a complement, we will explore the significance of the Tsallis entropy toward informational issues. Especially, we test measures of information based on the Tsallis entropy in order to assess an effect of stochastic resonance or noise-aided informational signal transmission in a reference model formed by a binary channel.

Stochastic resonance is a phenomenon originating in nonlinear physics, which is progressively gaining the status of a general paradigm to designate situations where the noise can reveal beneficial to some transmission or processing of information [29–31]. It is a specially significant concept at the intersection of physics and information science, which is still under active exploration. Stochastic resonance has been reported in a large variety of domains, including optical devices [32–34], electronic circuits [35–37], neural processes [38–41], nonlinear sensors [42–44]. Classi-

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cally, to characterize an effect of stochastic resonance, a measure of performance appropriate to the process under study is introduced, and conditions are investigated where this measure can be improved by the action of noise. In this way, several standard measures have been shown improvable by an increase of the noise. The signal-to-noise ratio was one of the earliest measures applied to characterize stochastic resonance [45,29,46]. However, it has repeatedly appeared that the signal-to-noise ratio does not necessarily provide a complete characterization of the impact of noise for all situations and purposes, and that in fact there is not one single measure universally significant for stochastic resonance. Other measures have progressively been shown relevant to characterize stochastic resonance, such as a cross-correlation coefficient [47–49], a probability of detection [50,30,51,52], or an estimation error [53–55]. Information-theoretic quantities, owing to their status of general information measures, have also been applied for assessment of stochastic resonance. Essentially Shannon-entropy-based information measures have been considered, with the demonstration of mutual information [56–59] or information capacity [60–62] improved by noise. Also, very recently, Ref. [63] developed a characterization of noise-aided transmission over a binary channel with information measures based on the Rényi entropy, generalizing the Shannon entropy, and uncovering nontrivial orders of the Rényi entropy that best exploit stochastic resonance. In the present Letter, a comparable analysis of stochastic resonance is carried out, yet with information measures based on the Tsallis entropy. The aim is two-fold: to evaluate the capabilities of general information measures based on the Tsallis entropy to assess meaningful informational processes like stochastic resonance; and to consolidate stochastic resonance as an informational process with broad applicability and significance. The present study especially develops a correspondence between Tsallis and Rényi information measures, which appears specifically relevant here to the characterization of stochastic resonance. This allows the novel analysis with the Tsallis entropy here, to closely parallel the approach of [63] with the Rényi entropy, and results in establishing that, despite the significant difference that the Tsallis entropy is nonextensive while the Rényi entropy is extensive, relevant properties in stochastic resonance are shared in common by both information measures.

Strong connections exist of the Tsallis entropy with statistical physics and of stochastic resonance with nonlinear physics. At the intersection, the present Letter seeks to contribute also to frontier physics formed by the physics of information [1,2]. Two important steps are useful to proceed in this direction: (i) to identify and assess the capabilities and properties of quantitative measures for information, and (ii) to involve them in the description of observable processes to which informational significance can be assigned. We perform here part (i) with Tsallis-entropy-based information measures, and part (ii) by means of a stochastic resonance phenomenon expressing a possibility of noise-aided informational signal transmission.

2. Tsallis entropy measures and properties

2.1. Tsallis entropy

The Tsallis entropy [24,25] represents a generalization of the Boltzmann–Gibbs or Shannon entropy. For an information source emitting symbols with probabilities P_i, for $i = 1$ to N, the Tsallis entropy of order q is defined as

$$H_q(P_i) = \frac{1}{\ln(a)} \frac{1}{q-1}\left(1 - \sum_{i=1}^{N} P_i^q\right). \tag{1}$$

At the limit $q = 1$, one has $H_1(P_i) = -\sum_{i=1}^{N} P_i \log_a(P_i)$, i.e. the Shannon entropy in the logarithm base a. The Tsallis entropy bears special importance since it has been postulated to form the ground of a nonextensive generalization to statistical mechanics [24,26–28,25].

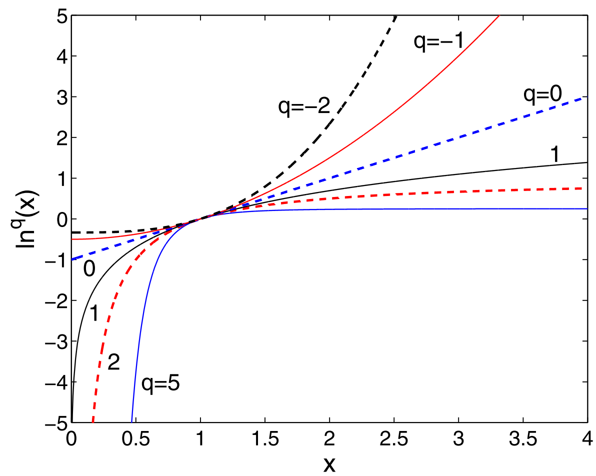

Fig. 1. The q-logarithm $\ln^q(x)$ versus x from Eq. (2) for various values of $q = -2, -1, 0, 1, 2, 5$. At $q = 1$ is the traditional natural logarithm $\ln^1(x) = \ln(x)$. At $q = 0$ is the linear function $x \mapsto x - 1$. All the curves intersect at point $(1, 0)$.

It is possible to express Eq. (1) through a generalization [64,25] of the natural logarithm, under the form of the q-logarithm defined as

$$\ln^q(x) = \frac{1 - x^{1-q}}{q-1}, \tag{2}$$

for a positive real argument x and q a real parameter. Inversion of $y = \ln^q(x)$ defines the q-exponential function

$$\exp^q(y) = \left[1 + (1-q)y\right]^{1/(1-q)}. \tag{3}$$

At $q = 1$, one recovers the traditional natural logarithm $\ln^1(x) = \ln(x)$ and exponential $\exp^1(y) = \exp(y)$. Interesting generalized properties exist such as

$$\frac{d}{dx}\ln^q(x) = \frac{1}{x^q} \quad \text{and} \quad \frac{d}{dy}\exp^q(y) = \left[\exp^q(y)\right]^q, \tag{4}$$

and a combination property

$$\ln^q(x_1 x_2) = \ln^q(x_1) + \ln^q(x_2) + (1-q)\ln^q(x_1)\ln^q(x_2). \tag{5}$$

Fig. 1 depicts the graphs of $\ln^q(x)$ for various q, in particular illustrating that $\ln^q(x)$ is a monotonically increasing function for any q, as provable from the first equation in Eq. (4).

The q-logarithm of base a is

$$\log_a^q(x) = \frac{1}{\ln(a)} \frac{1 - x^{1-q}}{q-1}. \tag{6}$$

Inversion of $y = \log_a^q(x)$ defines the q-exponential of base a as

$$\exp_a^q(y) = \left[1 + \ln(a)(1-q)y\right]^{1/(1-q)}. \tag{7}$$

At $q = 1$, one recovers the traditional logarithm $\log_a^1(x) = \log_a(x)$ and exponential $\exp_a^1(y) = a^y$. The Tsallis entropy of Eq. (1) then is an expectation of the q-logarithm as

$$H_q(P_i) = \mathrm{E}\left[\log_a^q(1/P_i)\right] = \sum_{i=1}^{N} P_i \log_a^q(1/P_i). \tag{8}$$

The Tsallis entropy $H_q(P_i)$ of Eq. (1) is nonnegative for any order q. Since the q-logarithm $\log_a^q(x)$ is an increasing function of its argument x, then $\log_a^q(1/P_i)$ increases for rare events with

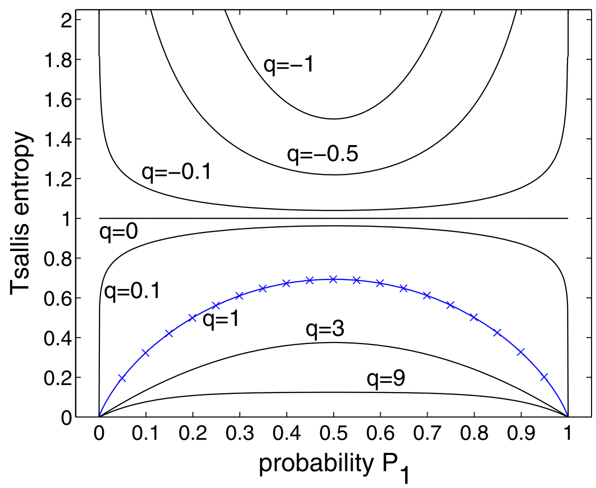

Fig. 2. Tsallis entropy $H_q(P_i)$ in nats of Eq. (1), as a function of the probability P_1 of a binary source $\{P_1, 1-P_1\}$, for several values of the order q. The value $q=1$ identified by crosses ($\times$) corresponds to the Shannon entropy, with the maximum $H_1(P_1=1/2)=H_1^{\text{extr}}=\ln(2)\approx 0.693$ nat.

smaller P_i, and its average value H_q in Eq. (8) keeps the status of a nonnegative measure of incertainty for any q, much like the Shannon entropy H_1. Also, $H_q(P_i)$ is concave ($\cap$) for $q>0$, constant for $q=0$, and convex ($\cup$) for $q<0$. The Tsallis entropy $H_q(P_i)$ reaches for $q>0$ its maximum (respectively for $q<0$ its minimum) of $H_q^{\text{extr}}=\log_a^q(N)$ with equiprobable $P_i=1/N$ for all $i=1$ to N. For a given probability distribution, $H_q(P_i)$ is a decreasing function of q.

For two sets A and B of independent events, the joint events (A,B) are involved in a nonadditive relation for the Tsallis entropies, following from Eq. (5),

$$H_q(A,B)=H_q(A)+H_q(B)+(1-q)\ln(a)H_q(A)H_q(B), \tag{9}$$

which forms the basis of the nonextensive character of the Tsallis entropy, and which takes part in a nonextensive generalization of statistical mechanics [24,26–28,25].

For illustration, Fig. 2 shows the Tsallis entropy $H_q(P_i)$ of Eq. (1) for a binary source (of interest to us in the sequel). Especially, Fig. 2 depicts a general property of H_q, that for $q<0$ the entropy H_q is not always bounded and can diverge to infinity in the presence of rare events with vanishing probabilities. Meanwhile, for $q>0$ the Tsallis entropy H_q always remains a finite bounded measure of incertainty, much like the Shannon entropy, and it will be studied essentially in this regime of $q>0$ in the sequel.

2.2. Tsallis relative entropy

It is possible to associate to the Tsallis entropy, a Tsallis relative entropy or divergence generalizing the Kullback–Leibler relative entropy; it refers to two probability distributions $\{P_i\}$ and $\{Q_i\}$, $i=1$ to N, over the same alphabet, and is defined as [25]

$$D_q(P_i\|Q_i)=-\sum_{i=1}^{N}P_i\log_a^q\left(\frac{Q_i}{P_i}\right)$$
$$=\frac{1}{\ln(a)}\frac{1}{1-q}\sum_{i=1}^{N}P_i\left[1-\left(\frac{Q_i}{P_i}\right)^{1-q}\right], \tag{10}$$

which is also

$$D_q(P_i\|Q_i)=\frac{1}{\ln(a)}\frac{1}{1-q}\left(1-\sum_{i=1}^{N}P_i^qQ_i^{1-q}\right). \tag{11}$$

At the limit $q=1$, one has $D_1(P_i\|Q_i)=\sum_{i=1}^{N}P_i\log_a(P_i/Q_i)$, i.e. the Kullback–Leibler relative entropy [65]. For any order $q\neq 0$, the Tsallis relative entropy $D_q(P_i\|Q_i)$ of Eq. (10) vanishes if and only if $P_i=Q_i$ for all $i=1$ to N. For any $q>0$, the Tsallis relative entropy $D_q(P_i\|Q_i)$ is always nonnegative. In this regime of $q>0$, the Tsallis relative entropy $D_q(P_i\|Q_i)$ therefore behaves much like the conventional Kullback–Leibler relative entropy, yet in a generalized form realized by the additional parameterization by q. Especially, the generalized family of relative entropy $D_q(P_i\|Q_i)$ allows one to obtain at $q=1/2$ a symmetric divergence

$$D_{1/2}(P_i\|Q_i)=\frac{2}{\ln(a)}\left(1-\sum_{i=1}^{N}\sqrt{P_iQ_i}\right), \tag{12}$$

which is directly (monotonically) related to the Bhattacharyya distance $B(P_i\|Q_i)=-\log(\sum_i\sqrt{P_iQ_i})=-\log(1-D_{1/2}/2)$ of two probability distributions. Additionally, when $q<0$, the Tsallis relative entropy $D_q(P_i\|Q_i)$ of Eq. (10) is always nonpositive; and when $q=0$, $D_{q=0}(P_i\|Q_i)$ is identically zero.

By choosing the reference probabilities $\{Q_i\}$ as the uniform distribution $\{Q_i=1/N\}$ for all $i=1$ to N, one obtains

$$D_q(P_i\|Q_i=1/N)=N^{q-1}\left[H_q^{\text{ext}}-H_q(P_i)\right], \tag{13}$$

expressing a connection between entropy and relative entropy at any Tsallis order q.

2.3. Tsallis transinformation

One now considers an input alphabet with N symbols, an output alphabet with M symbols, and over those two a joint probability distribution $\{P_{ij}\}$, for $(i,j)\in\{1,\dots,N\}\times\{1,\dots,M\}$, as would occur between the emitting and receiving ends of a communication channel. The N input symbols, indexed by i, have marginal probabilities $P_i=\sum_{j=1}^{M}P_{ij}$. The M output symbols, indexed by j, have marginal probabilities $Q_j=\sum_{i=1}^{N}P_{ij}$. A Tsallis transinformation or mutual information follows as

$$I_q=D_q(P_{ij}\|P_iQ_j)$$
$$=\frac{1}{\ln(a)}\frac{1}{1-q}\sum_{i=1}^{N}\sum_{j=1}^{M}P_{ij}\left[1-\left(\frac{P_{ij}}{P_iQ_j}\right)^{q-1}\right]. \tag{14}$$

At the limit $q=1$, the Tsallis transinformation I_1 from Eq. (14) is the Shannon transinformation.

When the input and output symbols are independent, the probabilities factorize as $P_{ij}=P_iQ_j$, for all $(i,j)\in\{1,\dots,N\}\times\{1,\dots,M\}$. As a result, the Tsallis transinformation I_q of Eq. (14) is identically zero, for any order q.

When applied to a communication channel with input X and output Y, transmission over the channel is characterized by the $N\times M$ conditional probabilities $P_{j|i}=\Pr\{Y=j\mid X=i\}$, for $(i,j)\in\{1,\dots,N\}\times\{1,\dots,M\}$. The $N\times M$ joint input–output probabilities result as $P_{ij}=P_{j|i}P_i$, and the M output probabilities $Q_j=\Pr\{Y=j\}=\sum_{i=1}^{N}P_{j|i}P_i$. The Tsallis information capacity C_q of the channel characterized by the $\{P_{j|i}\}$ is defined by the input probabilities $\{P_i^*\}$ that maximize the Tsallis input–output transinformation I_q from Eq. (14) as

$$C_q=\max_{\{P_i\}}I_q. \tag{15}$$

3. Relation to Rényi entropy measures

Another important generalization to the traditional Shannon entropy is provided by the Rényi entropy. For a probability distribution P_i, for $i=1$ to N, the Rényi entropy of order q is defined as [16]

$$\mathcal{H}_q(P_i) = \frac{1}{1-q}\log_a\left(\sum_{i=1}^{N} P_i^q\right). \tag{16}$$

At the limit $q=1$, the traditional Shannon entropy $\mathcal{H}_1(P_i) = -\sum_{i=1}^{N} P_i \log_a(P_i)$ is recovered.

From Eq. (16) one can extract the factor

$$\sum_{i=1}^{N} P_i^q = a^{(1-q)\mathcal{H}_q(P_i)} \tag{17}$$

and replacing in Eq. (1), a relation between Tsallis H_q and Rényi $\mathcal{H}_q$ entropies is obtained as

$$H_q = \frac{1}{\ln(a)}\frac{1}{q-1}\left[1 - a^{(1-q)\mathcal{H}_q}\right]. \tag{18}$$

From Eq. (6), another form is accessible for the relation of Eq. (18), as

$$H_q = \log_a^q\left(a^{\mathcal{H}_q}\right). \tag{19}$$

The function $H_q(\mathcal{H}_q)$ in Eq. (19), similar to $\log_a^q(\cdot)$, is a monotonic increasing function for any order q. At $q=1$, this function reduces to the identity function, when the two entropies H_1 and $\mathcal{H}_1$ in Eq. (19) coincide with the Shannon entropy. A relation between H_q and $\mathcal{H}_q$ similar to Eq. (18) is also given for instance in [25,66], although not written with the q-logarithm as in Eq. (19). When written with the q-logarithm as in Eq. (19), we have a formulation vividly manifesting the monotonic increasing character of the relation between H_q and $\mathcal{H}_q$, for any q. And such a character will play an important role in the sequel for the present study. Before, we extend here the relation between Tsallis and Rényi forms to other important information measures.

A Rényi relative entropy exists which generalizes the Kullback–Leibler relative entropy [65]. For two probability distributions $\{P_i\}$ and $\{Q_i\}$, $i=1$ to N, over the same alphabet, the Rényi relative entropy or divergence is defined as [16,23]

$$\mathcal{D}_q(P_i \| Q_i) = \frac{1}{q-1}\log_a\left(\sum_{i=1}^{N} P_i^q Q_i^{1-q}\right). \tag{20}$$

At the limit $q=1$, one gets $\mathcal{D}_1(P_i\|Q_i) = \sum_{i=1}^{N} P_i \log_a(P_i/Q_i)$, i.e. the Kullback–Leibler relative entropy [65]. For any order $q \neq 0$, the Rényi relative entropy $\mathcal{D}_q(P_i\|Q_i)$ of Eq. (20) vanishes if and only if $P_i = Q_i$ for all $i=1$ to N [23]. For any $q>0$, the Rényi relative entropy $\mathcal{D}_q(P_i\|Q_i)$ is always nonnegative.

From Eq. (20) one can extract the factor

$$\sum_{i=1}^{N} P_i^q Q_i^{1-q} = a^{(q-1)\mathcal{D}_q(P_i\|Q_i)} \tag{21}$$

and replacing in Eq. (11), a relation between Tsallis D_q and Rényi $\mathcal{D}_q$ relative entropies is obtained as

$$D_q = \frac{1}{\ln(a)}\frac{1}{1-q}\left[1 - a^{(q-1)\mathcal{D}_q}\right], \tag{22}$$

or equivalently,

$$D_q = \log_a^{2-q}\left(a^{\mathcal{D}_q}\right). \tag{23}$$

The function $D_q(\mathcal{D}_q)$ in Eq. (23), similar to $\log_a^{2-q}(\cdot)$, is a monotonic increasing function for any order q. At $q=1$, this function reduces to the identity function, when the two relative entropies D_1 and $\mathcal{D}_1$ in Eq. (23) coincide with the Kullback–Leibler relative entropy.

The Rényi transinformation or mutual information $\mathcal{I}_q$, paralleling I_q of Eq. (14), is

$$\mathcal{I}_q = \mathcal{D}_q(P_{ij}\|P_i Q_j), \tag{24}$$

and it is therefore related to the Tsallis transinformation I_q by the same monotonic increasing relation

$$I_q = \log_a^{2-q}\left(a^{\mathcal{I}_q}\right). \tag{25}$$

Due to the monotonic increasing character of Eq. (25), the Rényi channel capacity $\mathcal{C}_q$ realizing the maximum of $\mathcal{I}_q$ is related to the Tsallis capacity C_q in Eq. (15) similarly by

$$C_q = \log_a^{2-q}\left(a^{\mathcal{C}_q}\right), \tag{26}$$

and is achieved by the same input probabilities $\{P_i^*\}$ as in Eq. (15).

These relations between the Tsallis and Rényi information measures will be useful to us in the sequel. Both types of measures stand as possible generalizations that include the conventional Shannon information measures as a special case. As a useful application, these properties have formed the basis of an interesting generalization to the Shannon source coding theorem asserting an average coding length lower bounded by the Shannon entropy of the source. The work of [17] established an extended coding theorem where the Rényi entropy of the source forms a lower bound to some generalized average coding length. Based on the correspondence of Eq. (19), the possibility of another extended coding theorem based on the Tsallis entropy was announced in the publication of [21]. The theorem was explicitly worked out in [67], showing another generalized coding length lower bounded by the Tsallis entropy of the source. The correspondence of this section between Tsallis and Rényi information measures will appear specially significant also in the characterization of stochastic resonance.

4. A binary information channel

An information channel emits discrete input symbols X from the binary alphabet $\{0,1\}$. The successive input symbols are independent and identically distributed with the probabilities $P_1 = \Pr\{X=1\}$ and $P_0 = 1-P_1 = \Pr\{X=0\}$. At the receiving end of the channel, the discrete output symbols Y are in the binary alphabet $\{0,1\}$. Transmission over the channel is characterized by the four conditional probabilities $P_{j|i} = \Pr\{Y=j \mid X=i\}$, for $(i,j) \in \{0,1\}^2$. The joint input–output probabilities result as $P_{ij} = P_{j|i}P_i$, and the output probabilities $Q_j = \Pr\{Y=j\} = \sum_{i=0}^{1} P_{j|i}P_i$. For this discrete binary channel, the input–output Tsallis transinformation of Eq. (14) follows as

$$I_q(X;Y) = \frac{1}{1-q}\sum_{j=0}^{1}\left[P_0 P_{j|0}\left(1-\left(\frac{P_{j|0}}{Q_j}\right)^{q-1}\right) + P_1 P_{j|1}\left(1-\left(\frac{P_{j|1}}{Q_j}\right)^{q-1}\right)\right]. \tag{27}$$

Eq. (27) is the input–output Tsallis transinformation for any (memoryless) discrete binary channel characterized by the four transmission probabilities $P_{j|i}$. We now specify concrete physical conditions that determine a definite channel and its probabilities $P_{j|i}$. We consider the binary input X in the transmission corrupted by a white noise W to yield $X+W$, and then at the receiver $X+W$ is compared to a fixed response threshold θ to determine the binary output Y of the channel according to:

$$\text{If } X+W > \theta \quad \text{then } Y=1, \qquad \text{else } Y=0. \tag{28}$$

The noise W has the cumulative distribution function $F(w) = \Pr\{W \leqslant w\}$. The input X and the noise W are statistically independent.

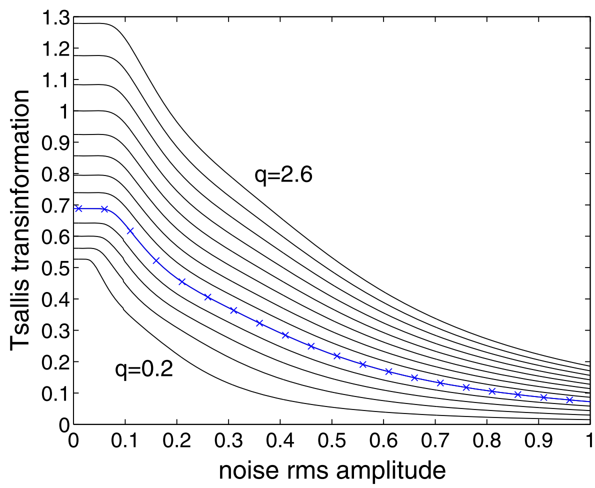

Fig. 3. Input–output Tsallis transinformation $I_q(X;Y)$ from Eq. (27), as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with input probability $P_1 = 0.45$ and threshold $\theta = 0.8$. The order q goes from 0.2 to 2.6 with step 0.2. The crosses ($\times$) identify $q = 1$ when $I_{q=1}(X;Y)$ is the Shannon transinformation. At $\sigma = 0$ is $I_q(X;Y) = H_{2-q}(P_0, P_1)$.

The input–output transmission probabilities of this discrete binary channel are readily derived. For instance, the probability $P_{0|1} = \Pr\{Y = 0 \mid X = 1\}$ is also $\Pr\{X + W \leqslant \theta \mid X = 1\}$ which amounts to $\Pr\{W \leqslant \theta - 1\} = F(\theta - 1)$. With similar rules one arrives at

$$P_{0|1} = \Pr\{Y = 0 \mid X = 1\} = F(\theta - 1), \tag{29}$$

$$P_{1|1} = \Pr\{Y = 1 \mid X = 1\} = 1 - F(\theta - 1), \tag{30}$$

$$P_{0|0} = \Pr\{Y = 0 \mid X = 0\} = F(\theta), \tag{31}$$

$$P_{1|0} = \Pr\{Y = 1 \mid X = 0\} = 1 - F(\theta). \tag{32}$$

These transmission probabilities $P_{j|i}$, $(i, j) \in \{0, 1\}^2$, define an asymmetric binary channel. We shall study on this channel the impact of the level of the noise W measured by its rms amplitude σ. Throughout the study, the level of signal is kept constant, only the level σ of the noise is varied. So this amounts to a signal-to-noise ratio which is a constant divided by σ; and the evolutions of the information measures we will study as a function of σ, as in Figs. 3, 4 and 6, can equivalently be seen as evolutions as a function of the signal-to-noise ratio. Also, as a classic picture inherent to the model of discrete channel, the bandwidth is not changed when the noise is varied. Bandwidth would come into play based on further assumptions on the rate at which discrete symbols are applied and transmitted through the channel. This aspect is decoupled and not addressed in the classic model of discrete channel, and assumed invariant in this respect.

For this asymmetric binary channel resulting from Eqs. (29)–(32), a typical evolution of the Tsallis transinformation $I_q(X;Y)$ of Eq. (27) is shown in Fig. 3, with the binary input $X = 0$ or 1 evolving on both sides of the response threshold $\theta = 0.8$. In such condition, the presence of the channel noise W in Eq. (28) hinders the recovery of the information signal at the receiving end. It results that the performance of the transmission, as measured by the input–output Tsallis transinformation $I_q(X;Y)$, decreases as the level of the noise W increases, as visible in Fig. 3.

In Fig. 3, a similar decreasing evolution of the Tsallis transinformation $I_q(X;Y)$ as the level of noise increases, is observed for any order q, especially, but not only, in the Shannon case $q = 1$. In this respect, this shows that the Tsallis transinformation $I_q(X;Y)$ at any order q, is capable of manifesting the detrimental action of the noise in the transmission of information through the channel.

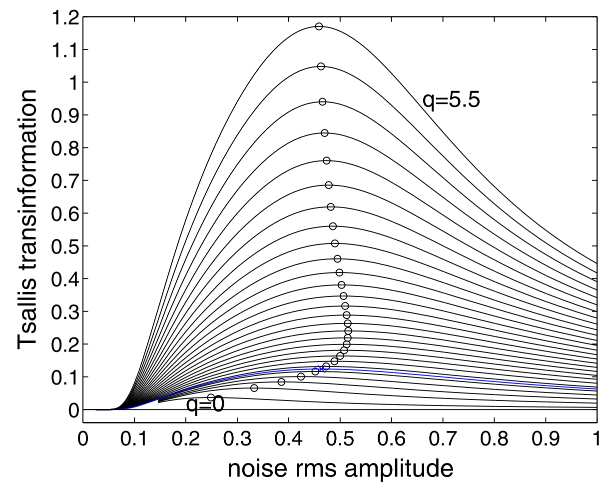

Fig. 4. Input–output Tsallis transinformation $I_q(X;Y)$ from Eq. (27), as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with input probability $P_1 = 0.45$ and threshold $\theta = 1.2$. The order is $q = 0$ then from $q = 0.1$ to 5.5 with step 0.2. In addition, the blue curve marked by the cross ($\times$) corresponds to $q = 1$ when $I_{q=1}(X;Y)$ is the Shannon transinformation. On each curve the maximum is indicated by a circle ($\circ$), except for $q = 1$ where the maximum is at the cross ($\times$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

We will now consider another regime of operation of the channel of Eq. (28), and show the possibility of a constructive action of the noise in the transmission of information assessed with the Tsallis transinformation $I_q(X;Y)$.

5. Noise-improved information transmission

5.1. Noise-improved Tsallis transinformation

For an information channel with a response threshold $\theta = 1.2$ in Eq. (28), the evolution of the input–output Tsallis transinformation $I_q(X;Y)$ from Eq. (27) is presented in Fig. 4.

The results of Fig. 4 clearly demonstrate a nonmonotonic action of the noise in the transmission. With the conditions of Fig. 4, the binary input X by itself is always below the response threshold $\theta = 1.2$ on the output. As a consequence, in the absence of noise at $\sigma = 0$ in Fig. 4, the channel output permanently stays at $Y = 0$. No information is transmitted through the channel, as expressed by the Tsallis transinformation $I_q(X;Y)$ which remains at zero when $\sigma = 0$, for any finite order q. However, as the noise level σ is progressively raised above zero in Fig. 4, a cooperative effect can take place, with the noise W assisting the subthreshold input X to overcome the response threshold θ. This elicits transitions in the output Y carrying statistical dependence with the input X. As a consequence, a nonzero input–output transmission of information occurs, as registered by the Tsallis transinformation $I_q(X;Y)$ which starts to increase in Fig. 4 as the noise level σ rises above zero. There exists a nonzero amount of noise for which the information transfer measured by $I_q(X;Y)$ is maximized, and such a maximum of $I_q(X;Y)$ occurs for any finite order q as visible in Fig. 4. This is the effect of stochastic resonance or noise-aided information transmission, registered by the Tsallis transinformation $I_q(X;Y)$ in Fig. 4 at any finite order q.

In Fig. 4, the order $q = 1$ corresponds to the situation where $I_{q=1}(X;Y)$ is the Shannon transinformation, and $I_{q=1}(X;Y)$ in Fig. 4 culminates at a maximum for a nonzero level of noise σ, as also reported in [60,63].

It is visible in Fig. 4 that the maximum of the Tsallis transinformation $I_q(X;Y)$ occurs at an optimal level $\sigma_{\rm opt}$ of the noise which varies with the Tsallis order q. This variation of $\sigma_{\rm opt}$ with q, i.e.

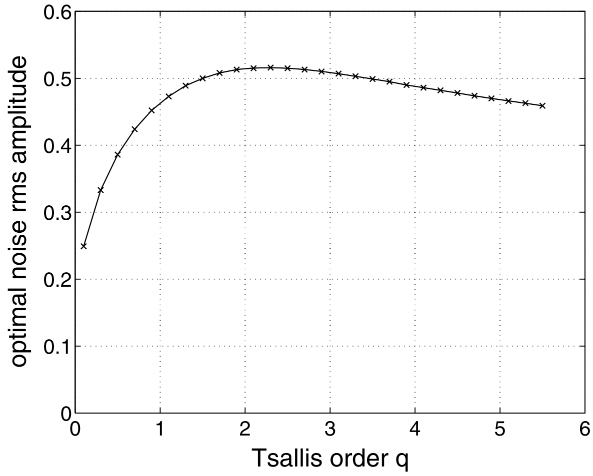

Fig. 5. Optimal noise rms amplitude $\sigma_{\rm opt}$ maximizing the input–output Tsallis transinformation $I_q(X;Y)$ of Fig. 4, as a function of the order q.

$$\sigma_{\rm opt}(q) = \arg\max_{\sigma} I_q(X;Y), \tag{33}$$

is depicted in Fig. 5 for the binary channel of Fig. 4.

A remarkable property observed in Fig. 5 is that the optimal noise level $\sigma_{\rm opt}$ maximizing $I_q(X;Y)$, experiences a nonmonotonic evolution with the order q. There exists in Fig. 5 an optimal value $q_{\rm opt} \approx 2.3$ of the Tsallis order where $\sigma_{\rm opt}$ is maximized. The nonmonotonic evolution of $\sigma_{\rm opt}$ maximized at $q_{\rm opt}$ in Fig. 5, demonstrates that the stochastic resonance effect selects a specific order $q_{\rm opt}$ of the Tsallis transinformation $I_q(X;Y)$. This optimal order $q_{\rm opt}$ identifies the Tsallis transinformation $I_{q_{\rm opt}}(X;Y)$ that is capable of drawing the most pronounced benefit from the added noise in stochastic resonance, since $I_{q_{\rm opt}}(X;Y)$ stands as the measure of input–output information transfer that gets maximized by the largest optimal noise level $\sigma_{\rm opt}$.

An interesting connection is that $\sigma_{\rm opt}$ defined by Eq. (33) is also the optimal noise level that maximizes the Rényi transinformation $\mathcal{I}_q(X;Y)$ with same order q. This is ensured by the monotonic increasing relation of Eq. (25) connecting Tsallis $I_q(X;Y)$ and Rényi $\mathcal{I}_q(X;Y)$ transinformations. Although the measures $I_q(X;Y)$ and $\mathcal{I}_q(X;Y)$ do not culminate at the same maximum value in stochastic resonance, they do so for the same optimal noise level $\sigma_{\rm opt}$. And since the curve $\sigma_{\rm opt}(q)$ is the same and unique for the Tsallis and Rényi measures, the optimal order $q_{\rm opt}$ selected by stochastic resonance is the same also. In this respect, such properties obtained in [63] when exploiting the Rényi entropy for characterizing stochastic resonance, are shared in common with the Tsallis entropy evaluated here.

5.2. Noise-improved Tsallis information capacity

The optimal Tsallis order $q_{\rm opt}$ selected by the stochastic resonance as in Fig. 5, is usually related to a given information source characterized by the input probability P_1. A point of view not impacted by this dependence with P_1 is accessible by considering the Tsallis information capacity of the channel, as defined in Eq. (15). For the binary channel according to Eq. (28), the Tsallis information capacity C_q is shown in Fig. 6, for different Tsallis orders q including the Shannon case $q = 1$.

Compared to the Tsallis transinformation $I_q(X;Y)$ at fixed P_1 as in Fig. 4, the same remarkable properties related to stochastic resonance are observed for the Tsallis information capacity C_q in Fig. 6. At any order q, the Tsallis information capacity C_q undergoes a nonmonotonic evolution as the noise level σ increases. In the regime $\theta > 1$ of a subthreshold binary input X, when no noise

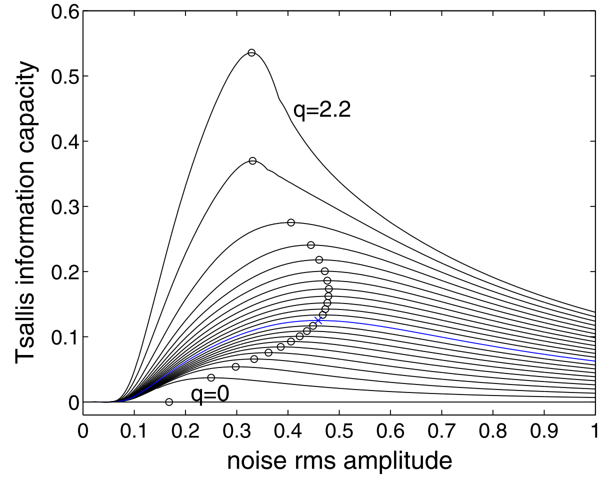

Fig. 6. Input–output Tsallis information capacity C_q, as a function of the rms amplitude σ of the zero-mean Gaussian noise W, for an information channel with response threshold $\theta = 1.2$. On each curve the maximum is indicated by a circle ($\circ$), except for $q = 1$ identified by a cross ($\times$) when $C_{q=1}$ is the Shannon information capacity. The order q goes from 0 to 2.2 with step 0.1.

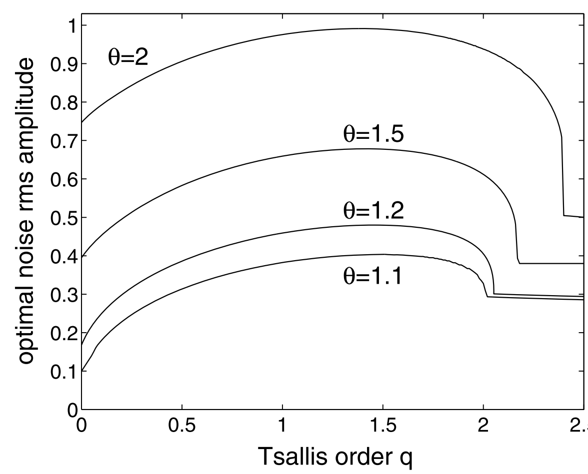

Fig. 7. Optimal noise rms amplitude $\sigma_{\rm opt}$ maximizing the Tsallis information capacity C_q, as a function of the order q. The information channel is with zero-mean Gaussian noise W and threshold θ.

is present no information is transmitted, as marked by a vanishing Tsallis capacity C_q at any order q when $\sigma = 0$ in Fig. 6. Adding noise then modifies the channel, in a way where the subthreshold input $X = 1$ has more chance to get across and be correctly decoded as $Y = 1$ at the output. This constructive action of the noise induces a capacity C_q rising above zero in Fig. 6. Moreover, a nonzero level of noise exists where the capacity C_q is maximized in Fig. 6. This is again a manifestation of stochastic resonance with a Tsallis information capacity C_q maximized at a nonzero optimal level of noise on the channel. Also, as in Fig. 4, the optimal noise level $\sigma_{\rm opt}$ maximizing C_q in Fig. 6, is found dependent upon the Tsallis order q, yet with a nonmonotonic dependence. This dependence of $\sigma_{\rm opt}$ with q is represented in Fig. 7.

The nonmonotonic dependence of $\sigma_{\rm opt}$ with q in Fig. 7, identifies an optimal Tsallis order $q_{\rm opt} = 1.44$ at which the optimal noise level $\sigma_{\rm opt}$ of stochastic resonance assumes its largest value. This optimal order $q_{\rm opt} = 1.44$ is observed in Fig. 7 invariant with the response threshold θ of the channel. The stochastic resonance selects a nontrivial Tsallis order $q_{\rm opt} = 1.44$ through the Tsallis information capacity $C_{q_{\rm opt}}$ that exploits stochastic resonance in the most pronounced way since $C_{q_{\rm opt}}$ gets maximized by the largest possible optimal noise level $\sigma_{\rm opt}$. The optimal Tsallis order

$q_{\rm opt} = 1.44$ selected by stochastic resonance in the capacity C_q, is now intrinsic to the binary channel and insensitive to the input probability. And the optimal order $q_{\rm opt}$, selected by stochastic resonance in the Tsallis information capacity, differs from the Shannon order $q = 1$.

It is interesting to notice also optimal orders $q \approx 1.4$ reported in other studies [68,69] concerned with nonextensivity in complex phenomena. It is known that the Tsallis entropy of order q is maximized by a probability distribution consisting in the q-Gaussian density [25,70]. The studies of [68,69] also find orders $q \approx 1.4$ as providing the best fits for describing fluctuations in financial data with q-Gaussian densities. There is at least in common with the results of Fig. 7 the exposition of specific processes that single out nontrivial orders q in relation to the Tsallis entropy.

Also, regarding the results of Fig. 7, an interesting connection is again established by the monotonic increasing relation of Eq. (26) between the Tsallis $C_q(X;Y)$ and Rényi $\mathcal{C}_q(X;Y)$ information capacities. Both capacities are simultaneously achieved by the same input probability P_1^*, and although $C_q(X;Y)$ and $\mathcal{C}_q(X;Y)$ do not assume the same values, they culminate for the same optimal noise level $\sigma_{\rm opt}$ in stochastic resonance as in Fig. 6. As a consequence, the evolutions of $\sigma_{\rm opt}$ with q shown in Fig. 7, are common to both $C_q(X;Y)$ and $\mathcal{C}_q(X;Y)$. And the nontrivial optimal order $q_{\rm opt} \neq 1$ selected by stochastic resonance is the same for both the Tsallis and the Rényi information measures. This justifies that in this respect, the characterization of stochastic resonance from the Tsallis entropy accomplished here, reproduces the results obtained in [63] from the Rényi entropy.

6. Discussion

The present results demonstrate that the Tsallis-entropy-based measures are able to register stochastic resonance taking place in a basic informational model formed by a binary channel. Quantitatively, the optimal noise level maximizing the Tsallis information measure is found to depend upon the order q of the measure. This confirms a general picture emerging from stochastic resonance studies that there is no one single noise condition adapted to all measures, but that the optimal noise level has to be determined depending upon the measure being optimized. Still, beyond the specific quantitative values of the optimal noise level, our study also reveals that, qualitatively, the feasibility of stochastic resonance identified by a nonzero optimal noise level, is preserved with all Tsallis measures, of any order q. At order $q = 1$, our analysis retrieves the traditional Shannon condition, where stochastic resonance was previously observed in [60], and where the channel is optimized in the sense of the standard (Shannon) measure of transmission efficacy. The extended conditions where stochastic resonance is obtained here consolidate the position of a generalized framework offered by the Tsallis entropy for quantitative measure of information. We come back in the last paragraph of this section on possible interpretations for such extended measures of transmission efficacy. Another outcome is that the observation of stochastic resonance allows us to single out optimal Tsallis order q. In definite conditions, when seeking the Tsallis measures that best exploit stochastic resonance, then nontrivial orders usually emerge, identifying in this context optimal information measures differing from the classic Shannon one. Stochastic resonance, with its intrinsic informational significance, acts here as a benchmark to assess quantitative information measures.

A previous paper [71] also proposed an evaluation of stochastic resonance with generalized measures based on the Tsallis entropy. This study of [71] considered stochastic resonance under its early form, for the transmission of a deterministic sinusoidal signal by a dynamic system governed by a double-well quartic potential, i.e. the original setting under which stochastic resonance was first introduced [29]. Since then, stochastic resonance has been extended to other forms of signals and systems for information transmission [31]. By contrast with [71], the form of stochastic resonance analyzed here concerns the transmission of a random information-carrying signal by a standard reference model of binary channel. The study of [71] assessed the noise-assisted transmission of the sinusoidal signal by means of the traditional measures of signal-to-noise ratio in the frequency domain, and spectral amplification factor, and also by means of generalized measures based on the Tsallis entropy, which especially were shown to allow enhanced sensitivity in specific conditions. The Tsallis measures that we used here are slightly different from those tested in [71], and stand as a direct generalization based on the Tsallis entropy, of the standard Shannon-entropy-based analysis of the reference model of communication formed by the binary information channel. Another connection with concepts from nonextensive statistical physics was discussed in [72–74], where for the early form of stochastic resonance with a sinusoidal signal transmitted by a double-well dynamics, the impact of non-Gaussian and especially q-Gaussian noises [70,25], was examined, with traditional measures like the signal-to-noise ratio in the frequency domain. Together with the present study, all these results contribute to substantiate the connections between Tsallis-entropy-based measures of information, and the emerging paradigm with strong informational significance constituted by stochastic resonance. Further elements in relation to stochastic resonance could be obtained by applying Tsallis information measures to more elaborate communication channels, for instance under the form of arrays of threshold devices supporting suprathreshold stochastic resonance and offering models for the transmission of information in neural networks [59,31].

The present study also involves a correspondence between Tsallis and Rényi information measures. This correspondence in Section 3 starts with the relation between the entropies expressed by Eq. (19), and is developed to all other quantities relevant for information measure, namely the divergence in Eq. (23), the transinformation in Eq. (25), and the information capacity in Eq. (26). These correspondence equations in Section 3, written using the q-logarithm, clearly manifest the monotonic increasing character of the relations between the Tsallis and Rényi information measures. The correspondence and its monotonic character are both specifically meaningful in the characterization of stochastic resonance worked out here. The essential step for characterizing stochastic resonance is to examine the evolution of the information measures as the level of noise is increased, and to identify conditions of a nonmonotonic peaky evolution where an information measure culminates at a maximum for a nonzero optimal level of noise. This step is passed in the same way by both the Tsallis and Rényi information measures connected by the monotonic increasing relations of Section 3. Stochastic resonance is then registered in the same conditions by both the Tsallis and Rényi information measures. And when stochastic resonance is observed, it occurs for an optimal noise level which simultaneously maximizes both the Tsallis and Rényi information measures. Moreover, owing again to the monotonic increasing relations of Section 3, the optimal orders that are singled out by stochastic resonance are found the same for the Tsallis and Rényi measures. In this respect, a strong parallel is established between the present study of stochastic resonance with Tsallis information measures and that of [63] with Rényi measures.

In a larger prospect, an essential difference is that the Rényi entropy of Eq. (16) is known to be extensive or additive for two independent sets of events, while the Tsallis entropy of Eq. (1) is nonextensive or nonadditive. This is consistent with the relation of Eq. (19) between the two entropies; since this relation is nonlinear in general, if the Rényi entropy is additive, then the Tsallis one cannot be. Nonextensivity of the Tsallis entropy is a specific feature, important to its implication in the nonextensive generalization of statistical mechanics. If a specific connection can be made of stochastic resonance with the nonextensive character of such generalization to statistical mechanics, then the characterization realized here with Tsallis measures could take special significance in this direction. Conversely, the present characterization of stochastic resonance with Tsallis-entropy-based information measures could enlarge the inventory of complex phenomena usually associated with the development of a nonextensive generalization to statistical mechanics. For this direction of development, the present study places a special emphasis at the intersection between statistical physics and the physics of information, since the Tsallis-entropy-based measures are here exploited and consolidated for primarily their informational significance. This suggests to envisage the possibility of a nonextensive generalization to statistical physics to include also some "nonextensive" generalization to statistical information theory.

As an illustration based on recent advances, a specific direction where the Tsallis entropy acquires additional significance at the intersection with information theory goes as follows. In traditional source coding, the overall coding cost is directly measured as the length of each codeword averaged according to the probability of occurrence of this codeword. Such an overall (linear) coding cost is lower bounded by the Shannon entropy of the source. A generalized approach to source coding exists that uses more flexible nonlinear coding costs. The coding cost can be taken as a nonlinear (nondecreasing) function of the length of each codeword averaged according to its probability of occurrence. This expresses that longer codewords are costly to encode in a way that is not linear with their length. With such a nonlinear way of measuring coding efficiency, it has been shown recently [21,67] that the Tsallis entropy of order q forms a lower bound to an overall coding cost based on a specific family, parameterized by q, of generalized (nonlinear) functions of the codeword length. This generalized source coding theorem based on the Tsallis entropy of order q, contains as a special case at $q = 1$, the traditional Shannon source coding theorem ruling the linear coding cost. Such generalized approach providing additional significance to the Tsallis entropy at the occasion of a source coding problem, could possibly be extended to a channel coding problem. In traditional channel coding, the rate of error-free information transmission over a channel is upper bounded by the input–output Shannon information capacity of the channel. This is based on a "linear" definition of the transmission rate as the ratio of the number of source symbols to the number of code symbols. A generalized rate could be envisaged based on nonlinear functions of the length of the codewords to measure the transmission cost. This would express for instance that longer codewords may be costly to transmit in a way that is not linear with their length. The Tsallis channel capacity could emerge as a bound to a generalized transmission rate in channel coding, much as the Tsallis entropy acts as a bound to a generalized coding length in source coding. If such conditions are realized, addition of noise would directly serve to improve the generalized transmission rate over the binary channel studied here, and different optimal levels of noise, as in Figs. 4–7, would maximize the generalized rates associated with different q. Such possibilities remain to be explored explicitly, but represent possible directions in which the Tsallis entropy could receive additional significance in relation to information science, especially in extended contexts with unconventional properties such as the beneficial role of noise in stochastic resonance.

References

[1] F.A. Bais, J.D. Farmer, The physics of information, in: P. Adriaans, J. van Benthem (Eds.), Philosophy of Information, Handbook of the Philosophy of Science, North-Holland, Amsterdam, 2008, chap. 5b, also arXiv:0708.2837v2.
[2] R. Landauer, Physics Letters A 217 (1996) 188.
[3] H. Nishimori, Statistical Physics of Spin Glasses and Information Processing, Oxford University Press, Oxford, 2001.
[4] M. Mézard, A. Montanari, Information, Physics, and Computation, Oxford University Press, Oxford, 2009.
[5] M.V. Volkenstein, Entropy and Information, Birkhäuser, Berlin, 2009.
[6] R.B. Frieden, Physics from Fisher Information: A Unification, Cambridge University Press, Cambridge, MA, 1998.
[7] G. Nicolis, C. Nicolis, Foundations of Complex Systems: Nonlinear Dynamics, Statistical Physics, Information and Prediction, World Scientific, Singapore, 2007.
[8] J.F. Traub, A.G. Werschulz, Complexity and Information, Cambridge University Press, Cambridge, 1998.
[9] G. Deco, An Information-Theoretic Approach to Neural Computing, Springer, Berlin, 1996.
[10] E. Desurvire, Classical and Quantum Information Theory, Cambridge University Press, Cambridge, 2009.
[11] F. Pennini, A. Plastino, B.H. Soffer, C. Vignat, Physics Letters A 373 (2009) 817.
[12] S. Kim, Physics Letters A 373 (2009) 1409.
[13] C.E. Shannon, W. Weaver, The Mathematical Theory of Communication, University of Illinois Press, Chicago, 1949.
[14] E.T. Jaynes, Physical Review 106 (1957) 620.
[15] E.T. Jaynes, Physical Review 108 (1957) 171.
[16] A. Rényi, On measures of entropy and information, in: Proceedings 4th Berkeley Symposium on Mathematical Statistics and Probability, Berkeley, USA, University of California Press, 1961, pp. 547–561.
[17] L.L. Campbell, Information and Control 8 (1965) 423.
[18] P.A. Humblet, IEEE Transactions on Information Theory 27 (1981) 230.
[19] A.C. Blumer, R.J. McEliece, IEEE Transactions on Information Theory 34 (1988) 1242.
[20] M.B. Baer, IEEE Transactions on Information Theory 54 (2008) 1273.
[21] J.F. Bercher, Physics Letters A 373 (2009) 3235.
[22] M. Ben-Bassat, J. Raviv, IEEE Transactions on Information Theory 24 (1978) 324.
[23] I. Csiszár, IEEE Transactions on Information Theory 41 (1995) 26.
[24] C. Tsallis, Journal of Statistical Physics 52 (1988) 479.
[25] C. Tsallis, Introduction to Nonextensive Statistical Mechanics, Springer, New York, 2009.
[26] S. Abe, S. Martínez, F. Pennini, A. Plastino, Physics Letters A 281 (2001) 126.
[27] A.S. Parvan, Physics Letters A 350 (2006) 331.
[28] A.S. Parvan, Physics Letters A 360 (2006) 26.
[29] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Reviews of Modern Physics 70 (1998) 223.
[30] F. Chapeau-Blondeau, D. Rousseau, Fluctuation and Noise Letters 2 (2002) L221.
[31] M.D. McDonnell, N.G. Stocks, C.E.M. Pearce, D. Abbott, Stochastic Resonance: From Suprathreshold Stochastic Resonance to Stochastic Signal Quantization, Cambridge University Press, Cambridge, 2008.
[32] B.M. Jost, B.E.A. Saleh, Optics Letters 21 (1996) 287.
[33] M. Misono, T. Kohmoto, Y. Fukuda, M. Kunitomo, Optics Communications 152 (1998) 255.
[34] S. Blanchard, D. Rousseau, D. Gindre, F. Chapeau-Blondeau, Optics Letters 32 (2007) 1983.
[35] M. Löcher, D. Cigna, E.R. Hunt, Physical Review Letters 80 (1998) 5212.
[36] D.G. Luchinsky, R. Mannella, P.V.E. McClintock, N.G. Stocks, IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing 46 (1999) 1205.
[37] D. Rousseau, F. Chapeau-Blondeau, Signal Processing 85 (2005) 571.
[38] A. Bulsara, A.J. Maren, G. Schmera, Biological Cybernetics 70 (1993) 145.
[39] X. Godivier, F. Chapeau-Blondeau, Europhysics Letters 35 (1996) 473.
[40] W.C. Stacey, D.M. Durand, Journal of Neurophysiology 83 (2000) 1394.
[41] G. Balázsi, L.B. Kish, Physics Letters A 265 (2000) 304.
[42] B. Andò, S. Baglio, S. Graziani, N. Pitrone, International Journal of Electronics 86 (1999) 86.
[43] L. Gammaitoni, A.R. Bulsara, Physical Review Letters 88 (2002) 230601.
[44] F. Chapeau-Blondeau, D. Rousseau, Physical Review E 70 (2004) 060101(R).
[45] D.C. Gong, G. Hu, X.D. Wen, C.Y. Yang, G.R. Qin, R. Li, D.F. Ding, Physical Review A 46 (1992) 3243.
[46] F. Chapeau-Blondeau, D. Rousseau, Physics Letters A 351 (2006) 231.
[47] J.J. Collins, C.C. Chow, T.T. Imhoff, Physical Review E 52 (1995) R3321.
[48] F. Chapeau-Blondeau, J. Rojas-Varela, International Journal of Bifurcation and Chaos 10 (2000) 1951.
[49] M.D. McDonnell, D. Abbott, C.E.M. Pearce, Fluctuation and Noise Letters 2 (2002) L205.
[50] M.E. Inchiosa, A.R. Bulsara, Physical Review E 53 (1996) R2021.
[51] J.C. Comte, S. Morfu, Physics Letters A 309 (2003) 39.
[52] F. Duan, D. Abbott, Physics Letters A 344 (2005) 401.
[53] M.D. McDonnell, D. Abbott, C.E.M. Pearce, Microelectronics Journal 33 (2002) 1079.
[54] F. Chapeau-Blondeau, Physical Review E 66 (2002) 032101 032101.

[55] H. Chen, P.K. Varshney, J.H. Michels, IEEE Transactions on Signal Processing 56 (2008) 5074.
[56] A.R. Bulsara, A. Zador, Physical Review E 54 (1996) R2185.
[57] G. Deco, B. Schürmann, Physica D 117 (1998) 276.
[58] D. Rousseau, J. Rojas Varela, F. Chapeau-Blondeau, Physical Review E 67 (2003) 021102.
[59] M.D. McDonnell, N.G. Stocks, C.E.M. Pearce, D. Abbott, Physics Letters A 352 (2006) 183.
[60] F. Chapeau-Blondeau, Physical Review E 55 (1997) 2016.
[61] X. Godivier, F. Chapeau-Blondeau, International Journal of Bifurcation and Chaos 8 (1998) 581.
[62] L.B. Kish, G.P. Harmer, D. Abbott, Fluctuation and Noise Letters 1 (2001) L13.
[63] F. Chapeau-Blondeau, D. Rousseau, A. Delahaies, Physical Review E 81 (2010) 051112.
[64] T. Yamano, Physica A 305 (2002) 486.
[65] T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, New York, 1991.
[66] J.D. Ramshaw, Physics Letters A 198 (1995) 119.
[67] F. Chapeau-Blondeau, A. Delahaies, D. Rousseau, Electronics Letters 47 (2011) 187.
[68] L. Borland, Europhysics News 36 (2005) 228.
[69] S. Drożdż, J. Kwapień, P. Oświęcimka, R. Rak, New Journal of Physics 12 (2010) 105003.
[70] W.J. Thistleton, J.A. Marsh, K. Nelson, C. Tsallis, IEEE Transactions on Information Theory 53 (2007) 4805.
[71] C.J. Tessone, A. Plastino, H.S. Wio, Physica A 326 (2003) 37.
[72] H.S. Wio, Europhysics News 36 (2005) 197.
[73] M.A. Fuentes, R. Toral, H.S. Wio, Physica A 295 (2001) 114.
[74] F.J. Castro, M.N. Kuperman, M. Fuentes, H.S. Wio, Physical Review E 64 (2001) 051105.

References with the titles of the cited papers:

[1] F. A. Bais and J. D. Farmer, "The physics of information," in *Philosophy of Information (Handbook of the Philosophy of Science)* (P. Adriaans and J. van Benthem, eds.), ch. 5b, Amsterdam: North Holland, 2008. (also arXiv:0708.2837v2).

[2] R. Landauer, "The physical nature of information," *Physics Letters A*, vol. 217, pp. 188–193, 1996.

[3] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing*. Oxford: Oxford University Press, 2001.

[4] M. Mézard and A. Montanari, *Information, Physics, and Computation*. Oxford: Oxford University Press, 2009.

[5] M. V. Volkenstein, *Entropy and Information*. Berlin: Birkhäuser, 2009.

[6] R. B. Frieden, *Physics From Fisher Information: A Unification*. Cambridge (MA): Cambridge University Press, 1998.

[7] G. Nicolis and C. Nicolis, *Foundations of Complex Systems: Nonlinear Dynamics, Statistical Physics, Information and Prediction*. Singapore: World Scientific, 2007.

[8] J. F. Traub and A. G. Werschulz, *Complexity and Information*. Cambridge: Cambridge University Press, 1998.

[9] G. Deco, *An Information-Theoretic Approach to Neural Computing*. Berlin: Springer, 1996.

[10] E. Desurvire, *Classical and Quantum Information Theory*. Cambridge: Cambridge University Press, 2009.

[11] F. Pennini, A. Plastino, B. H. Soffer, and C. Vignat, "Physical symmetries and Fisher's information measure," *Physics Letters A*, vol. 373, pp. 817–820, 2009.

[12] S. Kim, "Computability of entropy and information in classical Hamiltonian systems," *Physics Letters A*, vol. 373, pp. 1409–1414, 2009.

[13] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication*. Chicago: University of Illinois Press, 1949.

[14] E. T. Jaynes, "Information theory and statistical mechanics," *Physical Review*, vol. 106, pp. 620–630, 1957.

[15] E. T. Jaynes, "Information theory and statistical mechanics. II," *Physical Review*, vol. 108, pp. 171–190, 1957.

[16] A. Rényi, "On measures of entropy and information," in *Proceedings 4th Berkeley Symposium on Mathematical Statistics and Probability*, (Berkeley, USA), pp. 547–561, University of California Press, 1961.

[17] L. L. Campbell, "A coding theorem and Rényi's entropy," *Information and Control*, vol. 8, pp. 423–429, 1965.

[18] P. A. Humblet, "Generalization of Huffman coding to minimize the probability of buffer overflow," *IEEE Transactions on Information Theory*, vol. 27, pp. 230–232, 1981.

[19] A. C. Blumer and R. J. McEliece, "The Rényi redundancy of generalized Huffman codes," *IEEE Transactions on Information Theory*, vol. 34, pp. 1242–1249, 1988.

[20] M. B. Baer, "Optimal prefix codes for infinite alphabets with nonlinear costs," *IEEE Transactions on Information Theory*, vol. 54, pp. 1273–1286, 2008.

[21] J. F. Bercher, "Source coding with escort distributions and Rényi entropy bounds," *Physics Letters A*, vol. 373, pp. 3235–3238, 2009.

[22] M. Ben-Bassat and J. Raviv, "Rényi's entropy and the probability of error," *IEEE Transactions on Information Theory*, vol. 24, pp. 324–331, 1978.

[23] I. Csiszár, "Generalized cutoff rates and Rényi's information measures," *IEEE Transactions on Information Theory*, vol. 41, pp. 26–34, 1995.

[24] C. Tsallis, "Possible generalization of Boltzmann-Gibbs statistics," *Journal of Statistical Physics*, vol. 52, pp. 479–487, 1988.

[25] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics*. New York: Springer, 2009.

[26] S. Abe, S. Martínez, F. Pennini, and A. Plastino, "Nonextensive thermodynamic relations," *Physics Letters A*, vol. 281, pp. 126–130, 2001.

[27] A. S. Parvan, "Microcanonical ensemble extensive thermodynamics of Tsallis statistics," *Physics Letters A*, vol. 350, pp. 331–338, 2006.

[28] A. S. Parvan, "Extensive statistical mechanics based on nonadditive entropy: Canonical ensemble," *Physics Letters A*, vol. 360, pp. 26–34, 2006.

[29] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, "Stochastic resonance," *Reviews of Modern Physics*, vol. 70, pp. 223–287, 1998.

[30] F. Chapeau-Blondeau and D. Rousseau, "Noise improvements in stochastic resonance: From signal amplification to optimal detection," *Fluctuation and Noise Letters*, vol. 2, pp. L221–L233, 2002.

[31] M. D. McDonnell, N. G. Stocks, C. E. M. Pearce, and D. Abbott, *Stochastic Resonance: From Suprathreshold Stochastic Resonance to Stochastic Signal Quantization*. Cambridge: Cambridge University Press, 2008.

[32] B. M. Jost and B. E. A. Saleh, "Signal-to-noise ratio improvement by stochastic resonance in a unidirectional photorefractive ring resonator," *Optics Letters*, vol. 21, pp. 287–289, 1996.

[33] M. Misono, T. Kohmoto, Y. Fukuda, and M. Kunitomo, "Stochastic resonance in an optical bistable system driven by colored noise," *Optics Communications*, vol. 152, pp. 255–258, 1998.

[34] S. Blanchard, D. Rousseau, D. Gindre, and F. Chapeau-Blondeau, "Constructive action of the speckle noise in a coherent imaging system," *Optics Letters*, vol. 32, pp. 1983–1985, 2007.

[35] M. Löcher, D. Cigna, and E. R. Hunt, "Noise sustained propagation of a signal in coupled bistable electronic elements," *Physical Review Letters*, vol. 80, pp. 5212–5215, 1998.

[36] D. G. Luchinsky, R. Mannella, P. V. E. McClintock, and N. G. Stocks, "Stochastic resonance in electrical circuits – I: Conventional stochastic resonance," *IEEE Transactions on Circuits and Systems – II: Analog and Digital Signal Processing*, vol. 46, pp. 1205–1214, 1999.

[37] D. Rousseau and F. Chapeau-Blondeau, "Constructive role of noise in signal detection from parallel arrays of quantizers," *Signal Processing*, vol. 85, pp. 571–580, 2005.

[38] A. Bulsara, A. J. Maren, and G. Schmera, "Single effective neuron: Dendritic coupling effects and stochastic resonance," *Biological Cybernetics*, vol. 70, pp. 145–156, 1993.

[39] X. Godivier and F. Chapeau-Blondeau, "Noise-enhanced transmission of spike trains in the neuron," *Europhysics Letters*, vol. 35, pp. 473–477, 1996.

[40] W. C. Stacey and D. M. Durand, "Stochastic resonance improves signal detection in hippocampal CA1 neurons," *Journal of Neurophysiology*, vol. 83, pp. 1394–1402, 2000.

[41] G. Balázsi and L. B. Kish, "From stochastic resonance to brain waves," *Physics Letters A*, vol. 265, pp. 304–316, 2000.

[42] B. Andò, S. Baglio, S. Graziani, and N. Pitrone, "Optimal improvement in bistable measurement device performance via stochastic resonance," *International Journal of Electronics*, vol. 86, pp. 86–87, 1999.

[43] L. Gammaitoni and A. R. Bulsara, "Noise activated nonlinear dynamic sensors," *Physical Review Letters*, vol. 88, pp. 230601,1–4, 2002.

[44] F. Chapeau-Blondeau and D. Rousseau, "Enhancement by noise in parallel arrays of sensors with power-law characteristics," *Physical Review E*, vol. 70, pp. 060101(R),1–4, 2004.

[45] D. C. Gong, G. Hu, X. D. Wen, C. Y. Yang, G. R. Qin, R. Li, and D. F. Ding, "Experimental study of signal-to-noise ratio of stochastic resonance systems," *Physical Review A*, vol. 46, pp. 3243–3249, 1992.

[46] F. Chapeau-Blondeau and D. Rousseau, "Noise-aided SNR amplification by parallel arrays of sensors with saturation," *Physics Letters A*, vol. 351, pp. 231–237, 2006.

[47] J. J. Collins, C. C. Chow, and T. T. Imhoff, "Aperiodic stochastic resonance in excitable systems," *Physical Review E*, vol. 52, pp. R3321–R3324, 1995.

[48] F. Chapeau-Blondeau and J. Rojas-Varela, "Nonlinear signal propagation enhanced by noise via stochastic resonance," *International Journal of Bifurcation and Chaos*, vol. 10, pp. 1951–1959, 2000.

[49] M. D. McDonnell, D. Abbott, and C. E. M. Pearce, "A characterization of suprathreshold stochastic resonance in an array of comparators by correlation coefficient," *Fluctuation and Noise Letters*, vol. 2, pp. L205–L220, 2002.

[50] M. E. Inchiosa and A. R. Bulsara, "Signal detection statistics of stochastic resonators," *Physical Review E*, vol. 53, pp. R2021–R2024, 1996.

[51] J. C. Comte and S. Morfu, "Stochastic resonance: Another way to retrieve subthreshold digital data," *Physics Letters A*, vol. 309, pp. 39–43, 2003.

[52] F. Duan and D. Abbott, "Signal detection for frequency-shift keying via short-time stochastic resonance," *Physics Letters A*, vol. 344, pp. 401–410, 2005.

[53] M. D. McDonnell, D. Abbott, and C. E. M. Pearce, "An analysis of noise enhanced information transmission in an array of comparators," *Microelectronics Journal*, vol. 33, pp. 1079–1089, 2002.

[54] F. Chapeau-Blondeau, "Noise-aided nonlinear Bayesian estimation," *Physical Review E*, vol. 66, pp. 032101,1–3, 2002.

[55] H. Chen, P. K. Varshney, and J. H. Michels, "Noise enhanced parameter estimation," *IEEE Transactions on Signal Processing*, vol. 56, pp. 5074–5081, 2008.

[56] A. R. Bulsara and A. Zador, "Threshold detection of wideband signals: A noise-controlled maximum in the mutual information," *Physical Review E*, vol. 54, pp. R2185–R2188, 1996.

[57] G. Deco and B. Schürmann, "Stochastic resonance in the mutual information between input and output spike trains of noisy central neurons," *Physica D*, vol. 117, pp. 276–282, 1998.

[58] D. Rousseau, J. Rojas Varela, and F. Chapeau-Blondeau, "Stochastic resonance for nonlinear sensors with saturation," *Physical Review E*, vol. 67, pp. 021102,1–6, 2003.

[59] M. D. McDonnell, N. G. Stocks, C. E. M. Pearce, and D. Abbott, "Optimal information transmission in nonlinear arrays through suprathreshold stochastic resonance," *Physics Letters A*, vol. 352, pp. 183–189, 2006.

[60] F. Chapeau-Blondeau, "Noise-enhanced capacity via stochastic resonance in an asymmetric binary channel," *Physical Review E*, vol. 55, pp. 2016–2019, 1997.

[61] X. Godivier and F. Chapeau-Blondeau, "Stochastic resonance in the information capacity of a nonlinear dynamic system," *International Journal of Bifurcation and Chaos*, vol. 8, pp. 581–590, 1998.

[62] L. B. Kish, G. P. Harmer, and D. Abbott, "Information transfer rate of neurons: Stochastic resonance of Shannon's information channel capacity," *Fluctuation and Noise Letters*, vol. 1, pp. L13–L19, 2001.

[63] F. Chapeau-Blondeau, D. Rousseau, and A. Delahaies, "Rényi entropy measure of noise-aided information transmission in a binary channel," *Physical Review E*, vol. 81, pp. 051112,1–10, 2010.

[64] T. Yamano, "Some properties of q-logarithm and q-exponential functions in Tsallis statistics," *Physica A*, vol. 305, pp. 486–496, 2002.

[65] T. M. Cover and J. A. Thomas, *Elements of Information Theory.* New York: Wiley, 1991.

[66] J. D. Ramshaw, "Thermodynamic stability conditions for the Tsallis and Rényi entropies," *Physics Letters A*, vol. 198, pp. 119–121, 1995.

[67] F. Chapeau-Blondeau, A. Delahaies, and D. Rousseau, "Source coding with Tsallis entropy," *Electronics Letters*, vol. 47, pp. 187–188, 2011.

[68] L. Borland, "Long-range memory and nonextensivity in financial markets," *Europhysics News*, vol. 36, pp. 228–231, 2005.

[69] S. Drożdż, J. Kwapień, P. Oświęcimka, and R. Rak, "The foreign exchange market: Return distributions, multifractality, anomalous multifractality and the Epps effect," *New Journal of Physics*, vol. 12, pp. 105003,1–23, 2010.

[70] W. J. Thistleton, J. A. Marsh, K. Nelson, and C. Tsallis, "Generalized Box-Müller method for generating q-Gaussian random deviates," *IEEE Transactions on Information Theory*, vol. 53, pp. 4805–4810, 2007.

[71] C. J. Tessone, A. Plastino, and H. S. Wio, "Stochastic resonance and generalized information measures," *Physica A*, vol. 326, pp. 37–54, 2003.

[72] H. S. Wio, "Noise induced phenomena and nonextensivity," *Europhysics News*, vol. 36, pp. 197–201, 2005.

[73] M. A. Fuentes, R. Toral, and H. S. Wio, "Enhancement of stochastic resonance: The role of non Gaussian noises," *Physica A*, vol. 295, pp. 114–122, 2001.

[74] F. J. Castro, M. N. Kuperman, M. Fuentes, and H. S. Wio, "Experimental evidence of stochastic resonance without tuning due to non-Gaussian noises," *Physical Review E*, vol. 64, pp. 051105,1–3, 2001.