Extremum Cycle Times in Time Interval Models

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Abstract

In this paper, we analyze the 1-periodic schedule of a class of time interval models under the form of a polyhedron which can describe Timed Event Graphs and P-time Event Graphs. Using the duality and Stiemke's theorem, the main contribution is the determination of conditions where the extremum cycle times are finite and characteristic of a class of models.

I. Introduction

In the field of discrete event systems, the consideration of time is an important topic and the dater description can describe complex time phenomena. In this paper, we consider the class of time interval models which includes Timed Event Graphs, P-time Event Graphs and P-time Event Graphs with Affine-Interdependent Residence Durations (Chapter 3 in [1], [2]). Applications of P-time Event Graphs can be found in production systems [3] [4] [5], microcircuit design [6], transportation systems [7], and the food industry [8]. A simple application of P-time Event Graphs is cooking a product: The cooking time must not be too long, otherwise the product will be damaged; at the same time, the cooking time needs to be long enough. Time interval models are useful when some tasks must compensate for the undesirable effects of other operations such as the warming of a part or an incomplete achievement of a task [2] and a next paper will show that it can express a specific type of time macro-places which supervise the behavior of a set of places. A current aim is to obtain the best production of parts which is expressed by the

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production rate or the cycle time which is certainly one of the most important characteristics of time models.

A classical problem is to determine a 1-period schedule starting from a possible initial state with an optimal cycle time. In this paper, we focus on the analysis of the class of time interval models which can follow a 1-periodic trajectory which is a steady schedule. When Timed Event Graphs or P-time Event Graphs are considered, the determination of the extremum cycle times can be based on the analysis of the elementary circuits of the Timed Event Graphs [9] or an extended graph associated with the considered P-time Event Graph [10]. A main result is that the extremum cycle times only depend on the parameters (such as the time durations and the initial marking) and substructures of the model and do not depend on the state. It implies that the extremum cycle times which are constant for a given system represent fundamental characteristics of this system.

Therefore, the objective is to extend this property to the class of time interval models and we want to know if the system (which can be a P-time Event Graphs with additional constraints) presents extremum cycle times which are finite and characteristic of the system. In that case, the cycle time is said to be "intrinsic" (A formal description is given in Definition 2). If a cycle time depends on the state, there is no guarantee that the relevant value is constant; It can lead to a unexpected degradation of the optimal production rate and a possible loss of performance which naturally generates a cost and it implies that the value of the production rate is not guaranteed. The main contribution of the paper is the description of a class of models which avoids this drawback. With this aim in mind, we explore the duality which allows a deeper analysis of the connections between the concept of cycle time and the class of time interval models.

Let us put our contribution into a general context and give some related works. The papers [11] [6] [12] [13] [14] consider Timed Event Graphs while this proposed paper can consider P-time Event Graphs. Remember that P-time Event Graphs generalize Timed Event Graphs as they can describe lower and upper bounds on the token stays contrary to Timed Event Graphs. Considering a model named Negative Event Graph which corresponds to a P-time Event Graph, the papers [15] [16] examine the earliest and latest feasible steady firing schedules for each of the minimum and maximum cycle times and discuss the liveness. The analysis is based on the paths and exploits standard algebra but also max-plus algebra which is an elegant way to make graph theory. The article [17] analyzes a linear precedence constraints graph where the processing times present a minimum delay but not a maximum time duration like the P-time Event Graphs but the labels of the arcs bring a complex form of numbering of the starting times. This study focus on the minimization of the largest period of the tasks while we consider a unique period which is shared by all events. Strongly connected substructures are considered through the assumption of unitary graphs where every circuit has a weight equal to 1. Principally in graph theory, the papers [15] [16] [17] only consider relations between two events contrary to this proposed paper.

In this paper, since we consider a general algebraic model of a time interval model which is a polyhedron $A.x \leq b$ where A *is not always an incidence matrix* (a row of matrix A cannot be associated with an arc if it contains more than two entries), we cannot deduce an associated graph as for Timed Event Graphs, linear precedence constraint graphs [17] or P-time Event Graphs [10] [15]: It implies that the classical graph theory cannot be applied, and more advanced tools are necessary in the resolution and analysis of the problem. A presentation of the connections between graph theory and Linear Programming can be found in [2]. Linear Programming is a natural technique chosen by different authors [11] [12] [18][19] [13] [14] and some authors [6] [15] have exploited the principle of duality which provides a new possibility allowing the calculation of the cycle time. Contrary to these studies which principally are about the computation of the cycle time and resource optimization for Timed Event Graphs and P-time Event Graphs, this paper considers a more general class of models and focus on the connections between the class of time models and the concept of intrinsic cycle time. This objective implies the use of general theorems of linear programming such as Stiemke's theorem which seems to be an original point at the best of our knowledge. Note that the introduction of a new algorithm based on duality is only a necessary step in our approach.

In this paper, we assume that we can control the dates of the state. This assumption, which corresponds to the controllability of every transition when Petri nets are considered, is different from the assumption taken for the k-periodic case (out of the scope of the paper) where the rule is an immediate firing of each available transition. It implies that the 1-periodic trajectory starts without any transient period contrary to the k-periodic case where the Timed Event Graph reaches a periodical regime after a finite transient period (that can be extremely long even for small systems) [20]. We also make the assumption that the time model is consistent which corresponds to the time liveness when P-time Event Graphs are considered [21]. We assume the feasibility of the 1-periodic trajectory which is out of the scope of the paper (See [2] for more details). This paper generalizes [1] [2] by analyzing the finite character of the extremum cycle times. It also contains new material such as the introduction of the notion of forward- and backward-homogeneous system and a technique of calculation of a positive invariant.

The paper is organized as follows: Firstly, we give the algebraic model of the time interval model and describe the problem. Then the problems of maximization and minimization of the cycle time are rewritten under a synthetic form. The following section exploits the duality and the Stiemke's Theorem and analyzes the extremum cycle times. Different pedagogical examples illustrate the main concepts. By lack of place, the model of P-time Event Graphs is not presented and can be found in [4] [19] [21] [10] [15]. The reader is referred to [1] and [2] for a more detailed introduction of the formulation that bases the present paper.

II. Algebraic model

The notation $|E|$ stands for the cardinality of the set E while the notation $A_{i,.}$ corresponds to row i of matrix A. The transpose of the matrix A is denoted A^t. In this paper, we focus on the following algebraic model defined in the standard algebra

$$\begin{pmatrix} G^- \\ G^+ \end{pmatrix} \times \begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} \leq \begin{pmatrix} -T^- \\ T^+ \end{pmatrix}, \tag{1}$$

for $k \geq 0$ where: $x(k) \in \mathbb{R}^n$; T^- and $T^+ \in (\mathbb{R} \cup \{-\infty,+\infty\})^q$; $G^- = \begin{pmatrix} G_1^- & G_0^- \end{pmatrix}$ and $G^+ = \begin{pmatrix} G_1^+ & G_0^+ \end{pmatrix} \in \mathbb{R}^{q \ \times 2.n}$. This paper uses the "dater" representation well-known in (max, +) algebra : Each variable $x_i(k)$ over $\mathbb{R}$ represents the date of the k^{th} event associated with x_i.

Remark 1: When P-time Event Graphs are considered, this event is the firing of a transition belonging to the set of transitions TR. For the sake of simplicity, transition $x_i \in TR$ and the relevant date $x_i(k) \in \mathbb{R}$ are usually denoted with the same notation. The system (1) can always be obtained and corresponds to a P-time Event Graph where the initial marking of each place is equal to at most one. The correspondence is as follows: $n = |TR|$ and $q = |PL|$ where PL is the set of places. Each entry T_l^- and T_l^+ in column-vectors T^- and T^+ is respectively the lower bound and the upper bound of a time interval $[T_l^-, T_l^+]$ associated with each place $p_l \in PL$. When we consider a place p_l having a unitary (respectively, null) initial marking, the lower bound T_l^- of the time duration of place p_l linking its input transition x_j to its output transition x_i generates the

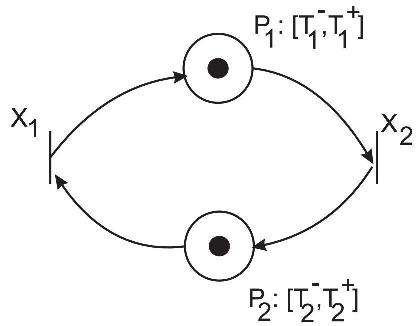

Fig. 1. Elementary P-Time Event Graph (Example 1)

inequality $x_j(k-1)-x_i(k)\leq -T_l^-$ (respectively, $x_j(k)-x_i(k)\leq -T_l^-$). So, $(G_1^-)_{l,j}=+1$ and $(G_0^-)_{l,i}=-1$ (respectively, $(G_0^-)_{l,j}=+1$ and $(G_0^-)_{l,i}=-1$). Similarly, the upper bound T_l^+ of the time duration of this place generates the inequality $-x_j(k-1)+x_i(k)\leq T_l^+$ (respectively, $-x_j(k)+x_i(k)\leq T_l^+$). So, $(G_1^+)_{l,j}=-1$ and $(G_0^+)_{l,i}=+1$ (respectively, $(G_0^+)_{l,j}=-1$ and $(G_0^+)_{l,i}=+1$).

Example 1.

Let us consider the elementary P-time Event Graph of Fig. 1. We obtain $n=|TR|=2$, $q=|PL|=2$, $T^-=\begin{pmatrix} T_1^- \\ T_2^- \end{pmatrix}$, $T^+=\begin{pmatrix} T_1^+ \\ T_2^+ \end{pmatrix}$, $G_1^-=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $G_0^-=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $G_1^+=-G_1^-$ and $G_0^+=-G_0^-$. Other examples can be found in [2] and Chapter 3 of [1].

Remark 2: The study [4] gives an algebraic model named implicit discrete model which is close to model (1). Indeed, assuming that $G^+=-G^-$, model (1) becomes $T^-\leq G^+.\begin{pmatrix} x(k-1) \\ x(k) \end{pmatrix}\leq T^+$ which is equivalent to $T^-\leq q'(k)\leq T^+$ where $q'(k)=G^+.\begin{pmatrix} x(k-1) \\ x(k) \end{pmatrix}$ which is system (3) in [4]. Model (1) is more general since it permits to consider the case $G^+\neq -G^-$ as in [2].

Remark 3: As illustrated by Example 1, an inequality relevant to a row in System (1) involves only two variables when an ordinary P-time Event Graph is considered. The model (1) is actually more general and also permits to handle affine inter-dependance where a residence duration of a token in a place determines the time duration of another place as explained in [2]. In that case, each inequality in model (1) can involve more than two variables and contains three or four variables; The components of G^- and G^+ depend not only on the elements of the incidence matrix in $\{-1,0,1\}$ but also on the coefficients of the affine inter-dependance over $\mathbb{R}$.

The problems solved in this paper directly consider model (1) without being specific to particular Time Petri nets.

III. Extremum cycle times

A. Objective 1

The aim in this section is to determine the minimum or maximum cycle time (if it exists) and the initial state $x(0)$ (i.e. the first firing dates of the transitions when time Petri nets are considered), such that the system (1) follows a 1-periodic behavior defined as follows.

***Definition** 1:* System (1) follows a *1-periodic behavior* when its trajectory satisfies equality $x(k)=\lambda.u+x(k-1)$ for $k\geq 1$ where λ is the cycle time and $u=(1\dots 1)^t$ with $|u|=|x(k)|$.

Let $M=\begin{pmatrix} G_1^- + G_0^- \\ G_1^+ + G_0^+ \end{pmatrix}$, $N=\begin{pmatrix} G_0^-.u \\ G_0^+.u \end{pmatrix}$ and $\theta=\begin{pmatrix} -T^- \\ T^+ \end{pmatrix}$. The dimensions of these matrices are respectively $(2.q\times n)$, $(2.q\times 1)$ and $(2.q\times 1)$. So, the System (1) following a 1-periodic behavior is described by

$$M.x(k)\leq \theta - N.\lambda \ . \tag{2}$$

We write below the two initial problems of minimization and maximization of the cycle time under a single form which is more synthetic than the description given in [1].

B. Problem I

The problems of maximization and minimization of the cycle time λ can be rewritten under a synthetic form if we consider $\min \rho.\lambda$ where the minimization of λ is given by $\rho=+1$ and the maximization by $\rho=-1$. As a finite optimal solution yields a realistic trajectory, we must start from a finite initial starting point $x(0)$ which is limited by the addition of the constraint $x(0)\geq L$ in the problem of minimization ($x(0)\leq L$ in the problem of maximization). A more synthetic form is the constraint

$$-\rho.x(0)\leq -\rho.L \tag{3}$$

where vector L is a lower bound of $x(0)$ when $\rho=1$ ($x(0)\geq L$) and an upper bound when $\rho=-1$ ($x(0)\leq L$).

Let us note that this relation describes a practical problem as we can consider the example of a company where the working day starts at 8.00. A current rule is that each task i must start *after* 8.00 as the company is closed before: We have $x_i(0) \geq L_i$ with $L_i = 8.00$. We can also imagine a symmetrical rule where each task i must start *before* 8.00 and we have $x_i(0) \leq L_i$.

Therefore, the resolution of the above linear programming problems denoted Problem I generates an initial state and an optimal cycle time:

Problem I

$\min \rho.\lambda$ under constraint

$$\begin{pmatrix} -\rho.I & 0 \\ M & N \end{pmatrix} . \begin{pmatrix} x(0) \\ \lambda \end{pmatrix} \leq \begin{pmatrix} -\rho.L \\ \theta \end{pmatrix} \tag{4}$$

where the minimization of λ is given by $\rho = +1$ and the maximization by $\rho = -1$.

For the sake of clarity, the rows corresponding to infinite time durations are kept. When a classical algorithm of linear programming is applied, the relevant rows can be removed or the infinite values can be replaced by an arbitrary large real number.

C. *Objective 2 and preliminary analysis*

As the relation (2) shows that the cycle time depends on the initial state $x(0)$, an aim is the elimination of $x(0)$ in the expression of the cycle time: An algebraic expression of the extremum cycle time which does not depend on $x(0)$ must be obtained. Therefore, the following definition highlights this possibility.

***Definition** 2:* An extremum cycle time is said to be *intrinsic* to the System (1) when it only depends on the matrices G^-, G^+, $T^-$and T^+ and not the initial state $x(0)$.

An intrinsic extremum cycle time is a characteristic of System (1) which can be reused as a known constant in any problem. Clearly, the minimum cycle time (the minimum and maximum cycle time, respectively) is intrinsic for Timed Event Graphs (for P-time Event Graphs, respectively) as the optimal cycle time only depends on the circuits, initial marking and time durations of the model [9] (of the associated graph, respectively [10]) if the relevant circuits exist. The following example shows that the elimination is not always possible.

Example 2. The matrices of the time interval model are as follows:

$G_1^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $G_0^- = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$, $G_1^+ = 0$, $G_0^+ = 0$, $(T^-)^t = \begin{pmatrix} T_1^- & T_2^- \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $T^+ = +\infty$. We denote $x(0) = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^t$ and $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}^t$. Let us apply the well-known Fourier-Motzkin algorithm to system composed of inequality (2) which is $\begin{cases} x_1 - 2.x_2 \leq -T_1^- + 2.\lambda \\ -x_1 + x_2 \leq -T_2^- + \lambda \end{cases}$ and $x(0) \geq L$. It is equivalent to $max(L_1, x_2 + T_2^- - \lambda) \leq x_1 \leq 2.x_2 - T_1^- + 2.\lambda$ with $L_2 \leq x_2$. As the existence of an interval on x_1 allows its elimination which gives $\begin{cases} L_1 \leq 2.x_2 - T_1^- + 2.\lambda \\ T_1^- + T_2^- \leq x_2 + 3.\lambda \end{cases}$, we finally obtain $max(L_2, (L_1 + T_1^- - 2.\lambda)/2, T_1^- + T_2^- - 3.\lambda) \leq x_2$. The elimination of x_2 is not possible as the relations do not define an interval. We finally obtained $max((L_1 + T_1^- - 2.x_2)/2, (T_1^- + T_2^- - x_2)/3) \leq \lambda$ and the minimum cycle time depends on x_2.

Let us consider the linear programming problem I. Its aim is not to eliminate $x(0)$ but to compute an optimal cycle time where $x(0)$ must satisfy the additional constraint (3) which reduces the space defined by (2). As the constraint (3) can possibly have an effect on the computed cycle time, the question is now if the limitation of the state space modifies the optimality: The computed cycle time is optimal for problem I for a given L but is not the best possible cycle time. If the optimal cycle time is intrinsic, it does not depend on $x(0)$ and a fortiori on a limitation of the state space. Therefore, the resolution of Problem I gives a unique optimal cycle time for any variation of L and, the computed value can be reused as a known constant in any problem. The following example illustrates the difficulty.

Example 3.

The matrices of the time interval model are as follows:

$$G_1^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}, G_0^- = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, G_1^+ = 0, G_0^+ = 0, (T^-)^t = \begin{pmatrix} T_1^- & T_2^- & T_3^- & T_4^- \end{pmatrix}$$

$= \begin{pmatrix} 1 & 2 & 3 & 4.5 \end{pmatrix}$ and $T^+ = +\infty$.

Table I gives the minimum cycle time and relevant initial state denoted x_{opt} and λ_{opt} computed by the linear programming problem I for different L. The tests show that the computed cycle time is finite but varies with L when the minimization of the linear programming problem I is applied. As the optimal cycle time depends on the lower bound L, it is not intrinsic to the

System (1).

TABLE I
COMPUTED SOLUTIONS BY PROBLEM (I) FOR EXAMPLE 3

L^t	x^t_{opt}	λ_{opt}
$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 8 & 0 & 1.5 \end{pmatrix}$	3
$\begin{pmatrix} 20 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 20 & 5.25 & 7.125 \end{pmatrix}$	3.375
$\begin{pmatrix} 0 & 20 & 0 \end{pmatrix}$	$\begin{pmatrix} 48 & 20 & 17.625 \end{pmatrix}$	6.875
$\begin{pmatrix} 0 & 0 & 20 \end{pmatrix}$	$\begin{pmatrix} 54.333 & 13.833 & 20 \end{pmatrix}$	7.666

Therefore, the optimal solution naturally depends on System (1) but can also depend on the values of L which have been introduced with constraint $-\rho.x(0) \leq -\rho.L$. We can conclude that problem I for a given L can compute an optimal solution where the optimal cycle time does not correspond to the best cycle time for any L. In the following sections, the analysis of the intrinsic characteristic is based on duality.

D. Dual Problem II

In the previous approach, the used variables are the cycle time λ and the initial state $x(0)$. As a consequence, a trajectory starting from $x(0)$ is generated. In this section, we apply the principle of duality which allows the replacement of the variables $x(0)$ and λ by a variable denoted y.

The references [22] [23] (Corollary 7.1g page 91) describe the following two dual forms:

Problem P: $\underset{y\in\mathbb{R}^n}{min}\ y.b$ under $y.A = c$ and $y \geq 0$
and

Problem D: $\max\limits_{x\in\mathbb{R}^m} c.z$ under $A.z \leq b$.

Therefore, we have a correspondence between Problem I and the second Problem D if we take $z = \begin{pmatrix} x(0) \\ \lambda \end{pmatrix}$, $A = \begin{pmatrix} -\rho.I & 0 \\ M & N \end{pmatrix}$, $b = \begin{pmatrix} -\rho.L \\ \theta \end{pmatrix}$ and $c = \begin{pmatrix} 0 & -\rho \end{pmatrix}$ where 0 is a zero row-vector with $|0|$=n . Similarly, Problem II is Problem P with the same matrix A and vectors b, c .

The following theorem makes the connection between the two problems and the corresponding optimal solutions. The set $Z^{ad} = \{z \in \mathbb{R}^{\mathrm{m}} | A.z \leq b\}$ is the set of the admissible solutions to the dual problem. In the same way, $Y^{ad} = \{y \in (\mathbb{R}^{+})^{\mathrm{n}} | y.A = c\}$. Let y_{opt} and z_{opt} be the optimal solutions to problems P and D, respectively.

Lemma 1: [22] (Chapter 4 in [24]).

1) If $y \in Y^{ad}$ and $z \in Z^{ad}$ then $y.b \geq c.z$
2) If problem P (respectively, D) presents a finite optimal solution, then the same assertion holds for problem D (respectively, P) and the optimal criteria are equal: $y_{opt}.b = c.z_{opt}$
3) If problem P (respectively, D) presents an infinite optimal solution, then $Y^{ad} = \emptyset$ (respectively, $Z^{ad} = \emptyset$).

Example 2 continued. The last inequality shows that $min(\lambda) \to -\infty$ if $x_2 \to +\infty$. So, the occurrence of an infinite solution for the optimization means that the concept of minimum cycle time is not pertinent to the problem under consideration and is not intrinsic. As the resolution of $y.A = c$ with $y \geq 0$ for $\rho = 1$ leads to incoherent results (Algebraically, it yields $1/4 \geq y_3 \geq 1/3$ where y_3 is the third component of y), it shows that this system has no solution: $Y^{ad} = \emptyset$ for $\rho = 1$. The admissibility can also be checked by linsolve() of Scilab. As the minimum cycle time is infinite, this result is coherent with the point 3 of Theorem 1. The same conclusion holds for the maximum cycle time which is clearly infinite and set $Y^{ad} = \emptyset$ for $\rho = -1$.

Remark 4: The non-decrease of the trajectories (guaranteed by the addition of the relation $x(k+1) \geq x(k)$ in the model) leads to a finite minimum cycle time as it implies $\lambda \geq 0$.

Note that, as the maximum cycle time of a Timed Event Graph is infinite, it implies that $Y^{ad} = \emptyset$ for $\rho = -1$ which can also be proved by analyzing the matrices. Used in the proof of Theorem 2, the point 3 of the following theorem focus on the finite aspect of the optimal cycle times.

***Theorem** 1:* 1) If $y \in Y^{ad}$ and $z \in Z^{ad}$ then $-\rho.\lambda \leq \; y.b$

2) If z_{opt} (respectively, y_{opt}) is finite, then y_{opt} (respectively, λ_{opt}) is finite and $-\rho.\lambda_{opt} = y_{opt}.b$.
3) If $Y^{ad} \neq \emptyset$, then z_{opt} and y_{opt} are finite.

Proof. As $c = \begin{pmatrix} 0 & -\rho \end{pmatrix}$, the first point of the duality theorem 1 indicates that $-\rho.\lambda \leq \; y.b$. The second point is directly deduced from point 2 of the same theorem 1 which also implies that the relevant criteria are equal. The third point is based on the negation of point 3 in theorem

1: Problem P does not present an infinite optimal solution, if $Y^{ad} \neq \emptyset$. As we assume the feasibility of the 1-periodic trajectory in the paper, the optimal solution z_{opt} exists and must be finite. Point 2 implies that y_{opt} is finite as z_{opt} is finite. ■

As the minimization of λ is expressed by $\rho = 1$, we have : $-y.b \leq \lambda$ and $\lambda_{min} = -y_{\min}.b$. Similarly, as the maximization of λ is expressed by $\rho = -1$, we have $\lambda \leq y.b$ and $\lambda_{max} = +y_{\max}.b$.

The condition $Y^{ad} \neq \emptyset$ of point 3 of Theorem 1 is now named *Condition 1*. Contrary to Condition 1 which does not depend on b which contains L, Point 2 shows that the optimal cycle time can depend on L and more results are necessary.

IV. Extremum cycle times intrinsic to System (1)

We now show that the concept of extremum cycle time is intrinsic to a large class of models under some general conditions. This result is based on Stiemke's theorem which also allows for a reduction of the size of Problem II.

A. *General case: Model (1)*

Lemma 2: (Stiemke Theorem [25] [23]). For a matrix Ω, the following cases are mutually exclusive from each other.

- case 1. $\Omega.\varkappa = 0$, $\varkappa > 0$ has a solution $\varkappa$.
- case 2. $y.\Omega \geq 0$ and $y.\Omega \neq 0$ has a solution y.

The following theorem defines a large class of models where the extremum cycle time is intrinsic. We denote $y = \begin{pmatrix} y^1 & y^2 \end{pmatrix}$ where y^1 and y^2 are two row-vectors such that $y^1 = y_{1,\ldots,n}$ and $y^2 = y_{n+1,\ldots,n+2.q}$. This theorem generalizes Property 3.3 in [1]: The Condition 1 of admissibility of a finite optimal cycle time is added; It simultaneously considers the minimization and the maximization, and specifies $(y^1)_{opt}$.

***Theorem** 2:* Let us assume that there exists a vector $\varkappa > 0$ such that $M.\varkappa = 0$ (*Condition 2*). If $Y^{ad} \neq \emptyset$ for a given ρ (*Condition 1*), then,

- The optimal cycle time is intrinsic to System (1).
- $(y^1)_{opt} = 0$ and $(y^2)_{opt}$ is the optimal solution to the problem $\min\limits_{y \in \mathbb{R}^n} y^2.\theta$ such that

$$y^2.\begin{pmatrix} M & N \end{pmatrix} = \begin{pmatrix} 0 & -\rho. \end{pmatrix} \text{ with } y^2 \geq 0 \tag{5}$$

and

$$\lambda_{opt} = \ (-\rho).(y^2)_{opt}.\theta \ . \tag{6}$$

Proof. If $Y^{ad} \neq \emptyset$, then there is a vector $y \geq 0$ satisfying system $y.A = c$ which gives $\begin{cases} -\rho.y^1 + y^2.M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix} \\ y^2.N = -\rho. \end{cases}$ or $\begin{cases} y^2.M = \rho.y^1 \\ y^2.N = -\rho. \end{cases}$ with $y^1 \geq 0$ and $y^2 \geq 0$. It implies that $y^2.M \geq 0$ for the minimization problem ($\rho = 1$), and that $y^2.M \leq 0$ for the maximization problem ($\rho = -1$). Considering the minimization, we shall prove that actually $y^2.M$ is equal to zero if the Condition 2 of Theorem 2 holds true.

Indeed, we can deduce from Stiemke's theorem that Case 2 has no solution as Case 1 holds: $M.\varkappa = 0$, $\varkappa > 0$ has a solution by assumption of the theorem (Condition 2). Therefore, the system composed of $y^2.M \geq 0$ and $y^2.M \neq 0$ has no solution y^2. In other words, it means that it is not possible to find y^2 whereby there is no entry i of the row-vector $y^2.M$ satisfying $(y^2.M)_i > 0$. Since the condition $y^2.M \geq 0$ must be satisfied, the remaining possibility is $y^2.M = 0$. The same result can be said for the maximization: The assumption of the theorem is equivalent to "$-M.\varkappa = 0$, $\varkappa > 0$ has a solution" and the reasoning is identical if we replace matrix M by $-M$.

Finally, equality $\rho.y^1 = y^2.M$ is equivalent to $y^1 = \rho.y^2.M$ as $\rho^2 = 1$. As $y^2.M = 0$, we have $y^1 = 0$ and only the optimization $\min\limits_{y \in \mathbb{R}^n} y^2.\theta$ is necessary. Moreover, points 2 and 3 of Theorem 1 imply that z_{opt}, y_{opt} are finite and $-\rho.\lambda_{opt} = \ y_{opt}.b = (y^2)_{opt}.\theta$. Therefore, $\lambda_{opt} = \ (-\rho).(y^2)_{opt}.\theta$ which means that the optimal cycle time does not depend on bound L and is intrinsic. If the Condition 2 of Theorem 2 is not satisfied, there is an entry i of the row-vector $y^2.M$ satisfying $(y^2.M)_i > 0$ in system $y^2.M \geq 0$, and the optimal cycle time can possibly depend on bound L. ■

So, expression (6) shows that the optimal cycle time does not depend on a limitation of the state space produced by L and is intrinsic to model (1). We now make the connection between the intrinsic property and general properties of the trajectories of System (1). The notion of homogeneous function [26] is extended to System (1) as follows.

Definition *3:* System (1) is said to be forward-homogeneous (respectively, backward-homogeneous) if any trajectory of System (1) is invariant by shifting with any positive delay (respectively, negative delay), that is: If a trajectory $x(k)$ satisfies (1), then the trajectory $x(k) + \omega.\Delta.u$ with

$\omega = 1$ (respectively, $\omega = -1$) satisfies (1) for any $\Delta > 0$. System (1) is said to be strictly-homogeneous when it is forward-homogeneous and backward-homogeneous.

***Theorem** 3:* System (1) is forward-homogeneous (respectively, backward-homogeneous) if and only if $M.\omega.u \leq 0$ with $\omega = 1$ (respectively, $\omega = -1$) . System (1) is strictly-homogeneous if and only if $M.u = 0$.

Proof. Considering a trajectory $x(k) + \omega.\Delta.u$, we obtain

$$\begin{pmatrix} G^- \\ G^+ \end{pmatrix} . \times \begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} + M.\omega.\Delta.u \leq \theta \tag{7}$$

As System (1) is satisfied by assumption, a sufficient condition of feasibility of (7) is $M.\omega.u \leq 0$ as $\Delta > 0$. The condition is also necessary as a positive component $(M.\omega.u)_i > 0$ implies that it is always possible to find a positive value of Δ such that the relevant row in (7) is not satisfied. The second part of the theorem is deduced from the equivalence: $M.u = 0 \Leftrightarrow M.\omega.u \leq 0$ for $\omega = 1, -1$. ■

Example 2 continued.

As $M.u \leq 0$, the model is forward-homogeneous.

Example 3 continued

The model is not forward-homogeneous as trajectory $x(0) + \Delta.u \rightarrow x(1) + \Delta.u$ with $x(0) = 0$ and $x(1) = \begin{pmatrix} 2 & 1.5 & 4.5 \end{pmatrix}$ satisfies System (1) for $\Delta = 0$ but not for $\Delta > 0$. Also, $M.u \nleq 0$.

If we consider P-time Event Graphs (respectively, Timed Event Graphs), we have $G_1^- + G_0^- = -(G_1^+ + G_0^+) = W$ (respectively, $G_1^- + G_0^- = W$) where W is the classical incidence matrix of an event graph (See relation (3) in Section III.A of [2]). As $M.u = 0$, these systems are clearly strictly-homogeneous. The following Corollary makes the connection with the intrinsic property.

***Corollary** 1:* In a strictly-homogeneous system, the optimal cycle time is intrinsic *if* $Y^{ad} \neq \emptyset$.

B. Checking Condition 2

We below sketch two techniques which allows the checking of Condition 2 of Theorem 2.

- Vector $\varkappa$ can be obtained from an algorithm to compute a set of generators, usually named ”minimal support invariants”, such that any right-invariant $\varkappa \geq 0$ satisfying $M.\varkappa = 0$ with the appropriate dimensions can be written as a linear combination of these generators [27]

[28] [29] [30]. Therefore, a positive $\varkappa > 0$ satisfying Condition 2 can be generated by a positive linear combination of the generators if the union of the set of indices corresponding to nonzero entries in each generator covers the set of indices of M. Note that the tests show that the application of the Martínez and Silva's algorithm [27] can be costly in terms of time and memory space.

- As a unique vector $\varkappa > 0$ must be determined, a more efficient technique is as follows: 1) Compute the kernel of M, denoted $\ker M$ (the set of solutions to $M.\varkappa' = 0$); 2) Take $\varkappa = Q.\delta$, where δ is an arbitrary vector of adequate dimension and Q is a generating matrix of the kernel, so that $\mathrm{Im}\, Q = \ker M$. Therefore, $M.\varkappa = M.Q.\delta = 0$. The kernel can be efficiently obtained by the gaussien elimination which gives a basis of the kernel. In addition, the condition $\varkappa > 0$ must be satisfied and the vectors of the basis can contain negative and null components. Under the assumption that the basis is non-empty otherwise Condition 2 is not satisfied, the determination of a positive $\varkappa$ can be made by solving $Q.\delta \geq \xi$ where $\xi > 0$ with $|\xi| = |\varkappa|$ is an arbitrary positive column vector. Typically, this resolution can be made by the Fourier-Motzkin algorithm but this approach is limited to small systems (The complexity is double exponential). We can also apply a standard algorithms of linear programming where an arbitrary criterion $\varsigma.\delta$ is optimized. The convergence needs the boundedness of the space which is guaranteed by the addition of the constraints $-\Phi \leq \delta \leq \Phi$ where the bounds are taken sufficiently large, so that the space is non-empty. This second technique is strictly more efficient than the above adaptation of the Martínez and Silva's algorithm and can consider practical problems as the application of this second technique with the function Karmarkar() or Linpro() (Simplex Algorithm) of Sclilab 5.5.2 on an Intel Core2 Duo 2.93 GHz needs approximately 1.2 seconds when a full integer matrix M with $n = 500$ is considered. Indeed, using well-known polynomial algorithms, it presents the complexity of the used algorithm of linear programming, that is, $O(n^4 \times U)$and $O(n^{3.5} \times U)$ in the worst case for the ellipsoid algorithm of Khashiyan and the interior point algorithm of Karmarkar where n is the number of variables and U is the number of bits necessary in the storage of the data [23] [31]). The simplex can also be used: Although some artificial examples show exponential running times, the simplex is efficient in practice as it has polynomial-time average-case complexity in some general cases [23].

C. Examples

Example 3 continued

We have $G_1^- + G_0^- = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ and $G_1^+ + G_0^+ = 0$. For $\rho = 1$, we can check that Condition 1 is satisfied as $y^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/4 & 1/2 \end{pmatrix} \in Y^{ad}$: The minimum cycle time is *finite*. For $\rho = -1$, as $Y^{ad} = \emptyset$ ($N.y^2 = 1$ with $y^2 \geq 0$ is not possible as $N < 0$), the maximum cycle time is *infinite*. The application of the techniques of Section IV-B on M shows that there is no solution $\varkappa$ satisfying Condition 2. Therefore, the optimal cycle times are not intrinsic to the considered model.

Let us verify these results with the analysis of inequality (2). Following the Fourier-Motzkin approach, the successive elimination of x_1, x_2 and x_3 gives the following inequalities

$$\begin{cases} 3.x_3 + T_2^- - \lambda \leq & x_1 & \leq 2.x_3 - T_1^- + 2.\lambda \\ x_3 + T_3^-/2 - \lambda \leq & x_2 & \leq x_3 - T_4^- + \lambda \\ & x_3 & \leq -T_1^- - T_2^- + 3.\lambda \\ (T_3^- + 2.T_4^-)/4 \leq & \lambda & \end{cases}$$

Let $\lambda^- = (2.T_4^- + T_3^-)/4$. So, the minimum cycle time is finite and, a priori the best minimum cycle time is λ^- if the relations of the three first rows are satisfied for $\lambda = \lambda^-$. In fact, the space of the initial state $x(0)$ presents an upper bound for a given cycle time and we can choose a lower bound L such that the space of initial states is empty for the case $\lambda = \lambda^-$: Therefore, the minimum solution is modified as a greater cycle time $\lambda > \lambda^-$ must be taken.

Example 4.

Example 4 is Example 3 where the second row of the matrices G_1^-, G_0^- and T^- are removed. As in Example 3, the minimum cycle time is *finite* ($Y^{ad} \neq \emptyset$ for $\rho = 1$ as $y^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/4 & 1/2 \end{pmatrix} \in Y^{ad}$) and the maximum cycle time is *infinite* ($Y^{ad} = \emptyset$ for $\rho = -1$). The Condition 2 of Theorem 2 is satisfied: $\varkappa = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}^t > 0$ gives $M.\varkappa = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^t$. We can conclude that the concept of minimum cycle time is intrinsic to the considered model.

Let us verify this assertion. From inequality (2), the successive elimination of x_1, x_2 and x_3 gives the following inequalities

$$\begin{cases} & x_1 \le 2.x_3 - T_1^- + 2.\lambda \\ x_3 + T_3^-/2 - \lambda \le & x_2 \le x_3 - T_4^- + \lambda \\ (T_3^- + 2.T_4^-)/4 \le & \lambda \end{cases}$$

Let $\lambda^- = (2.T_4^- + T_3^-)/4$. Therefore, the minimum cycle time is finite and, a priori the best minimum cycle time is λ^- if the initial state satisfies the relations of the two first rows for $\lambda = \lambda^-$. As the space is unbounded, we can add the constraint $L \le x(0)$ without restriction on the consistency for $\lambda = \lambda^-$. Indeed, contrary to Example 3, the system of example 4 is forward-homogeneous (Condition $M.u \le 0$ is satisfied): Any trajectory of System (1) such as an 1-periodic trajectory is invariant by shifting with a positive delay.

V. Conclusion

In this paper, we analyze the extremum cycle times for a large class of time interval models which includes Timed Event Graphs and P-time Event Graphs with some extensions. The study of the extremum cycle times leads to the writing of the primal and dual problems. Condition 1 in Theorem 1 implies that the extremum cycle time is finite. This theorem is reinforced by Theorem 2 based on Stiemke's theorem which shows that the concept of extremum cycle time is intrinsic to an important subclass of time interval models: Condition 2 guarantees that the computed cycle time can be reused as a known constant in any problem. We finally make the connection between the intrinsic property and the general properties of the trajectories of homogeneous systems. The Condition 2 can be checked by a standard technique which computes all the non-negative invariants. As this technique can be costly in terms of time and memory space, we also present a more efficient technique where the first step is based on the polynomial gaussian elimination and the second step exploits standard algorithms of the linear programming. Perspectives are the analysis of the robustness of the computed results with respect to variations of the parameters and the extension to more general time models such as T-time Petri nets and time stream Petri nets.

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