Constraint propagation for max-plus-linear discrete event systems : application to the state estimation

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Abstract— **Discrete event systems undergo perturbations such as failures that disrupt the control system and reduce the anticipation capacities of the future evolution of the process. Using the (max,+) algebra, processes modelled by a timed event graph may be represented by a linear model. The knowledge of the model and of the initial conditions make it possible to characterize the state vector with a state equation iteration but perturbations may generate a misappreciation of the state vector. The objective of this paper is to propose a state estimation using, not the classical theory of residuation but Kleene 's star. One considers the case that temporizations belong to known intervals and proposes a specific intitialization. The result is the trajectory of the least upper bound of the state.**

Keywords: discrete-event dynamic systems, supervisory control, estimation, Kleene's star, perturbation.

I. INTRODUCTION

Discrete Event Dynamic Systems (DEDS) can represent a great number of systems characterized as being concurrent, asynchronous, distributed or parallel such as flexible manufacturing systems, multiprocessor systems, and transportation networks. Among formalisms used to represent DEDS, Timed Event Graphs with constant firing form a subclass playing an important role because of its deterministic temporal behavior..

Its evolution is described by linear systems defined on a dioid. The interpretation of each variable is, for example, of "dater" type for (max,+) algebra: each function $x_i(k)$ represents the date of the kth firing of transition x_i ; $\oplus$ stands for the max. operation while the usual addition plays the role of the multiplication, denoted $\otimes$. In the dioid (max,+), the state variable representation of the system is of the form:

$x(k) = A \otimes x(k-1) \oplus B \otimes u(k)$
$y(k) = C \otimes x(k)$

where the control u, the state x and the output y and are defined on $\Re \cup \{-\infty\}$. In the (max,+) literature, $\otimes$ is usually replaced by . or is omitted.

When faced with the problem of controlling a system, the control theory designs are often based on the assumption that the state vector of the system to be controlled is available. For example, let us recall the classical problem. Let us suppose that some events are designated as controllable, meaning that their input transitions may be delayed from firing until some arbitrary time. The delayed enabling times $u_i(k)$ for the controllable events are to be provided by a supervisor. Let us suppose that we wish to slow the system down as much as possible without causing any event to occur later than some sequence of execution times Z. It can be proved that, for the system which dater equations give the least solution (the earliest times) of the process evolution, the solution (the latest times) of the control problem is explicitly given by the backward recursive equations where the co-state vector plays the role of the state vector.

The results can be given by two algorithms providing the earliest and the latest times of the tasks respectively. The differences between the co-state and the state represent the "spare time" or the "-margin" which is available for the firing of the transitions. A negative difference prevents the future deadlines from being achieved.

Thus, this approach requires the knowledge of the vector state values. The knowledge of the initial conditions and of the model enables us to characterize the state vector with a state equation iteration. However, the state vector is not always available from the information system if the process has for example an human aspect. Moreover, this solution must start from a known state and disregards unavoidable model errors produced by the following " internal " perturbations.

- the physical and human aspects of the process entails the variation of the holding times.

- this situation also occurs when the process undergoes a failure and must be recovered.

- the aim of the maintenance is to prevent any breakdowns. This pre-emptive maintenance produces a necessary perturbation of the production.

These perturbations can be described as "internal perturbations" in opposition to external perturbations like variations of desired outputs or supplying of products and parts. These internal perturbations produce variations of the model or even ruptures of the description of the model. A consequence can be a wrong determination of the state vector.

A past description rupture of the classical model which generates a misappreciation of the state vector is considered and, in this case, the problem is:

- to estimate the past values of the state from the known values of the input and output. The horizon of estimation is $[k_s, k_f]$.

- to compute the latest firing dates of the input transitions in such a way that the output events occur at the latest before the desired dates.

- to predict the future evolution of the output and to verify the existence of the optimal solution of the control synthesis.

In this paper, we only consider the estimation issue. The exact values of temporizations are unknown but belong to known intervals. A basic assumption that allows us to model the system, is that places are First In First Out (FIFO) channels. A place p_i is FIFO if the kth token to enter this place is also the kth token that becomes available in this place. The interpretation is that tokens cannot overtake one another which is a necessary numbering condition of the events.

The paper is structured in the following way: we give initially, the notations and some previous results[BAC, 92]; the second part relates to estimation using Kleene 's star.

II. Preliminaries

A monoid is a couple $(S, \oplus)$ where the operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive relatively to $\oplus$ and the element zero of $\oplus$ is the absorbing element of $\otimes$. A dioid D is a idempotent semi-ring. Let us notice that, contrary to the structures of group and ring, monoid and semi-ring do not have a property of symmetry on S.

The unit $\Re \cup \{-\infty\}$provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is usually called (max, +) algebra and is an example of dioid. We have: $\Re_{max} = (\Re \cup \{-\infty\}, \oplus, \otimes)$ with

$a \oplus b = \max(a, b)$; $\varepsilon = -\infty$ is the neutral element of $\oplus$

$a \otimes b = a + b$; $e = 0$ is the identity element of $\otimes$

$a \oplus a = a$ (idempotency of $\oplus$)

$a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$ (absorbing element ε)

Thereafter, a $\otimes$ b could be noted $a.b$ or simply ab . One recalls that in the $(max, +)$ notation , $[C_{i,j}]^{\otimes(-1)} = -[C_{i,j}]$ with ordinary minus sign. The sum and the product of the matrices operate as in the usual algebra:

$$(\mathrm{A} \oplus \mathrm{B})_{ij} = \mathrm{A}_{ij} \oplus \mathrm{B}_{ij}$$

$$(\mathrm{A} \otimes \mathrm{B})_{ij} = \bigoplus_{k=1}^{n} \mathrm{A}_{ik} \otimes \mathrm{B}_{kj}$$

Kleene's star is defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$

An induced graph to a square matrix A is deduced from this matrix by associating

- a node i to the column i and line i

- an arc from the node j towards the node i if $A_{ij} \neq \varepsilon$.

Theorem 1 *([BAC, 92] Theorem 3.17) For matrix A with induced graph G(A) , if the cycle weights in G(A) are all negative that there is a unique solution to the equation* $x = A \otimes x \oplus B$ *which is given by* $A^* \otimes B$.

III. Objective and models

A. Objective

The objective is to find the least upper bound of $x(k)$ from $k = k_s$ to k_f knowing the values of the input $u(k)$ and the output $y(k)$. The model is supposed to be known on the same horizon of observation. One can notice that this problem of estimation is thus different from the control synthesis which considers that the control and the output are the unknown data. The upper bound of the estimate and the co-state have a similar type but meet two different aims.

B. Model

From the event graph , one initially builds a new Petri net such as each place containing n tokens with n strictly higher than 1 is developed in the form of a chain of n places containing each one a token.

This new graph can be described by the following equations (1).

$$\left.\begin{array}{l} x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus B \otimes u(k) \\ y(k) = C \otimes x(k) \end{array}\right\}$$

The element $[A_0]_{ij}$ (respectively $[A_1]_{ij}$) represents the temporization of the place without token (respectively: with a token) forming the link between the transition x_j and the transition x_i. The element $[B]_{ij}$ (respectively $[C]_{ij}$) represents the temporization of the place without token forming the link between the input u_j and the transition x_i (respectively: between the transition x_j and the output y_i).

If the event graph does not present a circuit without token, the system can be reduced to a classical form by using the star of A_0 .

$x(k) = A_0^* \otimes A_1 \otimes x(k-1) \oplus A_0^* \otimes B.u(k)$

The components of $A_0^* \otimes A_1$and $A_0^* \otimes B$ are expressions of temporizations. As this last form is not essential, we will use the previous form (1) whose interest is that each component $[A_0]_{ij}$, $[A_1]_{ij}$, $[B]_{ij}$ or $[C]_{ij}$ represents exactly a simple temporization of the graph. This model is thus closer to the initial event graph and its conception. The following method can naturally be applied indiscriminately to the two types of equation after a light modification of matrices and equations.

We note $[A_0]_{i,j}^-$ and $[A_0]_{i,j}^+$ the lower and upper bounds of the interval relevant to $[A_0]_{i,j}$. So, $[A_0]_{i,j}$ belongs to the interval $[[A_0]_{i,j}^-$, $[A_0]_{i,j}^+]$ or $[A_0]_{i,j}^- \leqslant [A_0]_{i,j} \leqslant [A_0]_{i,j}^+$. The notation is identical for $[A_1]_{i,j}$, $B_{i,j}$ and $C_{i,j}$.

C. System of inequalities

In this part, the system is transformed into an inequality set of a special type which will be defined below. The equations (1) can be rewritten in the following way.

- For i from 1 to n (n dimension of the state, q dimension of the input),

$x_i(k) = \bigoplus_{j=1}^n [A_0]_{i,j} \otimes x_j(k) \oplus \bigoplus_{j=1}^n [A_1]_{i,j} \otimes x_j(k-1) \oplus \bigoplus_{j=1}^q B_{i,j} \otimes u_j(k)$

- For i from 1 to m (m dimension of the output)

$y_i(k) = \bigoplus_{j=1}^n C_{i,j} \otimes x_j(k)$

At first, as $\mathbf{a} \leqslant b \Rightarrow \mathbf{a} \oplus c \leqslant b \oplus c$, $\mathbf{c} \leqslant d \Rightarrow \mathbf{c} \oplus b \leqslant d \oplus b$ and $\oplus$ is commutative, we have $\mathbf{a} \leqslant b$ and $\mathbf{c} \leqslant d$ entails $\mathbf{a} \oplus c \leqslant b \oplus d$. Operation $\oplus$ is isotone ($\mathbf{a} \leqslant b\ \forall c$, $\mathbf{ac} \leqslant bc$). With these properties, we can deduce the following inequalities.

$[A_0]_{i,j}^- \leqslant [A_0]_{i,j} \leqslant [A_0]_{i,j}^+$

$\bigoplus_{j=1}^n [A_0]_{i,j}^- \otimes x_j(k) \leqslant \bigoplus_{j=1}^n [A_0]_{i,j} \otimes x_j(k) \leqslant \bigoplus_{j=1}^n [A_0]_{i,j}^+ \otimes x_j(k)$

$\bigoplus_{j=1}^n [A_1]_{i,j}^- \otimes x_j(k-1) \leqslant \bigoplus_{j=1}^n [A_1]_{i,j} \otimes x_j(k-1) \leqslant \bigoplus_{j=1}^n [A_1]_{i,j}^+ \otimes x_j(k-1)$

$\bigoplus_{j=1}^q B_{i,j}^- \otimes u_j(k) \leqslant \bigoplus_{j=1}^q B_{i,j} \otimes u_j(k) \leqslant \bigoplus_{j=1}^q B_{i,j}^+ \otimes u_j(k)$

Finally, we deduce

$\bigoplus_{j=1}^n [A_0]_{i,j}^- \otimes x_j(k) \oplus \bigoplus_{j=1}^n [A_1]_{i,j}^- \otimes x_j(k-1) \oplus \bigoplus_{j=1}^q B_{i,j}^- \otimes u_j(k) \leqslant x_i(k)$ (2)

and $x_i(k) \leqslant \bigoplus_{j=1}^n [A_0]_{i,j}^+ \otimes x_j(k) \oplus \bigoplus_{j=1}^n [A_1]_{i,j}^+ \otimes x_j(k-1) \oplus \bigoplus_{j=1}^q B_{i,j}^+ \otimes u_j(k)$ (3)

For the output,

$\bigoplus_{j=1}^n C_{i,j}^- \otimes x_j(k) \leqslant y_i(k)$ and $y_i(k) \leqslant \bigoplus_{j=1}^n C_{i,j}^+ \otimes x_j(k)$ (4)

From this system of $2(n+m)$ inequalities and n variables $x_i(k)$, given k, we will build a system of particular inequalities which is defined below.

Definition 2 *An upper bound constraint or UBC presents the following form* $x_{\tau(i)} \leqslant \bigoplus_{j=1}^s M_{i,j}\ x_j$ *where* $\tau(i)$ *is used to index the variable of the left hand side of the ith such UBC.*

Remark 1 *The variable* $x_{\tau(i)}$ *can be the left hand side of several UBCs and thus one can have* $x_{\tau(i)} = x_{\tau(j)}$ *with* $i \neq j$

- Let us consider the inequalities relative to the input where the term $[\bigoplus_{j=1}^q B_{i,j} \otimes u_j(k)\ ]$ represents a constant.

To come down the system (2)(3)(4) to an UBC form, we introduce a new variable x_0 which represents the neutral element e for the operation $\otimes$ and the zero in usual notation. For i ranging from 1 to n (n is the dimension of the state and q the dimension of the input),

$x_i(k) \leqslant \bigoplus_{j=1}^n [A_0]_{i,j}^+ \otimes x_j(k) \oplus \bigoplus_{j=1}^n [A_1]_{i,j}^+ \otimes x_j(k-1) \oplus [\bigoplus_{j=1}^q B_{i,j}^+ \otimes u_j(k)\ ] \otimes x_0$ and for j ranging from 1 to n, $[A_0]_{i,j}^- \otimes x_j(k) \leqslant x_i(k)$ or $x_j(k) \leqslant ([A_0]_{i,j}^-)^{\otimes(-1)} \otimes x_i(k)$ and also $x_j(k-1) \leqslant ([A_1]_{i,j}^-)^{\otimes(-1)} \otimes x_i(k)$

For the right hand term, $[\bigoplus_{j=1}^q B_{i,j}^- \otimes u_j(k)\ ] \otimes x_0 \leqslant x_i(k)$ or $x_0 \leqslant [\bigoplus_{j=1}^q [B_{i,j}^-] \otimes u_j(k)\ ]^{\otimes(-1)} \otimes x_i(k)$

- Let us consider the inequalities relative to the output. For i ranging from 1 to m (m is the dimension of the output), we have $y_i(k) \leqslant \bigoplus_{j=1}^n C_{i,j}^+ \otimes x_j(k)$ and $\bigoplus_{j=1}^n C_{i,j}^- \otimes x_j(k) \leqslant y_i(k)$. For the same reasons as above, one introduces the variable x_0 in the inequalities.

$y_i(k) \otimes x_0 \leqslant \bigoplus_{j=1}^n C_{i,j}^+ \otimes x_j(k)$ or $x_0 \leqslant [y_i(k)]^{-1}(\bigoplus_{j=1}^n C_{i,j}^+ \otimes x_j(k))$

and for j ranging from 1 to n, $C_{i,j}^- \otimes x_j(k) \leqslant y_i(k) \otimes x_0$ or $x_j(k) \leqslant [C_{i,j}^-]^{\otimes(-1)} \otimes y_i(k) \otimes x_0$

To sum up, the system can be come down to the following UBC system which is equivalent if $x_0 = e$:

- 2n(1+n) state/input relations

$(\forall i \in [1,n])$, $x_i(k) \leqslant \bigoplus_{j=1}^n [A_0^+]_{i,j} \otimes x_j(k) \oplus \bigoplus_{j=1}^n [A_1^+]_{i,j} \otimes x_j(k-1) \oplus [\bigoplus_{j=1}^q B_{i,j}^+ \otimes u_j(k)\ ] \otimes x_0$

$(\forall i \in [1,n])$ $x_0 \leqslant [\bigoplus_{j=1}^q B_{i,j}^- \otimes u_j(k)\ ]^{\otimes(-1)} \otimes x_i(k)$

$(\forall i \in [1,n])$ $(\forall j \in [1,n])$, $x_j(k) \leqslant [A_0^-]_{i,j}^{\otimes(-1)} \otimes x_i(k)$ and $x_j(k-1) \leqslant [A_1^-]_{i,j}^{\otimes(-1)} \otimes x_i(k)$

- m(1+n) state/output relations

$(\forall i \in [1,m])$, $x_0 \leqslant [y_i(k)]^{-1}(\bigoplus_{j=1}^{n} C_{i,j}^{+} \otimes x_j(k))$

$(\forall i \in [1,m])(\forall j \in [1,n])$, $x_j(k) \leqslant [C_{i,j}^{-}]^{\otimes(-1)} \otimes y_i(k) \otimes x_0$

- One adds moreover the n relations of iteration in order to integrate the previous step.

$(\forall k < k_f)$ $(\forall i \in [1,n])$ $[x(k)]_i \leqslant [x^{+}(k \mid [k+1,k_0])]_i \otimes x_0$

This system contains at the most $(1+n).(m+2n)+n$ UBC inequalities for $2.n+1$ variables, x_0 being a new variable added to the system. When x_0 equals zero , the maximum solution of the system will be found. To facilitate the reading of the document, we note r the number of inequalities and s the numbers of variables, x_0 not included. If we consider only a given number of event k, the system to be solved is thus a system of r UBC inequalities having $s+1$ variables. In fact, the exact resolution must consider the complete set of UBC inequalities for k belonging to the interval $[k_s, k_f]$.

IV. Resolution by Kleene 's star

Exploiting Kleene 's star, the technique of resolution of E.Walkup and G.Borriello make it possible to calculate the greatest solution of general systems of equalities and of inequalities. For example, it can solve linear (max, +) equations of the form $A \otimes x \oplus B = C \otimes x \oplus D$, inequalities of the type $Ax \leqslant B$ and make optimization of functions under constraints.

On the other hand, the solution appears in Walkup-Borriello's approach at the end of the decrease of an upper limit with a finite number of steps. Let us notice that this approach does not treat the (max, +) polynomial forms as the algorithm solving the ELCP problem (Extended Linear Complementarity Problem) of Bart de Schutter who generalizes the linear complementary problem by using the usual algebra [DES, 96] [DES, 01]. However, this last approach does not treat the equations of the type $A \otimes x \oplus B = C \otimes x \oplus D$ and is NP-complete. The two approaches share the fact of having an original idea to treat the equalities of the (max, +) equations while proposing different solutions. The broad outlines of Walkup-Borriello's approach a detailed description of which is in [WAL 95][WAL 98], is given below.

A. *Method*

The technique is composed of three stages:

- To choose a particular subset of UBCs
- To apply Kleene 's star to calculate the greatest solution
- To use the preceding solution which guides the choice of the new constraint and thus to minimize the greatest solution. On the assumption of existence of the solution, the convergence is obtained when a solution checks all the constraints.

B. *Resolution of a subset of UBCs*

The variables are reorganized such as x_0 is the first variable. The greatest solution will be found when $x_0 = e$.

Definition 3 *A targeting subset of a system of UBCs over $s+1$ variables is a set of s UBCs such as each variable x_i different from x_0 is the target of exactly one UBC.*

A system of UBCs may be expressed in matrix form as follows. We first make note of the fact that $a \leqslant b$ and $a \oplus b = b$ are equivalent. Consequently, the r inequalities $x_{\tau(i)} \leqslant \bigoplus_{j=0}^{s} M_{i,j}\, x_j$ may also be written as $x_{\tau(i)} \oplus \bigoplus_{j=0}^{s} M_{i,j}\, x_j = \bigoplus_{j=0}^{s} M_{i,j}\, x_j$. One obtains the matrix form $(J \oplus M) \bigotimes x = M \otimes x$ with J a Boolean matrix $r.(s+1)$ whose inputs are equal to e or ε . To choose s UBCs among r not targeting x_0, it is then enough to multiply by a Boolean matrix P of dimensions $s.r$ such as each row contains one e exactly and ε otherwise such that PJ (dimension: $s.s{+}1$) is a concatenation of a column of ε relative to x_0 and the identity matrix of dimension $s.s$. Inequalities which are not selected corresponds to columns of P containing only element ε. One obtains $(P.J \oplus P.M) \bigotimes x = P.M \otimes x$ (dimension of $P.M$: $s.s{+}1$). To be able to apply Kleene 's star, the matrix $P.M$ needs to be square. The addition of a top row containing only elements ε makes it possible to obtain a matrix P' of dimension $(s+1).r$ such that the matrix $P' \otimes J = E'$ is a identity matrix of dimension $(s{+}1).(s{+}1)$ except the element $(E')_{1.1} = \varepsilon$. The result is $(P'.(J \oplus M)) \bigotimes x = (P'.M) \otimes x$. The top row of the products $P'.J$ and $P'.M$ are always all ε-terms. The variable x_0 is not coupled with any inequality.

Definition 4 *A targeting subset is called a safe targeting if all circuits in the graph induced by the targeting subset $P'.M$ have a strictly negative weight.*

The previous form is solved by considering $(P.J) \bigotimes x = (P.M) \otimes x$ and the equality $x_0 = 0$ in order to obtain a square system. This system is equivalent to $x = (P'.M) \otimes x \oplus d$ with d a vector column in which only the first element $(d)_1$ is different from ε and equal to e. Theorem 1 indicates that its single solution is $(P'M)^{*}_{i,0}$. The following lemma declares that this solution is also the greatest solution of the targeting subset, i.e. of $(P'.(J \oplus M)) \bigotimes x = (P'.M) \otimes x$.

Lemma 5 (WALK 98) *Given a safely targeting subset of upper bound constraints, the vector*

$(l_0, l_1, ..., l_s)^t$ *where* $l_i = (P'M)^*_{i,0}$ *is the maximum solution to the constraint subset when* $x_0 = 0$.

C. *Resolution of the complete system*

As the solution of the preceding section comes from the resolution of a targeting subsystem, it is not guaranteed to be a solution of the complete system. One can however affirm that this solution is a upper limit of any solution of the system. If the solution l to the current safe targeting subsystem is not a solution to the complete system, then there must be at least one inequality UBC u_i which is not yet satisfied and is not in the current safe targeting subset. If one replaces the UBC currently targeting $x_\tau(i)$ with the new constraint u_i , one shows that this new system is also safe and that its resolution carries out a minimization. A new solution is obtained l' such as, for any $j \neq \tau(i)$, $l_j \geqslant l'_j$ and $l_{\tau(i)} > l'_{\tau(i)}$.

The found solution will have to also check the UBC inequalities coupled with the variable x_0 in order to verify all the inequalities. The system is without solution if an UBC inequality coupled with x_0 imposes a negative value on x_0. It is in addition possible to show the convergence in a finite number of steps.

D. *Application to the estimation : a specific initialization*

The algorithm must consider the full set of UBC inequalities for k belonging to the interval $[k_s, k_f]$ but also start from an initial upper bound. In this part, we consider the construction of an initialization which is specific to the estimation problem. We adopt here an iterative step which will enable us to limit the number of variables and equations to be treated simultaneously. The result is a realistic initial upper bound. Even though this intitialization gives the final solution in many cases, this step must be completed by the application of the algorithm on the full set of UBC inequalities for k belonging to the interval $[k_s, k_0]$. The initialization is the following.

Thus, we will initially solve the system (1) for $k = k_f$. It will enable us to give an upper limit for $x(k_f)$ and $x(k_f - 1)$. We note respectively $x^+(k \mid [k, k_f])$ and $x^+(k-1 \mid [k, k_f])$ the upper bound of $x(k)$ and $x(k-1)$ knowing the input and the output on the horizon $[k, k_f]$.

At the following rank $k = k_f - 1$, we have $x(k_f - 1) = A_0.x(k_f - 1) \oplus A_1.x(k_f - 2) \oplus B.u(k_f - 1)$

$y(k_f - 1) = C.x(k_f - 1)$

As it is necessary to take the preceding calculations into account and the particular case of the upper bound of $x(k_f - 1)$ already calculated, one must add the following inequality in the system to be solved : $x(k_f - 1) \leqslant x^+(k_f - 1 \mid [k_f, k_f])$.

One can thus repeat the procedure in an iterative and decreasing way until k_s. In short, the approach is the following one:

- To solve the system by calculating an upper bound for $x(k)$ and $x(k-1)$ by Kleene 's star.
- To start again the preceding step after the decrementation of k and the integration of the preceding result.

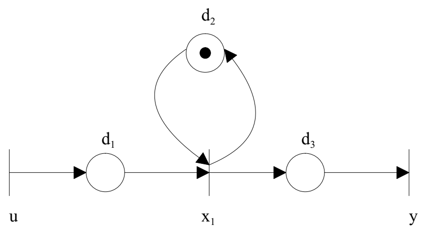

Figure 1 – mono-input / mono-output system

V. Example

Let an elementary system mono-input/mono-output (see figure) described by the following model ($B = d_1, A_1 = d_2, C = d_3$):

$$\begin{cases} x_1(k) = A_1 \otimes x_1(k-1) \oplus B \otimes u(k) \\ y(k) = C \otimes x_1(k) \end{cases}$$

with $(A_1^-, A_1^+) = (1,3)$, $(B^-, B^+) = (e, e)$ and $(C^-, C^+) = (1, 4)$. So, $A_0 = \varepsilon$, $1 \leqslant A_1 \leqslant 3$, $B = e$ and $1 \leqslant C \leqslant 4$,.

One deduces the following system of UBCs : 6 inequalities targeting 3 variables x_0, $x_1(k)$ and $x_1(k-1)$.

$$\begin{cases} x_0 \leqslant [y(k)]^{\otimes -1} \otimes C^+ \otimes x_1(k) \text{ (V-1)} \\ x_1(k) \leqslant [C^-]^{\otimes -1} \otimes y(k) \otimes x_0 \text{ (V-2)} \\ x_1(k) \leqslant [A_1^+] \otimes x_1(k-1) \oplus [B^+ \otimes u_1(k)] \otimes x_0 \text{ (V-3)} \\ x_0 \leqslant [B^- \otimes u(k)]^{\otimes -1} \otimes x_1(k) \text{ (V-4)} \\ x_1(k-1) \leqslant [A_1^-]^{\otimes -1} \otimes x_1(k) \text{ (V-5)} \\ x_1(k) \leqslant x_1^+(k \mid [k+1, k_0]) \otimes x_0 \text{ for } k < k_0 \text{ (V-6)} \end{cases}$$

Initially, we consider only the inequalities targeting $x_1(k)$ and $x_1(k-1)$. We choose a safe sub-system that is composed of the inequalities (V-2) and (V-5) for example. Let us notice that the system (V-3) (V-5) is not safe and cannot thus constitute an initial targeting subsystem. The calculation of the upper limit is achieved by the resolution of the following equalities.

$$\begin{cases} x_0 = 0 \\ x_1(k) = [C^-]^{\otimes -1} \otimes y(k) \otimes x_0 \\ x_1(k-1) = [A_1^-]^{\otimes -1} \otimes x_1(k) \end{cases}$$

The solution of the system "$x = Ax \oplus B$ " is given by $[A^*]_{i,1}$. One obtains $x_0 = 0$, $x_1(k) = [C^-]^{\otimes -1} \otimes y(k)$ and $x_1(k-1) = [A_1^-]^{\otimes -1} \otimes [C^-]^{\otimes -1} \otimes y(k)$. One replaces in the inequalities relatives to $x_1(k)$ and $x_1(k-1)$ and checks easily their validities except for the inequality (V-6) because one doesn't have $x_1^+(k \mid [k+1, k_f])$. On the assumption that the latter UBC is not checked, it will be necessary to replace by its corresponding equality the equation targeting $x_1(k)$ (V-2) and to carry out a new minimization. Lastly, the inequalities (V-1) and (V-4) targeting x_0 will have to be checked to confirm the existence of the solution. It is important to remark that the following steps of the algorithm can modify the values of $x_1(k)$ and $x_1(k-1)$. Consequently, the validity of the inequalities can also be modified.

Let us apply these results to the following simulation. Let us suppose that the input and the output are known from $k_s = 1$ to $k_f = 4$. We start from $x(1) = 1$ and the simulation of $x(k)$ uses arbitrary values of A and C given in the same column of rank k.

k	1	2	3	4
x	1	2	4	10
A		1	2	2
C	1	2	3	4
u	e	1	1	10
y	2	4	7	14

For $k = k_f = 4$ $y(4) = 14$ and $u(4) = 10$, the estimation gives $x_0^+(4[4,4] = 0$; $x_1^+(3[4,4]) = 12$; $x_1^+(4[4,4]) = 13$.

For $k = 3$ $y(3) = 7$ and $u(3) = 1$, the estimation gives $x_0^+(3[3,4]) = 0$; $x_1^+(2[3,4]) = 5$; $x_1^+(3[3,4]) = 6$.

For $k = 2$ $y(2) = 4$ and $u(2) = 1$, the estimation gives $x_0^+(2[2,4]) = 0$; $x_1^+(1[2,4]) = 2$; $x_1^+(2[2,4]) = 3$.

For $k = 1$ $y(1) = 2$ and $u(1) = 0$, the estimation gives $x_0^+(1[1,4]) = 0$; $x_1^+(0[1,4]) = 0$; $x_1^+(1[1,4]) = 1$.

The initialization produces the following result: $x_1^+(1) = 1$; $x_1^+(2) = 3$; $x_1^+(3) = 6$; $x_1^+(4) = 13$ which allows us the application of the procedure on the complete set of UBC inequalities: the following step is the verification of every UBC inequality with the previous upper bound. One can note the inequality (V.3) is not true for k= 4 . Consequently, a new minimization must start again : the only modified value is $x_1^+(4) = 10$. As this new upper bound verifies every UBC inequality, this new upper bound is the least upper bound of the state. So, the final result is: $x_1^+(1) = 1$; $x_1^+(2) = 3$; $x_1^+(3) = 6$; $x_1^+(4) = 10$. One can notice that the initialization gives an upper bound close to the final result.

VI. CONCLUSION

The aim of this study is the problem of the supervision without the knowledge of the state, this situation being caused for example, by the presence of perturbations. In this objective, we propose an estimation approach based on the calculation of an upper limit of the state and using not the classical theory of residuation but Kleene 's star. This approach is applied to time event graphs whose temporizations belong to known intervals and makes it possible to obtain this bound in a finite time. The initialization procedure gives an initial upper bound which allows the application of the algorithm on the complete set of inequalities. A simple example shows that the proposed initialization gives already a result which can be close to the final least upper bound. Future studies will show that this approach can also be applied in other field like process control. Some connections with other methods like interval method will be realized [JAU, 01].

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