From extremal trajectories to consistency in P-time Event Graphs

Technical report

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Abstract

Using (max, +) algebra, this paper presents a modeling and an analysis method of P-time Event Graphs whose behaviors are defined by lower and upper bound constraints. On the hypothesis of liveness of Event Graphs, consistency is defined by the existence of a temporal trajectory. The extremal trajectories starting from an initial interval are deduced by two dual polynomial algorithms based on a particular series of matrices. The analysis of the circuits introduces conditions of consistency.

Index Terms

P-time Petri Nets, Timed Event Graph, (max,+) algebra, token death, Kleene star, fixed point.

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I. Introduction

P-time Event Graphs can be described by a new class of systems called interval systems [7] [11] for which the time evolution is not strictly fixed but belongs to intervals. For interval systems, lower and upper interval bounds depend on the operations of maximization, minimization and addition, in general case. The algebraic model of P-time Event Graphs corresponds to the semantic "And" of Time Stream Event Graphs where the lower and upper bound constraints of P-time Event Graphs are (max, +) and (min, +) functions, respectively. P-time Event Graphs define a set of trajectories which follows the specifications given by the model for a nominal behavior. Firing dates of the transitions belong to the relevant time windows. Indeed, P-time Petri nets are convenient tools to model manufacturing systems whose operation times must be included between minimum and maximum values. A practical example is the electroplating lines where the minimum and maximum immersion times guarantee the quality of the chemical treatment [5]: Each chemical treatment must be sufficient otherwise the product will not be ready for the next task or treatment; On the other hand, each chemical treatment must not be too excessive, otherwise, the product would be damaged. Other practical examples can be given in food industry. In good bakery practice, the dough stays in the fermentation room from three to five hours, the time depending on room temperature and flour or gluten quality. The loaves need to be baked between a minimal and maximal time. If these times are too short or too long (e.g. a synchronization with another operation is not fulfilled), the product will be damaged (bad inner structure and grain in the finished loaf, insufficient or excessive baking). P-time Event Graphs can describe the losses of resources or parts and the failures to meet the time specifications.

However, even if the underlying graph of the Event Graph is live, the specifications can be too restrictive and some synchronizations cannot follow the desired model, producing the losses of resources. A process is composed of machines, resources, etc. and the issue is to know if they can work together following a schedule for a specified period. More particularly, a question is to know if the different tasks can be sufficiently repeated during a period such that a normal production can be performed without losses of parts. A practical problem can appear when a machine is changed: using a slower/quicker machine can affect the nominal behavior of the complete production line defined by a previous schedule. In this paper, we are interested in avoiding this situation. The relevant notion is the consistency, which can be defined by the existence of a time trajectory following the model. Many approaches as control, simulation, optimization, etc. are based on a model and they usually assume that the process follows a normal time evolution expressed by the considered model and not another one: the correctness of the model is clearly a major problem and it must be checked before the application of any approach.

An acceptable trajectory or a consistent trajectory can be defined as a time trajectory satisfying the model. A second natural aim is to determine acceptable trajectories starting from an initial state. In other words, the objectives are to check if there is an acceptable trajectory starting from a given interval and to calculate the corresponding extremal (lowest and greatest, see [17]) trajectories, that is to say, the relevant earliest and latest trajectories. Their existences confirm the consistency of the system. This also gives a model simulation for the earliest and latest functioning.

In [18], the determination of acceptable trajectories has been considered in the particular case of Timed Event Graphs. It has been shown that the initial state must satisfy a condition such that the trajectory is nondecreasing in the counter representation. Analysis of consistency of interval descriptor systems as Time Stream Event Graphs has been made by using the spectral vector

[11] which introduces the topic of consistency of P-time Event Graphs in the field of (max, +) algebra. [8] [14] and [9], introduce a new model of P-time Event Graph and a particular series of matrices whose evolution determines the system behavior and the existence of a trajectory without token deaths. The extremal trajectories obeying an interval of desired output are deduced. The present paper improves these studies by presenting a graphical interpretation. Circuits in a special associated graph will be highlighted.

In [1], the author considers a close model which is Timed Event Graphs with upper bounds on the temporizations of the places. After [8] and [14], consistency has been considered in [20] but without analysis of the generated graph and characterization of the circuits.

In this paper, the usual assumption of earliest behavior applied on the lower bound is not made as in [1], [20]. This assumption will particularly be discussed in the last part of this paper. Moreover, no hypothesis is made on the structure of Event Graphs. These need not be strongly connected. The initial marking must only satisfy the classical condition of liveness (no circuit without token), and the usual hypothesis First In First Out (FIFO) for tokens is made.

The paper is structured as follows: Notations and some previous results together with the definitions of P-time Event graph are first given. Then we introduce the modeling of P-time Event Graphs in the (max, +) algebra using the ”dater” form and we present the principle of the approach using a pratical example. We study the behavior of the new model with the help of a special series of matrices and deduce the extremal trajectories satisfying an initial condition defined on an interval. Last but not least, a general example illustrates the approach.

II. Preliminaries

A monoid is couple $(S, \oplus)$ where operation $\oplus$ is associative and presents neutral element ε. Semi-ring S is triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative, $\otimes$ is distributive in relation to $\oplus$ and zero element ε of $\oplus$ is the absorbing element of $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). Dioid D is an idempotent semi-ring (operation $\oplus$ is idempotent, that is $a \oplus a = a$). Unit $\mathbb{R} \cup \{-\infty\}$, provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid denoted $\mathbb{R}_{\max}$ = $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$, respectively. The absorbing element of $\otimes$ is ε. Isomorphic to the previous one by bijection: $x \longmapsto -x$, another dioid is $\mathbb{R} \cup \{+\infty\}$, provided with the minimum operation denoted $\wedge$ and the addition denoted $\odot$. The neutral elements of $\wedge$ and $\odot$ are represented by $T = +\infty$ and $e = 0$ respectively. The absorbing element of $\odot$ is ε. The following conventions are made: $T \otimes \varepsilon = \varepsilon$ and $T \odot \varepsilon = T$. Expressions $a \otimes b$ and $a \odot b$ are identical if at least either a or b is a finite scalar. The partial order denoted $\leqslant$ is defined as follows: $x \leqslant y \Longleftrightarrow x \oplus y = y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x_i \leqslant y_i$, for i from 1 to n in $\mathbb{R}^n$. Notation $x < y$ means that $x \leqslant y$ and $x \neq y$. Dioid D is complete if it is closed for infinite sums, and the distributivity of the multiplication with respect to addition applies to infinite sums : ($\forall\ c \in D$) ($\forall$ $A \subseteq D$) $c \otimes (\bigoplus\limits_{x \in A} x) = \bigoplus\limits_{x \in A} c \otimes x$. For example, $\overline{\mathbb{R}}_{max}$ = $(\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes)$ is complete. The set of $n.n$ matrices with entries in complete dioid D including the two operations $\oplus$ and $\otimes$ is also a complete dioid, which is denoted $D^{n.n}$. The elements of the matrices in the (max, +) expressions (respectively (min, +) expressions) are either finite or ε (respectively T). We can deal with nonsquare matrices if we complete them with rows or columns provided the entries equal ε (respectively T). The different operations obey the usual rules of algebra: notation $\odot$ refers to the multiplication of two matrices in which the $\wedge-$operation is used instead of $\oplus$. Mapping f is said to be residuated if for all $y \in D$, the least upper bound of subset $\{x \in D \mid f(x) \leq y\}$ exists and lies in this subset. Mapping $x \in (\overline{\mathbb{R}}_{max})^n \mapsto A \otimes x$, defined over $\overline{\mathbb{R}}_{max}$ is residuated (see [3]) and the left $\otimes-$residuation of B by A is denoted by: $A \backslash B = \max\{x \in (\overline{\mathbb{R}}_{max})^n$ such that

$A \otimes x \leqslant B\}$; moreover, $A \setminus B = (-A)^t \odot B$ or $A \odot B = (-A)^t \setminus B$ (see the proof of theorem 3.21 in part 3.2.3.2 of [3]) with convention $-\infty - (-\infty) = +\infty$ and $+\infty - (+\infty) = +\infty$.

The Kleene star is defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$. Denoted as $G(A)$, an associated graph of square matrix A is deduced from this matrix by associating node i with column i and line i and an arc from node j towards node i with $A_{ij} \neq \varepsilon$. Weight $\mid p \mid_w$ of path p is the sum of the labels (weights) on the edges in the path. Length $\mid p \mid_l$ of path p is the number of edges in the path. A circuit is a path which starts from and ends at the same node. Using the Kleene star, the two following theorems are dual and will be considered in the dioid of matrices.

***Theorem** 1:* (Theorem 4.75 part 1 in [3]) Consider equation

$$x = A \otimes x \oplus B \tag{1}$$

and inequality

$$x \geq A \otimes x \oplus B \tag{2}$$

with A and B in complete dioid D. Then, $A^* \otimes B$ is the least solution of (1) and (2). ■

***Theorem** 2:* (Theorem 4.73 part 1 in [3]) Consider equation

$$x = A \backslash x \wedge B \tag{3}$$

and inequality

$$x \leq A \backslash x \wedge B \tag{4}$$

with A and B in complete dioid D. Then, $A^* \backslash B$ is the greatest solution of (3) and (4). ■

For A_{ij} and B_i belonging to $\overline{\mathbb{R}}_{max}$, $A \setminus x \wedge B$ can be written $(-A)^t \odot x \wedge B$.

We shall need the following result in the sequel

***Corollary** 1:* Corollary $x \geq A \otimes x \oplus B \Longleftrightarrow x \geq A \otimes x$ and $x \geq A^* B$.

$x \leq A \backslash x \wedge B \Longleftrightarrow x \leq A \backslash x \otimes x$ and $x \leq A^* \backslash B$.

Proof. We only consider the first result. The proof is dual for the second part. a) $\Longrightarrow$: it is an application of Theorem 1. b) $\Longleftarrow$: $x \geq A^{*}B$ implies $x \geq B$. ■

III. Models and principle of the approach

In a first part, the definition of P-time Event Graphs is given and the firing interval of each transition is described. Using the dater form, the algebraic model is built. This model will be analyzed in the next parts. An elementary production system is described and the principle of the approach is presented.

A. P-time Event Graphs

Event Graphs constitute a subclass of Petri nets in which each place has exactly one upstream and one downstream transition. We shall use the following notations. The set of places is denoted P. The initial marking of place $p_l \in P$ is denoted m_l. Let ${}^{\bullet}p_l$ denote the set of input transitions of place $p_l \in P$ and $p_l^{\bullet}$ the set of output transitions of p_l. Similarly, ${}^{\bullet}x_i$ (respectively $x_i^{\bullet}$) denotes the set of the input (respectively, output) places of transition x_i.

In P-time Event Graphs [16], time constraints of the token stay are associated with each place. We associate with each place $p_l \in P$ temporal interval $[a_l, b_l]$ with $0 \leq a_l \leq b_l$ and $[a_l, b_l]$ $\in \mathbb{R}^{+} \times (\mathbb{R}^{+} \cup \{+\infty\})$

Interval $[a_l, b_l]$ is the static interval of residence time or duration of a token in place p_l belonging to the set of places P. The token must stay in place p_l during the minimal sojourn time a_l. Before this duration, the token is in unavailability state to firing unique transition $x_j \in p_l^{\bullet}$. Value b_l is the maximal sojourn time after which the token must leave place p_l. If not, the system is found in a token-dead state. So, the token is available to fire transition x_j in time interval $[a_l, b_l]$ with a_l the lower bound of the temporization (respectively, b_l the upper bound).

A consequence is a possible bad synchronization of each transition which is the outgoing transition of more than one place. This situation occurs when the firing dates of the ingoing transitions of at least two places are incoherent. This non-synchronization can be solved by a prediction of the evolution of the tokens in the places.

For instance, let us consider two places P_1 and P_2 associated with intervals $[a_1, b_1]$ and $[a_2, b_2]$ respectively, where $a_1 = 1$, $b_1 = 2$, $a_2 = 3$ and $b_2 = 4$. The two places share same outgoing transition x_2. If the firing date of ingoing transition x_1 of place P_1 is 100, the firing date of its outgoing transition x_2 must be between $100 + a_1$ and $100 + b_1$. However, transition x_2 cannot be fired and the token in P_1 will die if the firing date of ingoing transition x_3 of place P_2 is equal to 10. Therefore, the firing date of transition x_3 must be chosen such that transition x_2 can be fired: $x_2 \in [100 + a_1, 100 + b_1] \cap [x_3 + a_2, x_3 + b_2]$. As ingoing transition x_3 is before outgoing transition x_2, this choice must predict the future phenomena. Therefore, the main difference with usual Timed Event Graphs is that the evolution of P-time Event Graphs needs an anticipation of its trajectory. This characteristic will determine the form of following algebraic models and the results of this paper.

Now, we consider the dater form which will give an efficient description.

B. Dater form

We consider the "dater" description in the (max, +) algebra: each variable $x_i(k)$ represents the date of the k^{th} firing of transition x_i. If we assume a FIFO functioning of the places which guarantees that the tokens do not overtake one another, a correct numbering of the events can be carried out.

In an Event Graph, card($^\bullet p_l$)=card($p_l^\bullet$)=1 for each place $p_l \in P$ and we can associate only a pair (x_i, x_j) with each place $p_l \in P$, such that transition x_j is ingoing ($x_j \in^\bullet p_l$) and transition

x_i is outgoing ($x_i \in p_l^\bullet$). Time interval $[a_l, b_l]$ and initial marking m_l are also associated with place p_l. The evolution of the P-time Event Graph is described by the following inequalities expressing relations between the firing dates of transitions:

$\forall p_l \in P$ with $x_j \in^\bullet p_l$ and $x_i \in p_l^\bullet$, $a_l + x_j(k - m_l) \leq x_i(k)$ and $x_i(k) \leq b_l + x_j(k - m_l)$

Now, let us consider a pair of transitions (x_i, x_j) and a given marking m. These conditions define a set of places $P_{i,j,m}$ which can be empty or contain more than one place: Each place p_l of $P_{i,j,m}$ satisfies $\{x_j\} =^\bullet p_l$, $\{x_i\} = p_l^\bullet$ and $m_l = m$. $\forall p_l \in P_{i,j,m}$, we can take the maximum of lower bounds a_l and the minimum of upper bounds b_l and we denote the corresponding values $a^-_{i,j,m} \in \mathbb{R}^+$ and $a^+_{i,j,m} \in \mathbb{R}^-$. More formally, $a^-_{i,j,m} = \bigoplus_{\forall p_l \in P_{i,j,m}} a_l$ and $a^+_{i,j,m} = \bigwedge_{\forall p_l \in P_{i,j,m}} b_l$.

Remark. Naturally, if for each pair of transitions (x_i, x_j), there is a unique place $p_l \in P$ in the Event Graph, we can simplify the notation and replace $a^-_{i,j,m}$ by $a^-_{i,j}$ and $a^+_{i,j,m}$ by $a^+_{i,j}$. In the figures of the paper, each temporisation is directly indexed with the index l of the relevant place p_l.

Therefore, the system can be described as follows

$\forall P_{i,j,m} \subset P$, $a^-_{i,j,m} + x_j(k - m) \leq x_i(k)$ and $x_i(k) \leq a^+_{i,j,m} + x_j(k - m)$

After permutation of indexes i and j,

and application of simple transformations, the latter inequality is equivalent to $-a^+_{j,i,m} + x_j(k + m) \leq x_i(k)$

In short,

$a^-_{i,j,m} + x_j(k - m) \leq x_i(k)$ and $-a^+_{j,i,m} + x_j(k + m) \leq x_i(k)$

The system can now be expressed with matrices in (max, +) algebra. This allows the writing of a synthetic description on a horizon defined by the maximal initial marking $M = \bigoplus_{\forall p_k \in P} m_k$.

$$x(k) \quad \geq \bigoplus_{0 \leq m \leq M} A_m^- \otimes x(k-m) \tag{5}$$

$$x(k) \quad \geq \bigoplus_{0 \leq m \leq M} A_m^+ \otimes x(k+m) \tag{6}$$

with $(A_m^-)_{ij} = a_{i,j,m}^-$ if $a_{i,j,m}^-$ exists in $\mathbb{R}$ or ε otherwise,

$(A_m^+)_{ij} = -a_{j,i,m}^+$ if $a_{j,i,m}^+$ exists in $\mathbb{R}$ or ε otherwise.

For instance, in figure 1., $(A_0^-)_{3,1} = a_{3,1,0}^- = a_{p_3} = a_3$ and $(A_0^+)_{1,3} = -a_{3,1,0}^+ = -b_{p_3} = -b_3$ for place P_3.

This model completely describes the relevant P-time Event Graph by giving a lower bound of state $x(k)$. This lower bound depends on values $x(k-m)$ and $x(k+m)$ for $m = 0$ to M in respectively, inequalities (5) and (6). As inequality (5) corresponds to a classical Timed Event Graph (without assumption of earliest functioning), a P-time Event Graph can be seen as a Timed Event Graph (5) following additional specifications (6). The Timed Event Graph can express the physical limitations of the process as the minimal cooking time while the upper bounds describe quality criteria on the finished parts and products: The respect of these constraints needs an anticipation of the future behavior of the process. Therefore, the calculation of the lower bound trajectory cannot be made from only the past trajectory like a Timed Event Graph working in the earliest functioning, but must use a prediction of the future evolution. In the sequel, we will see that this remark also holds for the upper bound.

Remark. Some authors add the additional assumption of earliest behavior and replace the inequality in (5) by an equality. Therefore, they limit the possibility of modeling of P-time Event Graphs which does not describe a unique trajectory but a set of trajectories. Particularly, P-time Event Graphs can describe uncertainties on sequence time of the process [2] [5] [12] while the minimal and maximal times of each task are exactly known. For instance, the choice of

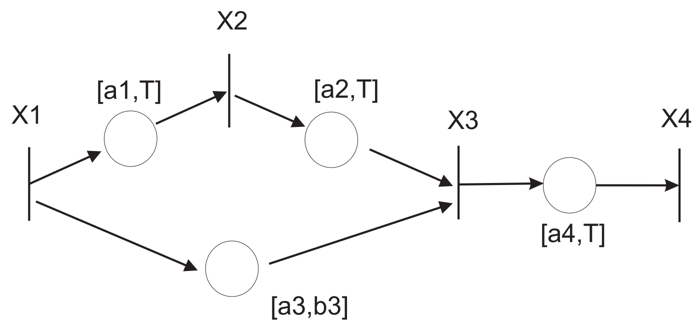

Fig. 1. Principle

a cooking time in the middle of interval $[T_{min}, T_{max}]$ guarantees the quality of the final product but other choices are possible. Choices T_{min} and T_{max} are risky (underdone, overdone) as the parameters of the oven can change.

The aim of the paper is the analysis of the implicit model defined by (5) and (6). Before considering the general model, we first introduce the principle of the general approach with a simple example. This part only uses usual algebra. A more general study will be given in the sequel.

C. Principle of the approach

Let us consider an elementary production system composed of two lines in parallel which start at the same time. The process is described by a P-time Event Graph in figure 1. The first line is composed of two tasks while the second one only corresponds to the cooking of a product (a_3). The tasks of line 1 are successively the making of a packet (a_1) and its moving (a_2). When the activities are completed, the finished product is packed (a_4). Naturally, the cooking time b_3 must not be too excessive, otherwise, the product would be damaged ($x_3 > b_3 + x_1$).

The following inequalities describe the two lines. $a_1 + x_1 \leq x_2$, $a_2 + x_2 \leq x_3$ and $a_3 + x_1 \leq x_3 \leq b_3 + x_1$.

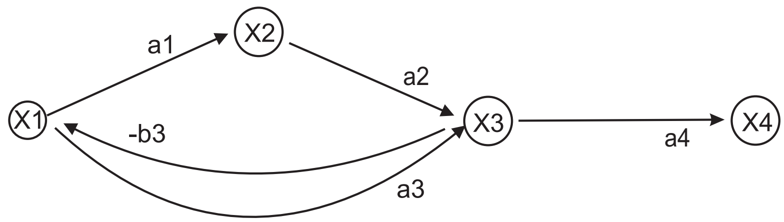

Fig. 2. Associated graph 1

Therefore, $a_1 + a_2 + x_1 \leq x_3$ for line 1, and $a_3 + x_1 \leq x_3 \leq b_3 + x_1$ for line 2. So, $a_1 + a_2 + x_1 \leq x_3 \leq b_3 + x_1$ and consequently, condition $a_1 + a_2 \leq b_3$ is necessary otherwise, the system is inconsistent. In other words, the process does not work if time b_3 is less than the sum of the temporizations a_1 and a_2.

Another explanation is as follows. Inequality $x_3 \leq b_3 + x_1$ can be written $-b_3 + x_3 \leq x_1$. From $a_1 + x_1 \leq x_2$; $a_2 + x_2 \leq x_3$; $-b_3 + x_3 \leq x_1$, we can deduce that $-b_3 + a_2 + a_1 + x_1 \leq x_1$. A necessary and sufficient condition of existence of a solution is $-b_3 + a_2 + a_1 \leq 0$.

These inequalities can be described by a graph (see Figure 2.) defined as follows. The vertices correspond to the transitions of the Petri net and a directed arc from j to i is associated with each inequality $a + x_j \leq x_i$.

This graph shows that term $-b_3 + a_2 + a_1$ is the weight of the circuit defined by transitions x_1, x_2, x_3 and x_1. We can say that the system will be consistent if any circuit of the graph has a negative or null weight. In this case, a sequence of firing dates meeting the consistent system can be found. If the process follows these dates, the production will be satisfactory as each cooked product never waits in oven after delay b_3.

Now, we generalize this first intuitive study and consider the case where the initial marking is null.

IV. Analysis in the static case

Let us assume that the process is static or, in other words that the marking is null: $M = 0$. Therefore, the model described by (5) and (6) is reduced to the following form.

$$x \geq (A_0^- \oplus A_0^+) \otimes x \tag{7}$$

Inequalities of this form are classical in the (max, +) context. The following well-known result clearly shows that the consistency analysis of (7) needs an analysis of the circuits in the static case.

Proposition There is a finite vector $x \in \mathbb{R}^{dim(x)}$ satisfying (7) if and only if the associated graph of matrix $A_0^- \oplus A_0^+$ has only circuits with only non-positive weight.

Recall that $(A^*)_{i,i}$ is the greatest weight of the circuits going by vertex i of the associated graph of matrix A. Another formulation of the proposition is that a necessary and sufficient condition for the existence of a state in $\mathbb{R}$ (not in $\mathbb{R}_{\max}$) is that $((A_0^- \oplus A_0^+)^*)_{i,i}$ converges on $\mathbb{R}_{\max}$ and not on $T = +\infty$ for any index i.

The following form makes the connection with the study in [22]. First, let us note that $x \geq A_0^+ \otimes x$ is equivalent to $x \leq A_0^+ \backslash x$. So, inequality $x \geq (A_0^- \oplus A_0^+) \otimes x$ is equivalent to $A_0^- \otimes x \leq x \leq A_0^+ \backslash x$. This system implies the following expression

$$A_0^- \otimes x \leq A_0^+ \backslash x = (-A_0^+)^t \odot x \tag{8}$$

which has been analyzed in the proposition below. It is given with a slightly modified notation.

Proposition [22]

There is a finite vector $x \in \mathbb{R}^{dim(x)}$ satisfying $A_0^- \otimes x \leq (-A_0^+)^t \odot x$ if and only if the associated graph of $A_0^+ \otimes A_0^-$ contains circuits with only non-positive weight.

The relation defined by (8) has been deduced from (7) or, in other words, the set defined by (8) includes the set defined by (7) but the relations are not mathematically equivalent as shown in the following counter-example.

Example $A_0^- = \begin{pmatrix} -10 & -20 \\ -15 & 0 \end{pmatrix}$ and $(-A_0^+)^t = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$x = \begin{pmatrix} 11 \\ 1 \end{pmatrix}$

$A_0^- \otimes x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq (-A_0^+)^t \odot x = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq \begin{pmatrix} 11 \\ 1 \end{pmatrix} \not\leq \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

The following results allow a comparison of the consistency of (7) and (8) based on the circuits of the associated graphs. The usual multiplication is denoted by a dot below.

Proposition $(\forall \mathbf{k} \in \mathbb{N})(A \oplus B)^{2.k} \geq (A \otimes B)^k$

Proof

The inductive proof is as follows. The hypothesis is H_k:$(A \oplus B)^{2.k} \geq (A \otimes B)^k$.

Initial Step. H_1 defined by $(A \oplus B)^2 \geq (A \otimes B)^1$ is true as $(A \oplus B)^2 = A^2 \oplus A \otimes B \oplus B \otimes A \oplus B^2$

Inductive Step. Let us assume that H_k is true for a given k$\in \mathbb{N}$. We must prove the formula is true for k+1.

$(A \oplus B)^{2.(k+1)} = (A \oplus B)^{2.k} \otimes (A \oplus B)^2 \geq (A \otimes B)^k \otimes (A \otimes B)^1 = (A \otimes B)^{k+1}$ and H_{k+1} is proved. ■

Proposition $(A \oplus B)^* \geq (A \otimes B)^*$

Proof

By definition, $(A \oplus B)^* = \bigoplus_{i \in \mathbb{N}} (A \oplus B)^i \geq \bigoplus_{k \in \mathbb{N}} (A \oplus B)^{2.k}$.

The previous proposition implies that $\bigoplus_{k \in \mathbb{N}} (A \oplus B)^{2.k} \geq \bigoplus_{k \in \mathbb{N}} (A \otimes B)^k = (A \otimes B)^*$ and the proposition is proved. ■

Therefore, the application of the previous result gives:

$(A_0^- \oplus A_0^+)^* \geq (A_0^- \otimes A_0^+)^* \oplus (A_0^+ \otimes A_0^-)^*$. Particularly,

$((A_0^- \oplus A_0^+)^*)_{i,i} \geq ((A_0^- \otimes A_0^+)^*)_{i,i} \oplus ((A_0^+ \otimes A_0^-)^*)_{i,i}$

Consequently, even if $\forall i \in ((A_0^+ \otimes A_0^-)^*)_{i,i} \leq 0$, term $((A_0^- \oplus A_0^+)^*)_{i,i}$ can be positive. Therefore, if the associated graph of $(A_0^+ \otimes A_0^-)$ contains circuits with only non-positive weight, the associated graph of matrix $(A_0^- \oplus A_0^+)$ can have circuits with positive weights. Therefore, application of the previous two propositions shows that inequality (7) can be inconsistent while inequality (8) is consistent.

In conclusion, this part shows that the consistency depends on the circuits in an associated graph. This analysis will now be generalized to an arbitrary initial marking in the sequel. The implicit model described by (5) and (6) will first be rewritten on a short horizon in order to simplify the analysis. Then this new form will be used to calculate extremal trajectories and to analyze the consistency in the following sections.

V. Dynamical model

Now, we consider an arbitrary initial marking. Recall that M is the maximal initial marking: $M = \bigoplus_{\forall p_k \in P} m_k$. The following proposition is about the existence of a state trajectory in $\mathbb{R}$ (and not in $\mathbb{R}_{\max}$).

Proposition. A necessary condition for the existence of a state trajectory in $\mathbb{R}$ is that the associated graph of matrix $A_0^- \oplus A_0^+$ has only circuits with only non-positive weight.

Proof. From inequalities (5) and (6) of the model, we have

$$x(k) \geq \bigoplus_{0\leq m\leq M} A_m^- \otimes x(k-m) \oplus \bigoplus_{0\leq m\leq M} A_m^+ \otimes x(k+m) =$$

$$(A_0^- \oplus A_0^+) \otimes x(k) \oplus \bigoplus_{1\leq m\leq M} A_m^- \otimes x(k-m) \oplus \bigoplus_{1\leq m\leq M} A_m^+ \otimes x(k+m) \quad (9)$$

We can deduce that $x(k) \geq (A_0^- \oplus A_0^+) \otimes x(k)$ and apply the first Proposition in part IV. ■

From (9), we deduce the following inequalities, where the right hand term of the first inequality represents the least solution of (9) .

$$\begin{cases} x(k) \geq (A_0^- \oplus A_0^+)^* \otimes [\bigoplus_{1\leq m\leq M} A_m^- \otimes x(k-m) \oplus \bigoplus_{1\leq m\leq M} A_m^+ \otimes x(k+m)] \\ x(k) \geq (A_0^- \oplus A_0^+) \otimes x(k) \end{cases} \quad (10)$$

The following property shows that (10) completely expresses the model.

Proposition. The inequalities (10) and the implicit model defined by (5) and (6) are equivalent.

Proof: Immediate from Corollary 1.■

Now, let us introduce the following notations.

$$\begin{cases} \mathbb{A}_0^= = A_0^- \oplus A_0^+ \\ \mathbb{A}_m^- = (A_0^- \oplus A_0^+)^* \otimes A_m^- \text{ , for } m = 1 \text{ to } M. \\ \mathbb{A}_m^+ = (A_0^- \oplus A_0^+)^* \otimes A_m^+ \text{ , for } m = 1 \text{ to } M. \end{cases}$$

Therefore, the model (10) can be rewritten as follows.

$$\begin{cases} x(k) \geq \mathbb{A}_0^= \otimes x(k) \\ x \geq \bigoplus_{1\leq m\leq M} \mathbb{A}_m^- \otimes x(k-m) \oplus \bigoplus_{1\leq m\leq M} \mathbb{A}_m^+ \otimes x(k+m) \end{cases} \quad (11)$$

System (11) can be simplified by defining an augmented state vector. The new state vector denoted $\mathcal{X}$, includes variables $x(k)$, $x_i^-(k)$ and $x_i^+(k)$, for $i = 1$ to $M-1$ defined as follows.

$x_1^-(k) = x(k-1)$, $x_1^+(k) = x(k+1)$,

$x_i^-(k) = x_{i-1}^-(k-1)$ and $x_i^+(k) = x_{i-1}^+(k+1)$, for $i = 2$ to $M-1$.

$\mathcal{X} = \begin{pmatrix} (x_{M-1}^-)^t & \dots & (x_2^-)^t & (x_1^-)^t & (x)^t & (x_1^+)^t & (x_2^+)^t & \dots & (x_{M-1}^+)^t \end{pmatrix}^t$ (t: transpose). The dimension of $\mathcal{X}$ is denoted n which is equal to the product of the dimension of x by $2.M-1$.

The following inequalities completely describe both static part and dynamic part of the system.

$x_1^-(k) = x(k-1) \Leftrightarrow x_1^-(k) \geq x(k-1)$ and $x(k) \geq x_1^-(k+1)$

$x_1^+(k) = x(k+1) \Leftrightarrow x_1^+(k) \geq x(k+1)$ and $x(k) \geq x_1^+(k-1)$

$x_i^-(k) = x_{i-1}^-(k-1) \Leftrightarrow x_i^-(k) \geq x_{i-1}^-(k-1)$ and $x_{i-1}^-(k) \geq x_i^-(k+1)$, for $i = 2$ to $M-1$.

$x_i^+(k) = x_{i-1}^+(k+1) \Leftrightarrow x_i^+(k) \geq x_{i-1}^+(k+1)$ and $x_{i-1}^+(k) \geq x_i^+(k-1)$, for $i = 2$ to $M-1$.

Finally, the simplified inequalities are as follows.

$$\begin{cases} \mathcal{X}(k) \geq \mathcal{A}^= \otimes \mathcal{X}(k) \\ \mathcal{X}(k) \geq \mathcal{A}^- \otimes \mathcal{X}(k-1) \\ \mathcal{X}(k) \geq \mathcal{A}^+ \otimes \mathcal{X}(k+1) \end{cases} \tag{12}$$

We shall now illustrate the procedure for the synthesis of matrices $\mathcal{A}^=$, $\mathcal{A}^-$ and $\mathcal{A}^+$ in inequalities (12). Let $M = 3$. So, $\mathcal{X} = \begin{pmatrix} (x_2^-)^t & (x_1^-)^t & (x)^t & (x_1^+)^t & (x_2^+)^t \end{pmatrix}^t$

$x(k) \geq \mathbb{A}_3^- \otimes x_2^-(k-1) \oplus \mathbb{A}_2^- \otimes x_1^-(k-1) \oplus \mathbb{A}_1^- \otimes x(k-1) \oplus (\mathbb{A}_0^- \oplus \mathbb{A}_0^+) \otimes x(k)$

$\oplus \mathbb{A}_1^+ \otimes x(k+1) \oplus \mathbb{A}_2^+ \otimes x_1^+(k+1) \oplus \mathbb{A}_3^+ \otimes x_2^+(k+1)$

$x_1^-(k) = x(k-1) \Leftrightarrow x_1^-(k) \geq x(k-1)$ and $x(k) \geq x_1^-(k+1)$

$x_1^+(k) = x(k+1) \Leftrightarrow x_1^+(k) \geq x(k+1)$ and $x(k) \geq x_1^+(k-1)$

$x_2^-(k) = x_1^-(k-1) \Leftrightarrow x_2^-(k) \geq x_1^-(k-1)$ and $x_1^-(k) \geq x_2^-(k+1)$

$x_2^+(k) = x_1^+(k+1) \Leftrightarrow x_2^+(k) \geq x_1^+(k+1)$ and $x_1^+(k) \geq x_2^+(k-1)$

$$\mathcal{A}^{=}=\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \mathbb{A}_0^{=} & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\end{pmatrix}, \mathcal{A}^{-}=\begin{pmatrix}\varepsilon & I & \varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & I & \varepsilon & \varepsilon\\ \mathbb{A}_3^{-} & \mathbb{A}_2^{-} & \mathbb{A}_1^{-} & I & \varepsilon\\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & I\\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\end{pmatrix}, \text{and } \mathcal{A}^{+}=\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\ I & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\ \varepsilon & I & \mathbb{A}_1^{+} & \mathbb{A}_2^{+} & \mathbb{A}_3^{+}\\ \varepsilon & \varepsilon & I & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \varepsilon & I & \varepsilon\end{pmatrix}$$

Also, the diagonal of $\mathcal{A}^{-}$ can be modified such that the nondecrease of state trajectory is guaranteed. This operation keeps expressions $x_1^-(k) = x(k-1)$, $x_2^-(k) = x_1^-(k-1)$, ... unchanged, because $x_2^-(k) \geq x_1^-(k-1) \oplus x_2^-(k-1) = x_1^-(k-1) \oplus x_1^-(k-2) = x_1^-(k-1)$, for instance.

These expressions describe the “lower” constraints on $\mathcal{X}$ produced by the model which can maximize the state estimation. The set of inequalities (12) clearly describes a forward part ($\mathcal{X}(k) \geq \mathcal{A}^{-} \otimes \mathcal{X}(k-1)$, a backward part and a static (i.e neither backward, nor forward) part ($\mathcal{X}(k) \geq \mathcal{A}^{=} \otimes \mathcal{X}(k)$). These relations lead to complex backward/forward interconnections which can produce inconsistencies in the model.

Symmetrically, as mapping $\mathcal{A}^{=} \otimes \mathcal{X}(k)$, $\mathcal{A}^{-} \otimes \mathcal{X}(k-1)$ and $\mathcal{A}^{+} \otimes \mathcal{X}(k+1)$ are residuated, the application of property f3 in [3] part 4.4.4) gives the following form: it expresses every “upper” constraint on $\mathcal{X}(k)$ which can minimize it.

$$\begin{cases}\mathcal{X}(k) \leq \mathcal{A}^{=} \backslash \mathcal{X}(k)\\ \mathcal{X}(k) \leq \mathcal{A}^{-} \backslash \mathcal{X}(k+1)\\ \mathcal{X}(k) \leq \mathcal{A}^{+} \backslash \mathcal{X}(k-1)\end{cases}$$

Each model can be deduced from the other one by duality and each lower (upper) matrix respectively corresponds to an upper (lower) matrix with the same notation: symbols $\geq$, $\oplus$ and $\otimes$, respectively correspond to $\leq$, $\wedge$ and $\backslash$; Number of events $k-1$ is replaced by $k+1$ and conversely.

In the following part, the time evolution of the model (12) is analyzed.

VI. Extremal acceptable trajectories by series of matrices

Unlike the class of Timed Event Graphs which define a unique trajectory on assumption of earliest behavior, P-time Event Graphs define a set of trajectories which depend on matrices $\mathcal{A}^{=}$, $\mathcal{A}^{-}$ and $\mathcal{A}^{+}$. The aim of this section is the determination of the lowest (respectively, greatest) acceptable trajectories satisfying an initial condition given by $\mathcal{X}(0) \in [\mathcal{X}_0^-, \mathcal{X}_0^+]$. In the sequel, we will show that the existence of a trajectory depends on special new matrices denoted w_k.

As a finite horizon $h \in \mathbb{N}$ is considered, the model behavior must be clarified. A realistic assumption is that the model operates on the same horizon. Therefore, the process starts at $k = 0$ and the constraints before zero are not considered. So, the only constraint on $\mathcal{X}(k)$ for $k = 0$ is $\mathcal{X}(0) \geq \mathcal{A}^{+} \otimes \mathcal{X}(1) \oplus \mathcal{X}_0^-$. Symmetrically, as the process can stop after the horizon denoted h, the only constraint on $\mathcal{X}(k)$ for $k = h$ is $\mathcal{X}(h) \geq \mathcal{A}^{-} \otimes \mathcal{X}(h-1)$.

A. *Lowest state trajectory*

The following algorithms give the lowest and greatest trajectory satisfying the objective. The first step a) is the forward calculation of parameters w_k which only depend on the model. Starting from the initial condition $\mathcal{X}_0^-$ (resp. $\mathcal{X}_0^+$), the second step b) is also based on a forward iteration. It expresses a first estimate of the lowest (resp. greatest) trajectory denoted β_k^- (resp. β_k^+), which is finally improved by a maximisation (respectively, a minimisation) in step c). The final result is the lowest (resp. greatest) trajectory denoted by $\mathcal{X}_k^-$ (resp. $\mathcal{X}_k^+$).

***Theorem** 3:* If the process operates on horizon $h \in \mathbb{N}$ and if matrices w_k defined below have no positive circuit, the lowest state trajectory in $\mathbb{R} \cup \{-\infty\}$ satisfying $\mathcal{X}(0) \geq \mathcal{X}_0^- \in (\mathbb{R} \cup \{-\infty\})^n$ is given by the following forward/backward algorithm.

Forward/backward algorithm

a) Coefficients of w_k by forward iteration

Initialization: $w_0 = \mathcal{A}^=$

for $k = 1$ to h, $\quad w_k = \mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_{k-1})^* \otimes A^+$

b) First estimate β_k^- by forward iteration

Initialization: $\beta_0^- = \mathcal{X}_0^-$

for $k = 1$ to h, $\beta_k^- = \mathcal{A}^- \otimes (w_{k-1})^* \otimes \beta_{k-1}^-$,

c) Trajectory $\mathcal{X}_k^-$ by backward iteration

Initialization: $\mathcal{X}_h^- = (w_h)^* \otimes \beta_h^-$

for $k = h-1$ to 0, $\mathcal{X}_k^- = (w_k)^* \otimes [\mathcal{A}^+ \otimes \mathcal{X}_{k+1}^- \oplus \beta_k^-]$

Proof Theorem 1 shows that the smallest solution satisfying $\mathcal{X} \geq (\gamma^0 \mathcal{A}^= \oplus \gamma^1.\mathcal{A}^- \oplus \gamma^{-1}.\mathcal{A}^+) \otimes \mathcal{X}$ with $\mathcal{X}(0) \geq \mathcal{X}_0^-$ also satisfies the corresponding equality. These can be written by the following equations.

$$\begin{cases} \mathcal{X}(0) = \mathcal{A}^= \otimes \mathcal{X}(0) \oplus \mathcal{A}^+ \otimes \mathcal{X}(1) \oplus \mathcal{X}_0^- \\ \mathcal{X}(k) = \mathcal{A}^= \otimes \mathcal{X}(k) \ \oplus \mathcal{A}^- \otimes \mathcal{X}(k-1) \oplus \\ \mathcal{A}^+ \otimes \mathcal{X}(k+1) \text{ for } k = 1 \text{ to } h-1 \\ \mathcal{X}(h) = \mathcal{A}^= \otimes \mathcal{X}(h) \oplus \mathcal{A}^- \otimes \mathcal{X}(h-1) \end{cases} \tag{13}$$

The following proposition $\mathcal{P}(k)$ is now proved by recursion.

$\mathcal{P}(k)$: $\mathcal{X}^-(k) = (w_k)^* \otimes [\mathcal{A}^+ \otimes \mathcal{X}^-(k+1) \oplus \beta_k^-]$

Base case: $\mathcal{P}(0)$

From the first equality of (13), we can write

$\mathcal{X}(0) = w_0 \otimes \mathcal{X}(0) \oplus \mathcal{A}^+ \otimes \mathcal{X}(1) \oplus \beta_0^-$ where $w_0 = \mathcal{A}^=$ and $\beta_0^- = \mathcal{X}_0^-$. Therefore, $\mathcal{X}(0) = (w_0)^*[\mathcal{A}^+ \otimes \mathcal{X}(1) \oplus \beta_0^-]$, which proves $\mathcal{P}(0)$.

Case: $\mathcal{P}(1)$

From the second equality of (13), we can write for $k = 1$

$\mathcal{X}(1) = \mathcal{A}^{=} \otimes \mathcal{X}(1) \oplus \mathcal{A}^{-} \otimes \mathcal{X}(0) \oplus \mathcal{A}^{+} \otimes \mathcal{X}(2)$

If $\mathcal{P}(0)$ is used,

$\mathcal{X}(1)$=$\mathcal{A}^{=} \otimes \mathcal{X}(1) \oplus \mathcal{A}^{-} \otimes [(w_0)^{*}[\mathcal{A}^{+} \otimes \mathcal{X}(1) \oplus \beta_0^{-}]] \oplus \mathcal{A}^{+} \otimes \mathcal{X}(2)$

The distributivity of $\otimes$ with respect to $\oplus$ leads to

$\mathcal{X}(1) = [\mathcal{A}^{=} \oplus \mathcal{A}^{-} \otimes (w_0)^{*} \otimes \mathcal{A}^{+}] \otimes \mathcal{X}(1) \oplus \mathcal{A}^{-} \otimes (w_0)^{*} \otimes \beta_0^{-} \oplus \mathcal{A}^{+} \otimes \mathcal{X}(2) =$

$w_1 \otimes \mathcal{X}(1) \oplus \beta_1^{-} \oplus \mathcal{A}^{+} \otimes \mathcal{X}(2)$ where $w_1 = \mathcal{A}^{=} \oplus \mathcal{A}^{-}(w_0)^{*}\mathcal{A}^{+}$ and $\beta_1^{-} = \mathcal{A}^{-}(w_0)^{*} \otimes \beta_0^{-}$. Therefore,

$\mathcal{X}(1) = (w_1)^{*} \otimes [\mathcal{A}^{+} \otimes \mathcal{X}(2) \oplus \beta_1^{-}]$ and $\mathcal{P}(1)$ is proved. Now, this approach is generalized for $k = 1$ to $h - 1$.

Case: $\mathcal{P}(k)$ for k from 1 to $h - 1$.

Let us assume $\mathcal{P}(k-1)$: $\mathcal{X}(k-1) = (w_{k-1})^{*} \otimes [\mathcal{A}^{+} \otimes \mathcal{X}(k) \oplus \beta_{k-1}^{-}]$. We will prove that $\mathcal{P}(k-1)$ entails $\mathcal{P}(k)$.

From the second equality of (13), we can write

$\mathcal{X}(k) = \mathcal{A}^{=} \otimes \mathcal{X}(k) \oplus \mathcal{A}^{-} \otimes \mathcal{X}(k-1) \oplus \mathcal{A}^{+} \otimes \mathcal{X}(k+1)$

As $\mathcal{X}(k-1) = (w_{k-1})^{*} \otimes [\mathcal{A}^{+} \otimes \mathcal{X}(k) \oplus \beta_{k-1}^{-}]$, the expression below is deduced:

$\mathcal{X}(k) = \mathcal{A}^{=} \otimes \mathcal{X}(k) \oplus \mathcal{A}^{-} \otimes (w_{k-1})^{*} \otimes [\mathcal{A}^{+} \otimes \mathcal{X}(k) \oplus \beta_{k-1}^{-}] \oplus \mathcal{A}^{+} \otimes \mathcal{X}(k+1)$

The distributivity of $\otimes$ with respect to $\oplus$ yields

$\mathcal{X}(k) = [\mathcal{A}^{=} \oplus \mathcal{A}^{-} \otimes (w_{k-1})^{*} \otimes \mathcal{A}^{+}] \otimes \mathcal{X}(k) \oplus \mathcal{A}^{-} \otimes (w_{k-1})^{*} \otimes \beta_{k-1}^{-} \oplus \mathcal{A}^{+} \otimes \mathcal{X}(k+1) =$

$w_k \otimes \mathcal{X}(k) \oplus \beta_k^{-} \oplus \mathcal{A}^{+} \otimes \mathcal{X}(k+1)$, where $w_k = \mathcal{A}^{=} \oplus \mathcal{A}^{-} \otimes (w_{k-1})^{*} \otimes \mathcal{A}^{+}$and $\beta_k^{-} = \mathcal{A}^{-} \otimes (w_{k-1})^{*} \otimes \beta_{k-1}^{-}$

Therefore, $\mathcal{X}(k) = (w_k)^*[\mathcal{A}^+ \otimes \mathcal{X}(k+1) \oplus \beta_k^-]$ and the desired expression is obtained: $\mathcal{P}(k)$ has been deduced from $\mathcal{P}(k-1)$. Moreover, as $\mathcal{P}(0)$ is true, $\mathcal{P}(k)$ has been proved for k from 1 to $h-1$: the recursion is finished. Knowing β_k^-, the calculation of $\mathcal{X}(k)$ uses a backward iteration, while the calculation of β_k^- is relevant to a forward iteration.

Now, the final case will be proved.

Case: $\mathcal{P}(h)$

The last equality of (13) can be considered like the second equality but without $\mathcal{A}^+ \otimes \mathcal{X}(k+1)$: the argument of case $\mathcal{P}(k)$ can be taken and we can write

$\mathcal{X}(h) = (w_h)^* \otimes \beta_h^-$ with $w_h = \mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_{h-1})^* \otimes \mathcal{A}^+$and $\beta_h^- = \mathcal{A}^- \otimes (w_{h-1})^* \otimes \beta_{h-1}^-$

Finally, as matrices w_k have no positive circuit and $\mathcal{X}_0^-$ belongs to $(\mathbb{R} \cup \{-\infty\})^n$, the state trajectory is defined in $\mathbb{R} \cup \{-\infty\}$. ■

B. Greatest state trajectory

The following theorem can be deduced from the previous one by duality. Steps a) are identical.

***Theorem** 4:* If the process operates on horizon $h \in \mathbb{N}$ and if matrices w_k defined below have no positive circuit, the greatest state trajectory in $\mathbb{R} \cup \{+\infty\}$ satisfying $\mathcal{X}(0) \leq \mathcal{X}_0^+ \in (\mathbb{R} \cup \{+\infty\})^n$ is given by the following forward/backward algorithm.

Forward/backward algorithm

a) Coefficients of w_k by forward iteration

Initialization: $w_0 = \mathcal{A}^=$

for $k = 1$ to h, $\quad w_k = \mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_{k-1})^* \otimes \mathcal{A}^+$

b) First estimate β_k^+ by forward iteration

Initialization: $\beta_0^+ = \mathcal{X}_0^+$

for $k = 1$ to h, $\quad \beta_k^+ = ((w_{k-1})^* \otimes \mathcal{A}^+) \backslash \beta_{k-1}^+$

c) Trajectory $\mathcal{X}_k^+$ by backward iteration

$\mathcal{X}_h^+ = (w_h)^* \backslash \beta_h^+$

for $k = h-1$ to 0, $\mathcal{X}_k^+ = (w_k)^* \backslash [\mathcal{A}^- \backslash \mathcal{X}_{k+1}^+ \wedge \beta_k^+]$

Proof The proof is omitted as it can be deduced by duality from the previous theorem. ■

To sum up, the two algorithms allow the determination of the lowest (respectively, greatest) acceptable trajectories satisfying $\mathcal{X}(0) \geq \mathcal{X}_0^-$ (respectively, $\mathcal{X}(0) \leq \mathcal{X}_0^+$). Also they allow the checking of the existence of a trajectory satisfying $\mathcal{X}(0) \in [\mathcal{X}_0^-, \mathcal{X}_0^+]$ if constraints $\mathcal{X}(0) \leq \mathcal{X}_0^+$ and $\mathcal{X}(0) \geq \mathcal{X}_0^-$ are respectively added in the corresponding algorithms.

***Remark** 1:* Defined on a box $[\mathcal{X}_0^-, \mathcal{X}_0^+]$, the initial condition is less restrictive than the more usual $\mathcal{X}(0) = \mathcal{X}_0$ which is a particular case. In a natural way, checking this case is made as follows. The determination of the lowest trajectory such as $\mathcal{X}(0) \in [\mathcal{X}_0, \mathcal{X}_0^+]$ with $\mathcal{X}_0 \leq \mathcal{X}_0^+$, allows checking the acceptability of $\mathcal{X}_0$ or in other words, if there is a solution $\mathcal{X}$ so that $\mathcal{X}(0) = \mathcal{X}_0$. Similarly, the determination of the greatest trajectory such as $\mathcal{X}(0) \in [\mathcal{X}_0^-,\ \mathcal{X}_0]$ with $\mathcal{X}_0^- \leq \mathcal{X}_0$ also allows checking the existence of a solution $\mathcal{X}$ so that $\mathcal{X}(0) = \ \mathcal{X}_0$.

***Remark** 2:* The calculation of the state trajectories starts from values $(w_h)^* \otimes \beta_h^-$ and $(w_h)^* \backslash \beta_h^+$ and consequently depends on horizon h. The calculation of w_k, β_k^- and β_0^+ depends on index k, but not on horizon h.

VII. CONSISTENCY

In this paper, an acceptable behavior of the considered P-time Event Graph is defined by any operation guaranteing the liveness of tokens. Therefore, it does not lead to any deadlock situation. As this behavior is represented by the algebraic model (12), the aim of this part is to study the existence of a state trajectory solution to these inequalities. Clearly, if we can calculate an arbitrary trajectory starting from box $[\mathcal{X}_0^-, \mathcal{X}_0^+]$, we can deduce that the system is consistent on the considered horizon. We introduce the following notation.

***Definition** 1:* A dynamic associated graph of square matrices A, B and C on horizon h, denoted by $G_h(A, B, C)$, is deduced from these matrices by associating for $k = 0$ to h, a node j_k with column j and a node i_k with row i . The pattern is as follows: a) An arc from node j_{k-1} towards node i_k if $A_{ij} \neq \varepsilon$; b) An arc from node j_k towards node i_k if $B_{ij} \neq \varepsilon$; c) An arc from node j_k towards node i_{k-1} if $C_{ij} \neq \varepsilon$.

In this paper, we consider $G_h(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$. An example is given in figure 4. The dimension of each column k is dimension n of the state. As in the static case presented in section IV, the system is consistent if the dynamic associated graph $G_h(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$ has no circuit with positive weight. These circuits can simply be situated in the associated graph of the static part ($\mathcal{A}^=$) or of the dynamic part ($\mathcal{A}^-$ and $\mathcal{A}^+$). Figure 5 shows that the circuits can present a complex form.

Assuming the liveness of the Event Graph, the following theorem considers the temporal consistency of P-time Event graphs. This theorem is about the existence of a state trajectory in $\mathbb{R}$, and not in $\mathbb{R}_{\max}$.

***Theorem** 5:* A live P-time Event Graph is consistent on arbitrary horizon h if and only if the dynamic associated graph $G_h(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$ contains circuits with only non-positive weight.

Proof The model can completely be represented by system (13) after replacing symbol $=$ with $\geq$. This system can be rewritten in $\mathbb{R}_{\max}$ under the global form $x \geq A \otimes x \oplus B$ which includes every inequality. The relevant dynamic associated graph is $G_h(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$. As the least solution is A^*B, this system has at least a solution in $\mathbb{R}_{\max}$ if the global matrix A has no strictly positive circuits. This gives a sufficient condition for the existence of a solution in $\mathbb{R}_{\max}$.

Moreover, $\mathcal{X}_0^-$ can be taken finite: as $\mathcal{X}(0) \geq \mathcal{X}_0^-$ and as the trajectory is nondecreasing by construction of $\mathcal{A}^-$, each component of the state trajectory is different from ε and the trajectory belongs to $\mathbb{R}$. Conversely, if finite $\mathcal{X}$ satisfies the model of the P-time Event Graph, it can only satisfy subsystems with non-positive circuits. ■

The following result is immediate.

***Corollary** 2:* A live P-time Event Graph with a null initial marking, is consistent if and only if the dynamic associated graph of matrix $\mathcal{A}^=$ contains circuits with only non-positive weight.

Remark. A live P-time Event Graph whose initial marking is null is without circuit (in the Event Graph), but the dynamic associated graph of matrix $\mathcal{A}^=$ can have circuits.

Now, we consider matrices w_k which allow a characterization of the circuits. The following property gives a graphical interpretation of the calculation of these matrices.

***Property** 1:* Entry $((w_h)^*)_{i_h,j_h}$ represents the maximum weight of all the paths from vertices j_h to vertices i_h for i, $j \in [1..n]$ in the dynamic associated graph $G_h(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$ except the paths containing an arc from index 0 node to index 0 node.

Proof

Let us consider the relations inside horizon [0, 1]. So, $\mathcal{X}(1)_i \geq (\mathcal{A}^-)_{i,l} \otimes \mathcal{X}(0)_l \geq (\mathcal{A}^-)_{i,l} \otimes (\mathcal{A}^+)_{l,j}\mathcal{X}(1)_j$ but also, $\mathcal{X}(1)_i \geq (\mathcal{A}^=)_{i,j}\mathcal{X}(1)_j$. So, $(w_1)_{i,j} = (\mathcal{A}^= \oplus \mathcal{A}^- \otimes \mathcal{A}^+)_{i,j}$ represents the greatest weight on the following paths:

- an arc ($j_1 \rightarrow i_1$) (matrix $\mathcal{A}^=$)

- or two successive arcs ($j_1 \rightarrow l_0$) and ($l_0 \rightarrow i_1$) (product $\mathcal{A}^- \otimes \mathcal{A}^+$).

Expression $((w_1)^*)_{i,j}$ represents the greatest weight on the following paths (by default, the weight is zero if there is no path between two vertices) going successively through,

- nodes of indexes j_1 to i_1 and again (expressed by $(\mathcal{A}^=)^*$),

- or nodes of indexes j_1 to l_0 and l_0 to i_1 and again (expressed by $(\mathcal{A}^- \otimes \mathcal{A}^+)^*$),

- or nodes of indexes j_1 to l_1, then l_1 to k_0 and k_0 to i_1 (expressed by $(\mathcal{A}^- \otimes \mathcal{A}^+)(\mathcal{A}^=)$),

, and so on.

Remark: as there is no term as $\mathcal{A}^- \otimes \mathcal{A}^= \otimes \mathcal{A}^+$, $(w_1)^*$ is not the result of paths containing an arc from node of index j_0 to node of index i_0 directly.

For horizon [0, 2], $(w_2)_{ij} = (\mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_1)^* \otimes \mathcal{A}^+)_{ij}$ represents the greatest weight on the following paths: :

- an arc $(j_2 \rightarrow i_2)$ (matrix $\mathcal{A}^=$)

- or, an arc $(j_2 \rightarrow l_1)$ (matrix $\mathcal{A}^+$), the previous paths from l_1 to m_1 expressed by matrix $(w_1)^*$ (described above) and an arc $(l_1 \rightarrow i_2)$ (matrix $\mathcal{A}^-$).

Consequently, $(w_2)^*$ represents the greatest weight of every path (and circuit) of the associated graph on horizon [0, 2] and defined by a path from i_2 to i_2 and going possibly to nodes of indexes 1 and 0, except the paths containing an arc from nodes of indexes 0 to 0.

The argument can be repeated until $k = h$. ■

The following Theorem improves Theorem 5 by giving a practical way to check the consistency.

***Theorem** 6:* A live P-time Event Graph is consistent if and only if the associated graph of each matrix w_k for any $k \geq 0$ contains circuits with only non-positive weight.

Proof. Property 1 says that matrices $(w_k)^*$ represent the greatest weight of almost every path and circuit in $G_h(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$. The weights of circuits which are not ”present” in $(w_k)^*$ are ”present” in $(w_{k+1})^*$ for $G_{h+1}(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$ as the associated graph is the repetition of a pattern. Consequently, each circuit is expressed and the proof can be deduced from Theorem 5. ■

Therefore, if there is an index k_1 such that an entry $((w_k)^*)_{i,j}$ is infinite, we can conclude that there is a path from j to i, containing a circuit with a positive weight in $G_{k_1}(\mathcal{A}^-, \mathcal{A}^=, \mathcal{A}^+)$. So, the system is not consistent on horizon greater than $h \geq k_1$. An example of circuit with positive circuit is given in figure 5.

The following result will facilitate the analysis of the consistency and its checking.

***Property** 2:* $w_k \geq w_{k-1}$ for $k \geq 1$

Proof

Let us suppose that $w_{k-1} \geq w_{k-2}$, $w_k = \mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_{k-1})^* \otimes \mathcal{A}^+ =$

$\mathcal{A}^= \oplus \mathcal{A}^- \otimes \mathcal{A}^+ \oplus \mathcal{A}^- \otimes (w_{k-1}) \otimes \mathcal{A}^+ \oplus \mathcal{A}^- \otimes (w_{k-1})^2 \otimes \mathcal{A}^+ \oplus ...$

As $w_{k-1} \geq w_{k-2}$ and by isotony of the product, $w_k \geq \mathcal{A}^= \oplus \mathcal{A}^- \otimes \mathcal{A}^+ \oplus \mathcal{A}^- \otimes (w_{k-2}) \otimes \mathcal{A}^+ \oplus \mathcal{A}^- \otimes (w_{k-2})^2 \otimes \mathcal{A}^+ \oplus ... = w_{k-1}$

Moreover, $w_1 \geq w_0 = \mathcal{A}^=$ as $w_1 = \mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_0)^* \otimes \mathcal{A}^+$. So, the series $w_0 = \mathcal{A}^=$ and $w_k = \mathcal{A}^= \oplus \mathcal{A}^- \otimes (w_{k-1})^* \otimes \mathcal{A}^+$ for $k \geq 1$ is nondecreasing. ■

Suppose that there is an index k_1 such that matrix w_k does not increase ($w_{k_1+1} = w_{k_1}$) with w_k belonging to $\mathbb{R}_{\max}$. From Property 2, we can conclude that no matrix w_k has positive circuit for any index k. Consequently, the P-time Event graph is consistent on an infinite horizon. In this case, the tests show that the convergence of consistent P-time Event Graphs is fast.

VIII. Examples

The following example illustrates the results about consistency and extremal trajectories. Computation tests are made using the max-plus toolbox in Scilab.

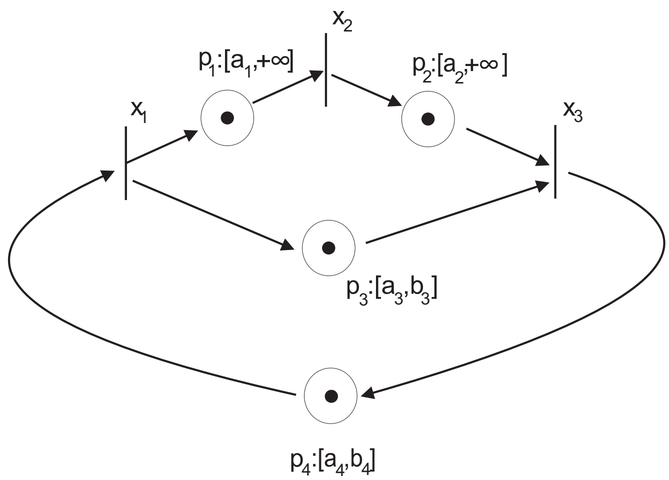

Fig. 3. P-time Event graph

A. Model

A slight modification of the example in figure 1 gives the closed-loop structure of figure 3 which describes a limitation of resources. An upper bound on packing (b_4) is added. The initial marking is $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t$

$a_1 = 3$, $a_2 = 3$, $a_3 = 1$, $a_4 = 3$, $b_3 = 2$, $b_4 = 11$. Therefore,

$$A_0^- = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} A_0^+ = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

$\mathbb{A}_0^= = A_0^- \oplus A_0^+ = \varepsilon$

$$A_1^- = \begin{pmatrix} \varepsilon & \varepsilon & 3 \\ 3 & \varepsilon & \varepsilon \\ 1 & 3 & \varepsilon \end{pmatrix} A_1^+ = \begin{pmatrix} \varepsilon & \varepsilon & -2 \\ \varepsilon & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon \end{pmatrix}$$

Matrices $\mathcal{A}^=$, $\mathcal{A}^-$ and $\mathcal{A}^+$ are now deduced.

$\mathcal{A}^= = \mathbb{A}_0^= = A_0^- \oplus A_0^+ = \varepsilon$, $\mathcal{A}^- = \mathbb{A}_1^- = (\mathbb{A}_0^=)^* \otimes A_1^- = A_1^-$ and $\mathcal{A}^+ = \mathbb{A}_1^+ = (\mathbb{A}_0^=)^* \otimes A_1^+ = A_1^+$ The relevant associated graph is in figure 4.

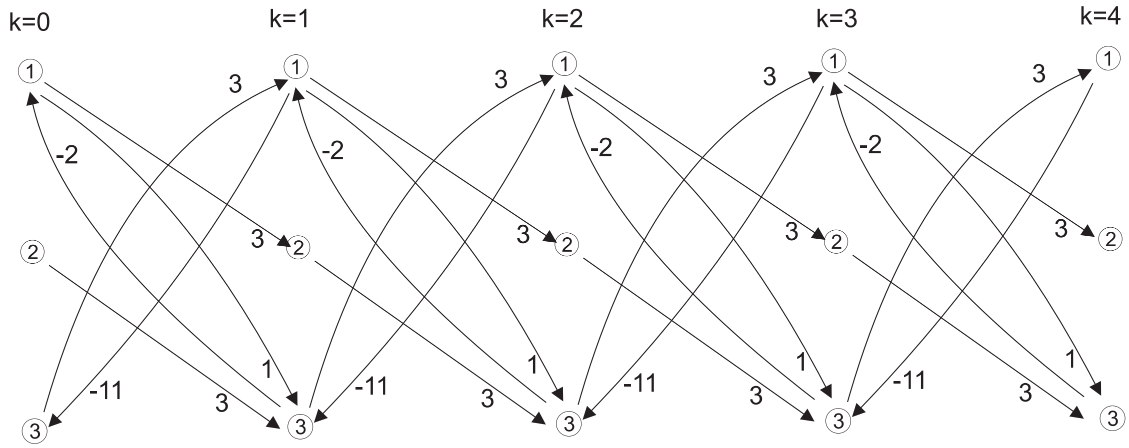

Fig. 4. Associated graph in consistent case

B. Series

$w_0 = \mathcal{A}^{=} = \varepsilon$

$$w_1 = \begin{pmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & -1 \end{pmatrix}, w_2 = \begin{pmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ -7 & \varepsilon & -1 \end{pmatrix}, w_3 = \begin{pmatrix} -8 & \varepsilon & -6 \\ \varepsilon & \varepsilon & 1 \\ -7 & \varepsilon & -1 \end{pmatrix}, w_4 = \begin{pmatrix} -8 & \varepsilon & -6 \\ -14 & \varepsilon & 1 \\ -7 & \varepsilon & -1 \end{pmatrix}$$

The calculation of matrices w_k shows that they are constant after a short transitory period ($w_k = w_4$ for $k \geq 5$) and that they have no positive circuit. Consequently, the system is consistent on an arbitrary horizon.

Now, assume that a failure appears in the moving of the packet whose normal duration a_2 associated with P_2 is equal to 3, which corresponds to $(A_1^-)_{3,2}$: the duration is now equal to 6. The relevant matrices w_k are as follows. We also give w_3^*.

$w_0 = \mathcal{A}^{=} = \varepsilon$

$$w_1 = \begin{pmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & -1 \end{pmatrix}, w_2 = \begin{pmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ -4 & \varepsilon & -1 \end{pmatrix}, w_3 = \begin{pmatrix} -8 & \varepsilon & -3 \\ \varepsilon & \varepsilon & 1 \\ -4 & \varepsilon & 1 \end{pmatrix}, w_3^* = \begin{pmatrix} T & \varepsilon & T \\ T & 0 & T \\ T & \varepsilon & T \end{pmatrix}$$

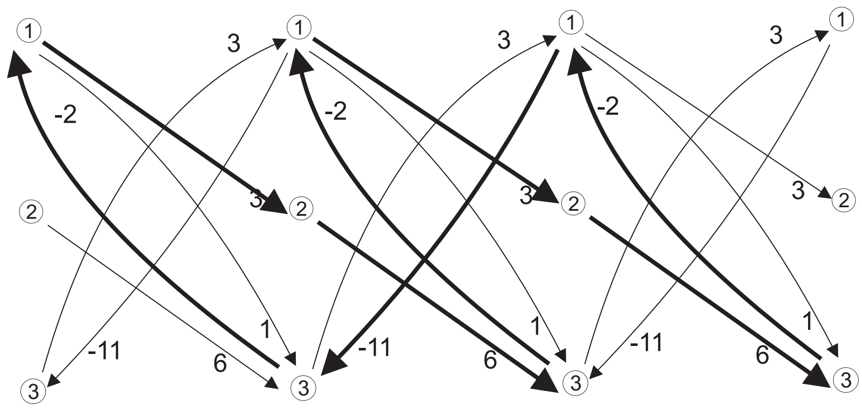

Fig. 5. Circuit with positive weight in inconsistent case

$$, w_4 = \begin{pmatrix} T & \varepsilon & T \\ T & \varepsilon & T \\ T & \varepsilon & T \end{pmatrix}$$

As some coefficients of w_3^* and also, w_4 are equal to $T = +\infty$, the system is not consistent. Therefore, even if the underlying graph of the Event Graph is live in the usual sense, it is not consistent for $a_2 = 6$. So, as $(w_3)_{3,3} = 1$, $\chi_3^-(k) \geq 1 \otimes \chi_3^-(k)$ which is inconsistent. This incoherence comes from the following inequalities. Figure 5 shows the relevant circuit with positive weight.

$\chi_3^-(k) \geq 6 \otimes \chi_2^-(k-1)$

$\chi_2^-(k-1) \geq 3 \otimes \chi_1^-(k-2)$

$\chi_1^-(k-2) \geq (-2) \otimes \chi_3^-(k-1)$

$\chi_3^-(k-1) \geq 6 \otimes \chi_2^-(k-2)$

$\chi_2^-(k-2) \geq 3 \otimes \chi_1^-(k-3)$

$\chi_1^-(k-3) \geq (-2) \otimes \chi_3^-(k-2)$

$\chi_3^-(k-2) \geq (-11) \otimes \chi_1^-(k-1)$

$\chi_1^-(k-1) \geq (-2) \otimes \chi_3^-(k)$

The trials show that the tolerance margin of a_2 where the system is consistent, is $[0, 5.5]$.

C. Lowest and greatest state trajectories

Now, we apply the algorithms of Theorems 3 and 4 which provide lowest and greatest state trajectories, χ^+ and χ^+ respectively. The arbitrary initial conditions are $\chi_0^- = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t$ and $\chi_0^+ = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t$. In step b), intermediate values β^- and β^+, which are given in the following tables, are deduced from matrices w_k and initial conditions χ_0^- and χ_0^+ by a forward approach. They give a first estimate of the lowest and greatest trajectories.

k	0	1	2	3	4	5	6	7	8	9	10
β_1^-	1	3	6	10	14	18	22	26	30	34	38
β_2^-	0	4	6	9	13	17	21	25	29	33	37
β_3^-	0	3	7	11	15	19	23	27	31	35	39

k	0	1	2	3	4	5	6	7	8	9	10
β_1^+	1	11	14	24	27	36	40	49	53	62	66
β_2^+	0	T	T	T	T	T	T	T	T	T	T
β_3^+	0	3	13	16	25	29	38	42	51	55	64

Intermediate values β^- and β^+ are now improved by the backward step c). The following tables contain lowest and greatest state trajectories, χ^- and χ^+ respectively.

k	0	1	2	3	4	5	6	7	8	9	10
$\mathcal{X}_1^-$	1	5	9	13	17	21	25	29	33	37	38
$\mathcal{X}_2^-$	0	4	8	12	16	20	24	28	32	36	40
$\mathcal{X}_3^-$	0	3	7	11	15	19	23	27	31	35	39

k	0	1	2	3	4	5	6	7	8	9	10
$\mathcal{X}_1^+$	1	10	14	23	27	36	40	49	53	62	66
$\mathcal{X}_2^+$	0	9	13	22	26	35	39	48	52	61	T
$\mathcal{X}_3^+$	0	3	12	16	25	29	38	42	51	55	64

The following table is the state trajectory of the Timed Event Graph using the lower bound of the temporisations of the P-time Event Graph (see figure 3). With the assumption of earliest behavior,

$x(k) = A \otimes x(k)$ with $A = A_1^-$. As $x(0) = \chi_0^- = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t$, a comparison can be made.

k	0	1	2	3	4	5	6	7	8	9	10
x_1	1	3	6	10	12	15	19	21	24	28	30
x_2	0	4	6	9	13	15	18	22	24	27	31
x_3	0	3	7	9	12	16	18	21	25	27	30

Figure 6 shows the trajectories of transition x_1: The lowest and greatest trajectories for the P-time Event Graph ; The trajectory of the relevant Timed Event Graph. The three trajectories are clearly different ($x(k) \leq \chi^-(k) \leq \chi^+(k)$) as their rates (the calculation of the different cycle times is made in [10]).

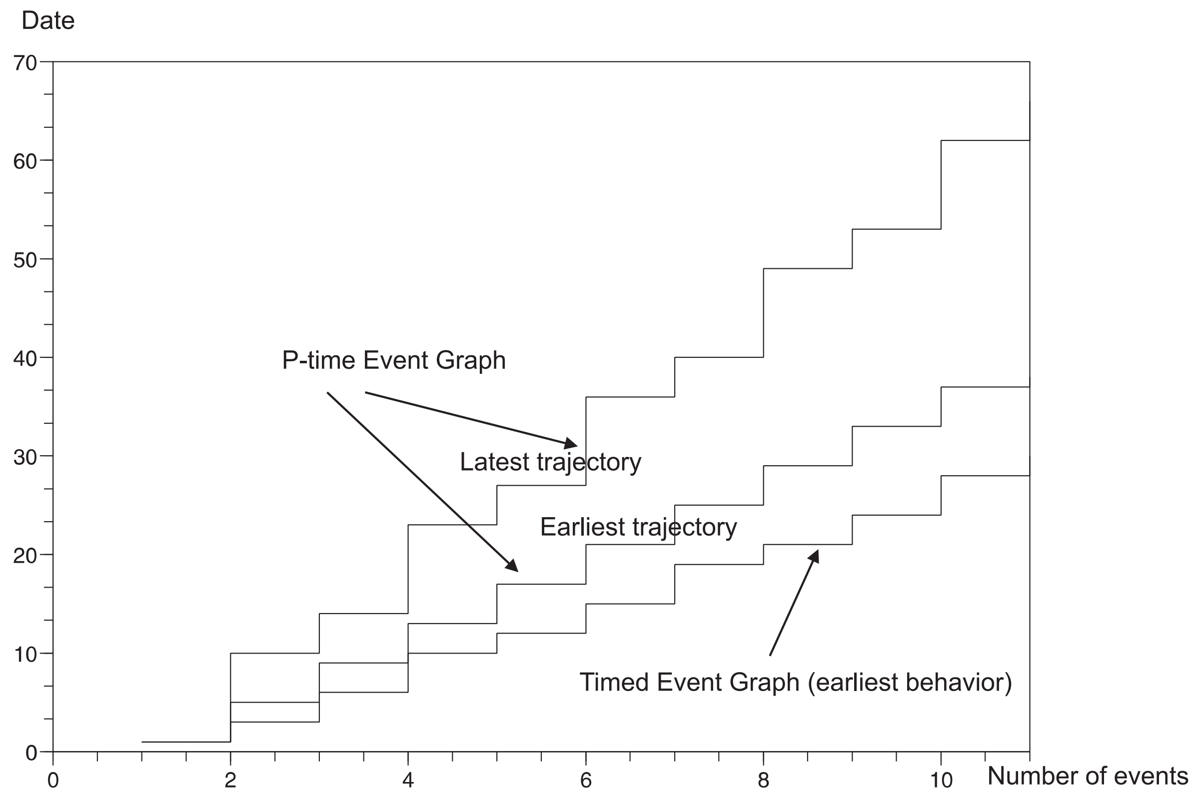

Fig. 6. Trajectories

Remark. It is important to note that each extremal trajectory depends on the lower and upper bounds of the temporisations and not only, on one of those. The calculation of the minimal trajectory naturally requires not only the inequalities corresponding to Timed Event Graphs (5) but also the upper constraints (6): the example in figure 6 shows that the earliest functioning of a Timed Event Graph using the relevant equality of (5) does not satisfy the inequalities of the P-time Event Graph in the autonomous case. This fact entails that the trajectories of P-time Event Graphs cannot always be deduced by a direct forward iteration like in the state equation in Timed Event Graphs. Note that in this example, the P-time Event Graph is consistent.

IX. Computational complexity

The following curve gives indications on the possible CPU times needed to compute the different matrices w_k, and the lowest and greatest trajectories on an ordinary $Pentium$ 1.3 GHz for a horizon h= 100. Computation tests are made using maxplus toolboxes under Scilab. The matrices $\mathcal{A}^-$ and $\mathcal{A}^+$ are completely full: there is a place containing a token between each couple of transitions. For instance, if $n = 50$, the relevant Petri net contains 50 transitions and 2500 places. The matrix $\mathcal{A}^-$ is generated randomly and $\mathcal{A}^+$ is deduced from $\mathcal{A}^-$ such that the system is temporally live on the desired horizon: the complete calculations are made, therefore. In that objective, we also take $\mathcal{A}^= = \varepsilon$ which do not effect significantly the time. At the moment, the code is not completely optimized and contains redundant operations.

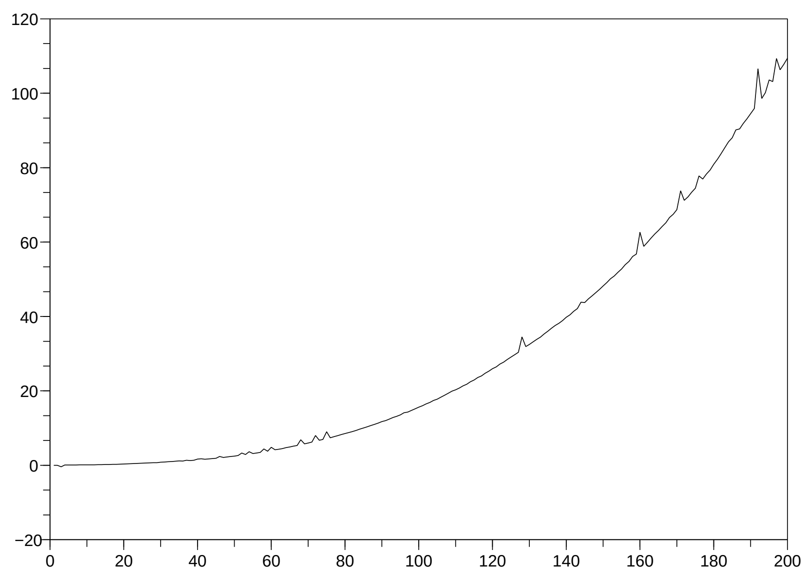

Fig. 7. CPU time for different dimensions from 3 to 200 and h=1000

The algorithms use elementary operations on matrices as $\otimes$, $\oplus$, $\backslash$, $\wedge$ and the more complex operation Klenne Star $*$. The last one determines the computational complexity of each step and the complexities of the different known algorithms are polynomial. Therefore, the complexity of calculation of the greatest trajectory is about $O(h.n^2)$ with h the horizon and n the dimension of the matrices. The space needed for the matrices w_k is $l.n^2$ with l the minimum of the horizon h and the length of the transient period. In short, the algorithm can consider important sizes of Event Graphs and horizon of calculation. Future papers will also consider sparse matrices.

X. Conclusion

P-time Event Graphs express a temporal behavior defined by lower and upper limits. This paper shows that they can be modeled using a model in (max, +) algebra. The complexity of the state equation resolution in P-time Event Graph depends on the paths and circuits of the associated graph generated by matrices $\mathcal{A}^{=}$, $\mathcal{A}^{-}$ and $\mathcal{A}^{+}$.

The introduction of a nondecreasing series of matrices enables the determination of the extremal state trajectories satisfying an initial condition defined on an interval. The circuit weights of these matrices define natural conditions of existence. A possible convergence to a constant matrix after a transitory period, facilitates the analysis. The resolution is based on a unique forward/backward iteration, requiring the calculation of Kleene star; consequently, the calculation time is polynomial for a given horizon. As the size of the matrices corresponds to the size of the forward/backward model, which depends on the number of transitions and on the initial marking, these series give an efficient way to calculate the extremal trajectories and to solve the consistency problem.

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