Optimal control synthesis of timed event graphs with interval model specifications

Philippe Declerck and Mohamed Khalid Didi Alaoui

Abstract

The purpose of this paper is the optimal control synthesis of a Timed Event Graph when the state and control trajectories should follow the specifications defined by an interval model. The problem is reformulated in the fixed point form and the spectral theory gives the conditions of existence of a solution.

Index Terms

Timed Event Graphs, P-time Petri nets, Time Stream Petri nets, (min, max, +) functions, cycle-time vector, fixed point.

I. Introduction

In [7] and [6], we have shown that P-time Event Graphs and Time Stream Event Graphs can be modeled by a new class of systems called interval systems, for which the time evolution is not strictly deterministic, but is described by intervals which use the operations of maximization, minimization and addition to define the lower and upper bound constraints. The consistency of P-time Event Graphs can also be studied in tropical algebra using the spectral vector [6]. In this paper, we focus on the following problem.

Some events are stated as controllable, meaning that the firing of the corresponding transitions (input) may be delayed until some arbitrary time provided by a supervisor. Considering a desired behavior of some transitions (output) of the Timed Event Graph such as a sequence of execution

P. Declerck and M.K Didi Alaoui are with LISA EA4014, University of Angers, 62 avenue Notre-Dame du Lac, 49000 Angers, France (tel: +33(0)2.41.22.65.60; fax: +33(0)2.41.22.65.61; e-mail: philippe.declerck@univ-angers.fr; mohamed.didi.alaoui@univ-angers.fr)

times (desired output), we wish to slow down the system as much as possible ensuring all output events to occur before their desired execution time.

Moreover, we assume that the Timed Event Graph must follow specifications defined by a second model. Expressed by an interval model, this desired behavior is expressed by inequalities which introduce lower and upper bounds on the dates of firing of all the transitions. As the specifications can be incoherent or too restrictive with respect to the Timed Event Graph, a first problem is to determine whether there exist control actions which will restrict the system to that behavior. If the trajectories of the Timed Event Graph can follow additional specifications, a second objective is to determine the greatest input in order to obtain the desired behavior defined by the static constraints (the desired output) and the dynamic constraints (expressed by an interval model).

In this field, a first class of approaches analyzes the state space and develops controllers in order to keep trajectories inside a space deduced from a given specification. The aim is the extension of the concept of (A,B)-invariant subspace to linear systems over the max-plus semiring. The computation of the maximal set of the initial states is analyzed in [2] [11]. Another group of methods [13] [7] [6] considers extremal points of the state space and develops optimal control in order to keep trajectories close to a reference trajectory following additional constraints. The aim is the extension of the principle of the well-known model predictive control.

Let us point out two main differences between these two classes of methods. In feedback approaches, the technique is based on the addition of new structures such as loops which modifies the initial Timed Event Graph. This technique can improve the boundedness of the Petri net but also reduce its production rate and modify its liveness. In predictive approaches, the Petri Net and all its characteristics are kept as we can assume that a preparation composed of scheduling, resources optimisation,... has established an optimized model. Another difference is that predictive approaches can be applied to a large class of models. Particularly, approaches based on a feedback defined by a Petri net are limited by the condition that the temporisation and initial marking of each added place are non-negative. The existence of a linear state feedback is discussed in [11] (see part V and in particular example 4): this problem is reminiscent of difficulties of the theory of linear dynamical systems over rings [10].

In [7] and [6], the control synthesis has been considered by the authors for P-time Event Graphs and Time Stream Event Graphs. Extending these studies, the present paper proposes the control synthesis of a Timed Event Graph (plant) following an interval system (specifications) which can express a set of Event Graphs. Based on a fixed point technique, the main advantage of the approach is that it allows adding (min, max, +) lower and upper bound constraints on all the transitions. It contains new material including a new fixed-point algorithm and an everyday example.

In this paper, the usual hypothesis that places of the Event Graphs should be First In First Out (FIFO) is taken. No hypothesis is taken on the structure of the Event Graphs, which does not need to be strongly connected. No closed-loop structure of control is given a priori. The initial marking should only satisfy the classical liveness condition.

In the next part, we present the optimal control problem and we analyze the existence of a finite solution through spectral theory. This section is followed by an algorithm which determines the greatest control. The notations and a brief review of preliminary results are presented in the Appendix.

II. Control problem

By usual (max,+) algebraic notation, maximization, minimization and addition operations are noted respectively $\oplus$, $\wedge$ and $\otimes$. Variable $x_i(k)$ (respectively, $u_i(k)$) is the date of the k^{th} firing of internal transition x_i (respectively, of input transition u_i). Assuming that the following models are available for $k \geq k_s + 1$ with $k \in \mathbb{Z}$ ('s' for 'start'), we consider the control of an event graph modeled as a (max, +) system

$$x(k) = A_1 \otimes x(k-1) \oplus A_0 \otimes x(k) \oplus B_0 \otimes u(k) \quad , \tag{1}$$

with state x subject to the following interval constraints

$$f^-(x(k-1), x(k), u(k)) \leq x(k) \leq f^+(x(k-1), x(k), u(k)) \quad , \tag{2}$$

and the desired output $y(k) \leq z(k)$ where

$$y(k) = C \otimes x(k) \quad ,$$

is the output of the event graph and $z(k)$ for $k \in [k_s+1, k_f]$ ('f' for 'final') is the desired output. The goal is to find the greatest control $u(k)$ for $k \in [k_s+1, k_f]$.

In this paper, the following usual assumptions on the plant defined by Timed Event Graph (1) are made. The Timed Event Graph is structurally observable and controllable [1]: every

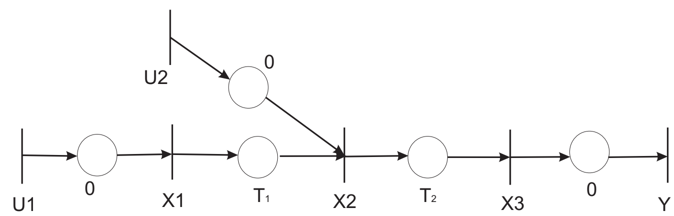

Fig. 1. Timed Event Graph (plant: tasks of the professor)

internal transition can be reached by a path from one input transition at least and, from every internal transition, there exists a path to one output transition at least. The structural observability (respectively controllability) gives a condition to observe an effect in the output (resp. transition) whose origin comes from one internal transition (resp. input) at least. We also suppose that the system works at the earliest time and the Timed Event Graph is defined by equality (1).

Moreover, the following assumption on additional specifications defined by interval model (2) is made: the lower bound f^- is max-only (see appendix) and is defined by $f^-(x(k-1), x(k), u(k)) = A_1^- \otimes x(k-1) \oplus A_0^- \otimes x(k) \oplus B^- \otimes u(k)$. No assumption is made on f^+ which is a (min, max, +) function. Except the form of functions in (2), there is no assumption on the interval model which can describe a set of non-connected Event Graphs.

Example. A simple real-world problem: education system

Plant. Described by the Timed Event Graph in Figure 1, the plant corresponds to the following work of a professor: the lesson is composed of a lecture (duration: T_1) followed by practical work (duration: T_2). Transitions express the following events. x_1: beginning of the lecture; x_2: beginning of practical work; x_3: end of practical work; u_1: decision to start the lecture; u_2: decision to start practical work. From Figure 1, we can deduce the following matrices. $A_0 = \begin{pmatrix} -\infty & -\infty & -\infty \\ T_1 & -\infty & -\infty \\ -\infty & T_2 & -\infty \end{pmatrix}$, $B_0 = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \\ -\infty & -\infty \end{pmatrix}$ and $C = \begin{pmatrix} -\infty & -\infty & 0 \end{pmatrix}$. As the state trajectory is non-decreasing ($x(k) \geq x(k-1)$), $A_1 = I$.

Interval system. Moreover, the teacher must follow the official instructions: a lesson must not exceed T_3^+ and not be less than T_3^-. These specifications can be described by a new Event Graph which can be a simple P-time Event Graph. As $T_3^- \otimes x_1(k) \leq x_3(k) \leq T_3^+ \otimes x_1(k)$, the corresponding interval system is as follows.

$$A_1^- = -\infty,\ A_0^- = \begin{pmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \\ T_3^- & -\infty & -\infty \end{pmatrix},\ B^- = -\infty \text{ and } f^+(x(k-1), x(k), u(k)) = \begin{pmatrix} +\infty \\ +\infty \\ T_3^+ \otimes x_1(k) \end{pmatrix}.$$

Problem. The lesson must stop before the daily closing time of school which corresponds to the desired output z. A problem can be the determination of the latest times to begin the lesson such that each specification is satisfied. If the teacher begins the lesson after this date which corresponds to the greatest control u, the practical work cannot be finished ($y \not\leq z$).

III. Approach

In the sequel, the problem is reformulated as a fixed point problem which describes all the relations between components of state trajectory and control trajectory. A relaxation presented in part III-B allows the analysis of existence of a finite control while the determination of the greatest control $u(k)$ for $k \in [k_s + 1, k_f]$ is made in part III-C. The resolution also proposes a state trajectory $x(k)$ for $k \in [k_s, k_f]$ which can be followed by the system if the corresponding control $u(k)$ is applied.

A. ***Fixed point form***

Any feasible control trajectory $u(k)$ to the control problem is considered in the following theorem.

***Theorem** 1:* Given the desired output $z(k)$, the system composed of models (1), (2) and constraint $y(k) = C \otimes x(k) \leq z(k)$ where $k \in [k_s + 1, k_f]$, is equivalent to the following inequality system:

$$\begin{cases} x(k) \leq C \backslash z(k) \\ \wedge [A_0 \oplus A_0^-] \backslash x(k) \wedge g^+(x(k-1), x(k), u(k)) \\ x(k-1) \leq [A_1 \oplus A_1^-] \backslash x(k) \\ u(k) \leq [B_0 \oplus B^-] \backslash x(k) \end{cases}, \tag{3}$$

where $k \in [k_s+1, k_f]$ and, $g^+(x(k-1), x(k), u(k)) =$

$$[A_1 \otimes x(k-1) \oplus A_0 \otimes x(k) \oplus B_0 \otimes u(k)] \wedge f^+(x(k-1), x(k), u(k)) \ . \tag{4}$$

Proof:

Equality (1) is equivalent to

$$\begin{cases} A_1 \otimes x(k-1) \oplus A_0 \otimes x(k) \oplus B_0 \otimes u(k) \leq x(k) \\ x(k) \leq A_1 \otimes x(k-1) \oplus A_0 \otimes x(k) \oplus B_0 \otimes u(k) \end{cases} , \tag{5}$$

and using the residuation (see Appendix), we obtain

$$\begin{cases} x(k-1) \leq A_1 \backslash x(k) \\ x(k) \leq A_0 \backslash x(k) \wedge [A_1 \otimes x(k-1) \oplus A_0 \otimes x(k) \\ \oplus B_0 \otimes u(k)] \\ u(k) \leq B_0 \backslash x(k) \end{cases} . \tag{6}$$

From (2), we similarly obtain

$$\begin{cases} x(k-1) \leq A_1^- \backslash x(k) \\ x(k) \leq A_0^- \backslash x(k) \wedge f^+(x(k-1), x(k), u(k)) \\ u(k) \leq B^- \backslash x(k) \end{cases} . \tag{7}$$

Finally, system (3) is found after adding the classical relation $x(k) \leq C \backslash z(k)$ which is directly deduced from $C \otimes x(k) \leq z(k)$. ■

The above equivalences are based on well-known properties of residuation (see Appendix).

Note that we can deduce an upper bound on the state by using $x(k) \leq C \backslash z(k)$ knowing $z(k)$. Using this calculated upper bound, we can calculate an upper bound on the control by using the last inequality of (3). Let us develop system (3) algebraically on horizon $[k_s, k_f]$.

$$\begin{cases} x(k_s) \leq [A_1 \oplus A_1^-]\backslash x(k_s+1) \\ x(k) \leq C\backslash z(k) \wedge [A_0 \oplus A_0^-]\backslash x(k) \wedge [A_1 \oplus A_1^-]\backslash x(k+1) \\ \wedge g^+(x(k-1), x(k), u(k)) \text{ for } k \in [k_s+1, k_f-1] \\ u(k) \leq [B_0 \oplus B^-]\backslash x(k) \text{ for } k \in [k_s+1, k_f-1] \\ x(k_f) \leq C\backslash z(k_f) \wedge [A_0 \oplus A_0^-]\backslash x(k_f) \\ \wedge g^+(x(k_f-1), x(k_f), u(k_f)) \\ u(k_f) \leq [B_0 \oplus B^-]\backslash x(k_f) \end{cases} \tag{8}$$

We now introduce the following notation. Denoted X_l, vector $(x(k_s)^t, x(k_s+1)^t, u(k_s+1)^t, x(k_s+2)^t, u(k_s+2)^t, ..., x(k_s+l)^t, u(k_s+l)^t)^t$ is the concatenation of state trajectory $(x(k_s)^t, x(k_s+1)^t, x(k_s+2)^t, ..., x(k_s+l)^t)^t$ and input trajectory $(u(k_s+1)^t, u(k_s+2)^t, ..., u(k_s+l)^t)^t$ where $l = k_f - k_s$ denotes the length of the horizon. System (8) can be rewritten as the following fixed point form

$$X_l \leq h_l(X_l) \quad , \tag{9}$$

where h_l is clearly a (min, max, +) function. Therefore, we must analyze and solve a fixed-point problem of type $x \leq f(x)$ (if x exists) over horizon $[k_s, k_f]$. System (9) contains a "Backward" part as $x(k-1) \leq [A_1 \oplus A_1^-]\backslash x(k)$ but also, a "Forward" part with expression $x(k) \leq g^+(x(k-1), x(k), u(k))$. This fact forbids the immediate writing of forward or backward recurrences such as the state equations or the classical backward equations used in control for Timed Event Graphs. Let us note that g is a (min, max, +) function even if there is no function $f^+(x(k-1), x(k), u(k))$ in the specifications.

B. Existence of a finite solution

The aim of this part is to verify the existence of a finite solution (not just the greatest solution) in the control synthesis. Presented in the appendix, the property of homogeneity of (min, max, +) functions belonging to $F(n, n)$ is necessary to use Theorem 3. However, function h_l in (9) is not homogeneous as h_l contains desired output z which is a datum of the problem (see terms $C\backslash z(k)$ in (8)). To apply the spectral theory, we will use a relaxation by associating a new variable x_0 with every constant which leads to slight modifications of (8). All terms of system (8) are kept except terms $C\backslash z(k)$ which become $C\backslash(z(k) \otimes x_0)$ for $k \in [k_s+1, k_f]$. Moreover, inequality $x_0 \leq x_0$ which is always satisfied, is added and consequently, the system defined by (9) can also be defined by following system (10) with condition $x_0 = 0$. Therefore, with nonhomogeneous function h_l is associated a homogeneous function H_l, denoted with the same letters but in capitals.

$$\begin{pmatrix} x_0 \\ X_l \end{pmatrix} \leq H_l \begin{pmatrix} x_0 \\ X_l \end{pmatrix} \tag{10}$$

If $\begin{pmatrix} x_0 \\ X_l \end{pmatrix}$ is an arbitrary solution of (10) in $\mathbb{R}$ (without condition $x_0 = 0$), then

$$(x_0)^{-1} \otimes \begin{pmatrix} x_0 \\ X_l \end{pmatrix} = (-x_0) \otimes \begin{pmatrix} x_0 \\ X_l \end{pmatrix} = \begin{pmatrix} 0 \\ (-x_0) \otimes X_l \end{pmatrix} \quad , \tag{11}$$

is a solution of (10) with condition $x_0 = 0$. We can interpret variable x_0 as a possibly negative period added to the desired output which delays or anticipates all calculated dates with respect to the origin of time. Particularly, all the input dates can be postponed or anticipated with the same duration. A similar technique is used in part V of [11] and the relevant variable can be interpreted as an increase or decrease of every temporisation of the Timed Event Graph.

Using the cycle time vector χ (see Appendix), the following theorem analyzes the existence of a finite vector X_l. Therefore, it gives the conditions such that the plant can follow a finite state trajectory obeying the specifications.

Theorem 2: There exists a finite vector X_l satisfying models (1), (2) and constraint $y(k) = C \otimes x(k) \leq z(k)$ on horizon $[k_s, k_f]$ if and only if $\chi(H_l) \geq 0$.

Proof: As H_l is a homogeneous function, spectral vector $\chi(H_l)$ can be calculated and Theorem 3 (see appendix) applies. If the cycle time satisfies the corresponding condition of existence, system (10) has a solution $\begin{pmatrix} x_0 \\ X_l \end{pmatrix}$. For any solution, an obvious translation can be applied in such a way that the first component x_0 equals zero. If $x_0 = 0$, then this solution satisfies (8) which is equivalent to system which is made up of (1), (2) and constraint $y(k) = C \otimes x(k) \leq z(k)$. ■

Example (education system continued)

The calculation of the spectral vector of the above function denoted H_1 leads to $\chi(H_1) = 0$ for the following values: $T_1 = 60; T_2 = 90; T_3^- = 165; T_3^+ = 175$. Therefore, the above system is consistent. The following subsystem can be deduced from (8).

$$\begin{cases} x_1(1) \leq -T_1 + x_2(1) \\ x_2(1) \leq -T_2 + x_3(1) \\ x_3(1) \leq +T_3^+ + x_1(1) \end{cases} \tag{12}$$

Therefore, $x_1(1) \leq -T_1 - T_2 + T_3^+ + x_1(1)$. If $T_1 = 60$, $T_2 = 90$ and $T_3^+ = 175$, the inequality becomes $x_1(1) \leq -60 - 90 + 175 + x_1(1) = 25 + x_1(1)$ which is consistent.

Now, if we take $T_3^- = 135$ and $T_3^+ = 145$, we obtain $-T_1 - T_2 + T_3^+ = -5 < 0$. An incoherency appears as inequality $x_1(1) \leq -T_1 - T_2 + T_3^+ + x_1(1)$ gives $x_1(1) \leq -5 + x_1(1)$: the interpretation is that the lesson time of the professor is inconsistent with the official instructions. The calculation shows that several components of spectral vector $\chi(H_1)$ are negative: $\chi(H_1) = (0, -\frac{5}{3}, -\frac{5}{3}, ..., -\frac{5}{3})^t$. Consequently, $\chi(H_1) \ngeq 0$ and the above system (10) has no solution. ■

C. *Determination of the greatest solution*

The previous part considers the existence of an arbitrary finite solution X_l. Let us now consider a particular solution which is the greatest solution. The greatest control trajectory and also, the greatest state trajectory are clearly found if the greatest solution is determined.

The existence of the greatest solution on complete lattices can be proved by using the famous fixed point theorem of Knaster-Tarski [14] whose conditions are already satisfied: in our problem, $h_l\,(.)$ is an isotone function defined on a complete lattice $\overline{\mathbb{R}}_{max}$=($\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \leq$). If an algorithm gives the greatest solution of $x \leq f(x)$, this solution also satisfies the relevant equality. Function $h_l\,(.)$ is also discontinuous but Knaster-Tarski Theorem does not require continuity of the function.

The effective calculation of the greatest control of inequality (9) can be made by a classical iterative algorithm of Mc Millan and Dill [12] which particularizes the algorithm of Kleene to (min, max, +) expressions. The general resolution of $x \leq f(x)$ is given by the iterations of $x_i \leftarrow x_{i-1} \wedge f(x_{i-1})$ if the finite starting point is greater than the final solution. Here, number i represents the number of iterations and not the number of components of vector x. Following this framework, we provide an algorithm specific to the determination of the greatest state and control. Described below, algorithm 1 uses a decomposition of system (8) in its backward part and forward part. For instance, the second relation of (8) is divided into two parts

$$\begin{cases} x(k) \leq C\backslash z(k) \wedge [A_0 \oplus A_0^-]\backslash x(k) \wedge [A_1 \oplus A_1^-]\backslash x(k+1) \\ x(k) \leq g^+(x(k-1), x(k), u(k)) \end{cases}, \quad (13)$$

for $k \in [k_s+1, k_f-1]$. In the following algorithm, the input is the desired output trajectory $z(k)$ from $k = k_s+1$ to k_f. The outputs are the greatest control trajectory $u(k)$ from $k = k_s+1$ to k_f and the greatest state trajectory $x(k)$ from $k = k_s$ to k_f . Term $x(k,i)$ is the state estimate at event number k and iteration i.

Algorithm 1

Step 0 (initialization): $i = 1$; $x(k, i-1) \leftarrow T$ for $k \in [k_s, k_f]$

Repeat

- Step 1: backward calculation of the state

$x(k_f, i) \leftarrow x(k_f, i-1) \wedge C\backslash z(k_f) \wedge [A_0 \oplus A_0^-]\backslash x(k_f, i-1)$

$x(k,i) \leftarrow x(k,i-1) \wedge C\backslash z(k) \wedge [A_0 \oplus A_0^-]\backslash x(k,i-1) \wedge [A_1 \oplus A_1^-]\backslash x(k+1,i)$ from $k = k_f - 1$ to $k_s + 1$

$x(k_s, i) \leftarrow x(k_s, i-1) \wedge [A_1 \oplus A_1^-]\backslash x(k_s+1, i)$

- Step 2: backward calculation of the control

$u(k) \leftarrow (B_0 \oplus B^-)\backslash x(k,i)$ from $k = k_f$ to $k_s + 1$

- Step 3: forward calculation of the state

$x(k,i) \leftarrow x(k,i) \wedge g^+(x(k-1,i), x(k,i), u(k))$ from $k = k_s + 1$ to k_f

Until no $x(k,i)$ changes for $k_s \leq k \leq k_f$

Using a "Backward" approach, the first iteration of step 1 allows the determination of the starting state trajectory of the general algorithm. In step 2, the control is directly calculated by a unique relation from the state and the memorization of their previous calculated values is useless as $x(k,i)$ is minimized at each step i. In step 3, state minimization improves the calculated values of step 1 by a forward recurrence. These steps are repeated until convergence. When the minimization of the state stops, the algorithm gives the optimal state and control which satisfy the inequalities of the plant (described by a Timed Event Graph (1)) following the specifications (expressed by an interval system (2)).

Step 1 corresponds to the well-known backward equality described in part 5.6 in [1] if we consider the case of a Timed Event Graph without specification. Classical handlings can reduce the writing of step 1 and a relation similar to (5.62) in [1] can be obtained. Let us recall that the greatest solution (the latest times) of the control problem is explicitly given by the "Backward" recursive equations. The development of the algorithm is easy and only requires the memorization of the matrices of the different models and the estimated trajectory $x(k, i)$. In the general case, it is difficult to carry out a theoretical analysis of the number of iterations as in many algorithms in this field [3]. The general algorithm of Mc Millan and Dill [12] is known to be pseudo-polynomial in practice.

Example (education system continued)

The lesson must now stop before the daily closing time of school from Monday (first day: $k = 1$) to Friday ($k = 5$): Assume that desired output sequence $z(k)$ from $k = 1$ to 5 is 1140, 2580, 3600, 5460, 6480. So, $k_s = 0$ and $k_f = 5$. The corresponding output sequence is as follows.

k	1	2	3	4	5
u_1	975	2415	3435	5295	6315
u_2	1050	2490	3510	5370	6390

■

IV. Conclusion

This paper solves the problem of optimal control synthesis of a Timed Event Graph when the state and control trajectories are constrained by specifications defined by an interval model. The interval descriptor system can describe the time behavior of a lot of models such as Timed Event Graphs, P-time Petri nets and Time Stream Event Graphs for semantic rules "And" and "Weak-And" [4]. The problem is reformulated in a fixed point form. The spectral theory gives conditions of existence of a solution while a proposed algorithm makes it possible to determine the greatest state and control. The application of the calculated control generates a state trajectory obeying the specifications.

V. Appendix

In this section, we shall review a few basic theoretical notions about dioids. For more extensive presentations, the reader is invited to consult the following references: [1] and [5].

A monoid is a couple $(S, \oplus)$ where operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$ is commutative,

$\otimes$ is distributive in relation to $\oplus$ and the zero element ε of $\oplus$ is the absorbing element of $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). A dioid D is an idempotent semi-ring (operation $\oplus$ is idempotent, that is $a \oplus a = a$). Let us note that unlike the structures of group and ring, monoids and semi-rings do not have a property of symmetry on S. The set $\mathbb{R} \cup \{-\infty\}$ provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid which is usually denoted $\mathbb{R}_{max}$ = ($\mathbb{R} \cup \{-\infty\}, \oplus, \otimes$). The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$, respectively.

A dioid D is complete if it is closed for infinite sums and the distributivity of the multiplication with respect to addition extends to infinite sums: ($\forall\ c \in D$) ($\forall\ A \subseteq D$) $c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$. For example, ($\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes$) usually denoted $\overline{\mathbb{R}}_{max}$, is complete. The set of nxn matrices with entries in a complete dioid D included with the two operations $\oplus$ and $\otimes$ is also a complete dioid, which is denoted D^{nxn}. Nonsquare matrices can be considered if they are completed with rows or columns with entries equal to ε. The sum and product of matrices are defined conventionally from the sum and product in D.

Let Γ be a subset of vectors over $\overline{\mathbb{R}}_{max}$. The partial order denoted $\leq$ is defined as follows: $v \leq w \Longleftrightarrow v \oplus w = w$. It is also a componentwise order which allows the comparison of any pair of vectors (v, w) i.e. $v \leq w \Longleftrightarrow v_i \leq w_i$, for each component i. In the paper, this concept is applied to control and state trajectories. The element v of subset Γ is called greatest element or maximum element if and only if $w \leq v$ for all $w \in \Gamma$. In other words, it is greater than any other element of the subset: (see part 4.3.1 of [1] for more details). If this greatest element exists, it is unique as the existence of two different maximum elements v and w implies $w \leq v$ and $v \leq w$. Let $v \wedge w$ denote the lower bound of v and w.

A mapping f is monotone or isotone if $x \leq y$ implies $f(x) \leq f(y)$. Let f: $E \rightarrow F$ be an isotone mapping, where (E, $\leq$) and (F,$\leq$) are ordered sets. Mapping f is said to be residuated if for all $y \in F$, the least upper bound of subset $\{x \in E \mid f(x) \leq y\}$ exists and belongs to this subset. The corresponding mapping, denoted $f^d(y)$ is called the residual of f. When f is residuated, f^d is the only isotone mapping, such that $f \circ f^d \leq Id_F$ and $f^d \circ f \geq Id_E$ where Id_F and Id_E are identity mappings. Mapping $x \in (\overline{\mathbb{R}}_{max})^n \mapsto A \otimes x$, defined over $\overline{\mathbb{R}}_{max}$ is residuated [1] and the left $\otimes-$ residual of b by A is denoted by: $A \backslash b = \max\{x \in (\overline{\mathbb{R}}_{max})^n$ such that $A \otimes x \leq b\}$. Moreover, $(A \backslash b)_i = \bigwedge_{j=1}^{m} A_{ji} \backslash b_j$ where A is an $m \times n$ matrix.

A (min, max, +) function of set $F(n,1)$ is any function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^1$, which can be written as a term in the following grammar: $f = x_1, x_2, \ldots, x_n \mid f \otimes a \mid f \wedge f \mid f \oplus f$ where a is an arbitrary real number ($a \in \mathbb{R}$). The vertical bars separate the different ways in which terms can be recursively constructed. A (min, max, +) function of set $F(n,m)$ is any function f : $\mathbb{R}^n \longrightarrow \mathbb{R}^m$, such that each component f_i is a (min, max, +) function of $F(n,1)$.

Let $f \in F(n,1)$. If f can be represented by a term that does not use $\wedge$, it is said to be max-only or (max, +). If f can be represented by a term that does not use $\oplus$, it is said to be min-only or (min, +). If f can be represented by a term that does not use $\wedge$ and $\oplus$, it is said to be simple. As the type of the interval model is defined by the types of functions f^- and f^+ $\in F(n,m)$, we can characterize the model by the following pair (type of f^-, type of f^+) which defines different types of systems. Type ((min, max, +), (min, max, +)) naturally represents the more general mathematical case.

The following iterative form where number i represents the number of iterations, is considered: $x(i) = f(x(i-1))$, $\forall i \geq 1$ and $x(0) = \xi \in \mathbb{R}^n$, where f is a (min, max, +) function of $F(n,n)$. Every function of $F(n,n)$ has the property of homogeneity which is defined as follows: $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{R}^n \ f(\lambda \otimes x) = \lambda \otimes f(x)$ in the usual vector-scalar convention: $(\lambda \otimes x)_i = \lambda \otimes x_i$.

In the following fundamental theorem, the notion of cycle time makes it possible to verify the existence of a solution of different inequalities and equalities. The cycle time vector is classically defined by $\chi(f) = \lim_{i \to \infty} x(i)/k$ and always exists in $F(n,n)$. It does not depend on ξ.

***Theorem** 3:* [9] Let f be a function of $F(n,n)$. The following two conditions are equivalent:

(i) There is a finite x such that $x \leq f(x)$ (respectively, $x \geq f(x)$)

(ii) $\chi(f) \geq 0$ (respectively, $\chi(f) \leq 0$) . ■

REFERENCES

[1] F. Baccelli, G. Cohen, G.J. Olsder and J.P. Quadrat, *Synchronization and Linearity. An Algebra for Discrete Event Systems*, available from http://maxplus.org, New York, Wiley, 1992.

[2] P. Butkovič and G. Hegedüs, *An elimination method for finding all solutions of the system of linear equations over an extremal algebra*, Ekonomicko-Matematicky Obzor, Vol. 20, no 2, pp. 203-215, 1984.

[3] Y. Cheng and D-Z. Zheng, *Min-max inequalities and the timing verification problem with max and linear constraints*, Discrete Event Dynamic Systems : Theory and Applications, Springer Science and Business Media, Vol 15, pp. 119–143, 2005.

[4] J-P. Courtiat, M. Diaz, R.C. De Oliviera and P. Sénac, *Formal models for the description of timed behaviors of multimedia and hyper media distributed systems*, Computer Communications 19, pp. 1134-1150, 1996.

[5] B. Heidergott, G. J. Olsder and J. van der Woude, *Max Plus at Work*, Princeton University Press, 2006.

[6] P. Declerck and M.K. Didi Alaoui, *Optimal Control Synthesis in Interval Descriptor Systems, Application to Time Stream Event Graphs*, available from http://www.istia.univ-angers.fr/∼declerck, IFAC Congress Praha, 2005.

[7] P. Declerck and Didi Alaoui, M.K., *Extremal trajectories in P-time Event Graphs: application to control synthesis with specifications*, available from http://www.istia.univ-angers.fr/∼declerck, *Proc. 44th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC'05*. Seville, Spain, pp.7621-7626, 2005.

[8] V. Dhingra and S. Gaubert, *How to solve large scale deterministic games with mean payoff by policy iteration*, *ACM International Conference* Proceeding Series; Vol. 180, article No.12 in the Proceedings of the 1st international conference on Performance evaluation methodolgies and tools (VALUETOOLS), Pisa, Italy, 2006.

[9] S. Gaubert and J. Gunawardena, *The duality theorem for min-max functions*, CRAS, t. 326, Série I, pp. 43-48, 1998.

[10] M. L. J. Hautus, *Controlled invariance in systems over rings*, In Feedback Control of Linear and Nonlinear Systems. Berlin: Springer-Verlag, Vol. 39, pp 107-122, 1982.

[11] R. D. Katz, *Max-Plus (A,B)-Invariant Spaces and Control of Timed Discrete-Event Systems*, IEEE Transactions on Automatic Control, Vol.52, NO. 2, pp. 229-241, February 2007.

[12] K. Mc Millan and D. Dill, *Algorithms for interface timing verification*, Proceedings of the IEEE, International Conference on Computer Design: VLSI in Computers and Processors, 1992.

[13] B. De Schutter and T. van den Boom, *Model predictive control for max-plus-linear discrete event systems*, Automatica, Vol. 37(7), pp. 1049-1056, 2001.

[14] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific Journal of Mathematics, Vol. 5, pp. 285-309, 1955.