From Extremal Trajectories to Token Deaths in P-time Event Graphs

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Abstract— **In this paper, we consider the (max, +) model of P-time Event Graphs whose behaviors are defined by lower and upper bound constraints. The extremal trajectories of the system starting from an initial interval are characterized with a particular series of matrices for a given finite horizon. Two dual polynomial algorithms are proposed to check the existence of feasible trajectories. The series of matrices are used in the determination of the maximal horizon of consistency and the calculation of the date of the first token deaths.**

Index Terms— **P-time Petri Nets, (max,+) algebra, token death, Kleene star, fixed point.**

I. Introduction

Petri Nets (PNs) with time can express the time behavior of Discrete Event Systems with their specifications. Two main behaviors of the transitions can be distinguished: firing as soon as possible in Timed PNs and firing in given time intervals for Time PNs. Time can be associated with places, transitions and arcs of the PNs. In Time Stream PNs, temporal intervals are associated with arcs outgoing from places and the firing interval of transitions is defined by different semantics ([5] [11]). For Timed PNs, durations can be associated with places (P-Timed PNs) or transitions (T-Timed PNs) and the relevant subclasses are equivalent. For Time PNs, temporal intervals can similarly be associated with places or transitions but the corresponding subclasses (P-Time PNs and T-Time PNs) are fundamentally different. In Time PNs, a temporal interval of firing is associated with each transition enabled by the marking while a temporal interval of availability is associated with each token which enters a place in P-Time PNs. In this paper, we focus on P-time Event Graphs [13] whose evolution can undergo token deaths which express the loss of resources or parts and failures to meet time specifications. Applications of P-time Event Graphs can be found in production systems [10], food industry [6] and transportation systems [9].

A natural aim is to characterize the trajectories followed by the system starting from an initial state. In P-time Event Graphs, it is well-known that a simple forward simulation does not guarantee the correct synchronization of the transitions and often leads to token deaths. A first objective is the determination of possible trajectories without token deaths. The concept of extremal (lowest and greatest, see [14]) trajectories is relevant for the class of P-time Event Graphs and corresponds to the earliest and latest trajectories. In this paper, the objective is to express the *relations* of these extremal trajectories from the model and the initial condition, for a given horizon.

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An important notion is the consistency, which can be defined by the existence of a time trajectory following the model. A second question is to know if the different tasks can be sufficiently repeated during a period such that a given production plan can be performed. More precisely, our objective is to know if the different tasks can be repeated infinitely or during a finite period, and to determine the *maximal horizon* (maximal number of events) where the synchronizations of the transitions can be made.

A consequence is that the end of this horizon is also the limit of consistency which leads to a non-synchronization of the transitions: at least, a token death happens. The last objective is the determination of the *date of the first token deaths*.

In (max, +) algebra, other studies naturally analyse the trajectories. Using a fixed point approach, [8] considers the control of Timed Event Graphs with specifications defined by an interval model. Analysis of the consistency of interval descriptor systems as Time Stream Event Graphs is made by using the spectral vector for a given horizon while the greatest state and control trajectories are numerically calculated by an algorithm. In this paper, in-depth analysis of P-time event graphs is performed and algebraic expressions of extremal trajectories are derived. Polynomial algorithms are proposed for the determination of the maximal horizon of temporal consistency and the calculation of the first date of token deaths. This improves the pseudo-polynomial algorithm of [8] for similar problems.

Another possible approach is to rewrite the system in the form of a polyhedron in conventional algebra [6]. A priori, an application of the algorithms of linear programming can check the existence of an arbitrary trajectory. But, recall that the best algorithms of linear programming are polynomial *in the weak sense*. Contrary to these generic algorithms, we propose here algorithms specific to the considered problem whose complexity is polynomial *in the strong sense*.

If we only consider the problem of consistency, a possible technique is the model-checking which is an enumerative method based on the construction of a state class graph and its analysis. Some authors [2] apply this approach to T-Time Petri nets where each state class is defined by its marking and a set of firing times of the transitions. Starting from a given class, the firing of each enabled transition generates another class and a procedure establishes the list of the different classes and its connections. Generally speaking, model checking faces a combinatorial blow up of the state-space, commonly known as the state explosion problem, even for small systems [15]: the elementary event graph of the example of Figure 4 in [2] which is composed of two places and two transitions, illustrates this fact. As a consequence, these approaches generally consider a class of models where the graph is finite, that is, the Time Petri Nets are *bounded* and the bounds of the time intervals are defined in the *rational numbers*. In this paper, these assumptions are not taken as the considered models are non-bounded Event Graphs where the values of the temporisations are defined in $\mathbb{R}^+$. Contrary to the polyhedra of the classes which generally are approximations of the possible dates [3], the spaces considered in this paper are exact because the concept of lattice is relevant in the event graphs.

In this paper, no hypothesis is made on the structure of the Event Graphs which do not need to be strongly connected. The initial marking must only satisfy the classical condition of liveness (no circuit without token), and the usual hypothesis First In First Out (FIFO) for tokens is made.

The paper is structured as follows: We first define P-time Event Graphs and briefly introduce their algebraic model. Then, we study the time behavior of this model with the help of a special series of matrices. Notations and some previous results in the (max, +) algebra are given in Appendix I while the proof of Theorem 1 is given in Appendix II.

II. (MAX, +) MODEL OF P-TIME EVENT GRAPHS

Consider the following notations. The set of places is denoted P. The initial marking of a place $p_l \in P$ is denoted m_l. Let ${}^\bullet p_l$ denote the set of input transitions of place $p_l \in P$ and $p_l^\bullet$ the set of output transitions of p_l. Similarly, ${}^\bullet x_i$ (respectively $x_i^\bullet$) denotes the set of the input (respectively, output) places of transition x_i for $i \geq 1$. In an Event Graph, card(${}^\bullet p_l$)=card($p_l^\bullet$)=1 for each place $p_l \in P$ and we can associate only a pair (x_i, x_j) with each place $p_l \in P$, such that transition x_j is ingoing ($x_j \in^\bullet p_l$) and transition x_i is outgoing ($x_i \in p_l^\bullet$). Initial marking m_l is also associated with place p_l.

Moreover, we associate with each place $p_l \in P$ a temporal interval $[a_l, b_l]$ with $0 \leq a_l \leq b_l$ and $[a_l, b_l] \in \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{+\infty\})$. Time constraints can be defined as follows. After the arrival of a token into a place p_l at time t, it is available to fire its unique downstream transition $x_j \in p_l^\bullet$ in a given interval $[t + a_l, t + b_l]$ and dies if the firing of transition x_j does not occur before $t+b_l$. In other words, the token must stay in place p_l during a duration between a_l and b_l. Before the minimal sojourn time a_l, the token is unavailable for firing transition $x_j \in p_l^\bullet$. After the maximal sojourn time b_l, the token dies.

Example.

Let us consider the P-time Event Graph of Fig. 1. The initial marking is $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t$ and the temporal intervals are: $[a_1, +\infty] = [3, +\infty]$, $[a_2, +\infty] = [6, +\infty]$, $[a_3, b_3] = [1, 2]$ and $[a_4, b_4] = [3, 11]$. Let us consider the following simulation for $k = 0$, 1 and 2.

k	0	1	2
x_1	4	11	11
x_2	0	7	14
x_3	0	6	13

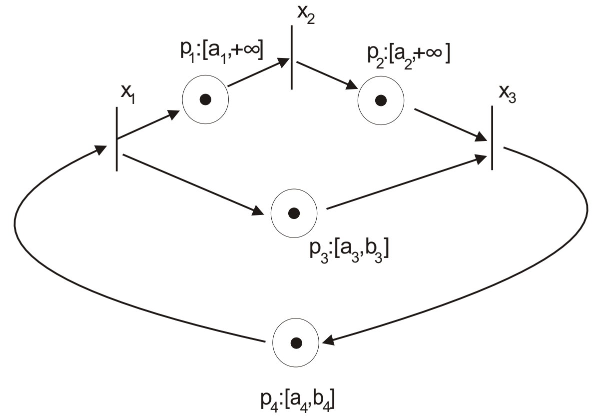

Fig. 1. P-time Event graph

k	0	1	2
p_1	$[7, +\infty]$	$[14, +\infty]$	$[14, +\infty]$
p_2	$[6, +\infty]$	$[13, +\infty]$	$[20, +\infty]$
p_3	$[5, 6]$	$[12, 13]$	$[12, 13]$
p_4	$[3, 11]$	$[9, 17]$	$[16, 24]$

The first table contains the firing dates while each column k of the second table is the bounds of the sojourn time (in absolute time) of the tokens, produced by the k^{th} firing of the transitions x_1, x_2 and x_3 in each place. We assume that the tokens of the initial marking are available immediately at $k = 0$. Let us consider the firing of transition x_3 for $k = 3$ which needs to use the tokens present in its upstream places p_2 and p_3 produced at $k = 2$. However, this synchronization does not occur because the interval $[20, +\infty] \cap [12, 13]$ is empty. A consequence is the death of the token in place p_3 at time $t = 13$.

However, transition x_1 can be fired at $t = 18$ because the interval of sojourn time of the token in place p_4 is $[16, 24]$. Therefore, a token is added in place p_3 with time interval $[19, 20]$ and the firing of transition x_3 can occur at time $t = 20$ because the interval $[20, +\infty] \cap [12, 13]$ is replaced by the interval $[20, +\infty] \cap [19, 20]$. ■

We now consider the "dater" description in the (max, +) algebra: each variable $x_i(k)$ represents the date of the k^{th} firing of transition x_i for $i \geq 1$. If we assume a FIFO functioning of the places which guarantees that the tokens do not overtake one another, a correct numbering of the events can be carried out. The evolution of the P-time Event Graph is described by the following inequalities expressing relations between the firing dates of transitions:

$\forall p_l \in P$ with $x_j \in^\bullet p_l$ and $x_i \in p_l^\bullet$, $a_l + x_j(k - m_l)$ $\leq x_i(k)$ and $x_i(k) \leq b_l + x_j(k - m_l)$

From these relations, we can derive an equivalent description of the system in (max, +) algebra [7]. By usual (max,+) algebraic notation, maximization and addition operations are denoted respectively $\oplus$ and $\otimes$. The notations and a brief review of preliminary results are presented in Appendix I. Without loss of generality, we assume that the initial marking of each place is equal to zero or one. Hence the (max, +) algebra model is as follows

$$\begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} \geq \begin{pmatrix} A^{=} & A^{+} \\ A^{-} & A^{=} \end{pmatrix} \otimes \begin{pmatrix} x(k) \\ x(k+1) \end{pmatrix} \quad (1)$$

for $k \geq 0$, where the initial condition is $x(0) = x_0$, matrix A^- (respectively, A^+) contains the temporizations a_l (respectively, b_l with minus sign) associated with each place $\forall p_l \in P$ with $m_l = 1$. We have $A^= = A_0^- \oplus A_0^+$ where matrix A_0^- (respectively, A_0^+) is defined as A^- (respectively, A^+) but with $m_l = 0$.

Example continued.

$A^= = \varepsilon$, $A^- = \begin{pmatrix} \varepsilon & \varepsilon & 3 \\ 3 & \varepsilon & \varepsilon \\ 1 & 6 & \varepsilon \end{pmatrix}$ and $A^+ = \begin{pmatrix} \varepsilon & \varepsilon & -2 \\ \varepsilon & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon \end{pmatrix}$. ■

Now, we analyze the time evolution of model (1).

III. Extremal acceptable trajectories by series of matrices

Unlike a Timed Event Graph which defines a unique trajectory according to the earliest firing rule, a P-time Event Graph defines a set of trajectories which depend on matrices $A^=$, A^- and A^+. The aim of this section is to give the relations of the extremal (lowest and greatest) trajectories satisfying an initial condition given by $x(0) \in [x_0^-, x_0^+]$ (that is, box $[x_0^-, x_0^+]$ defines a set of initial conditions) and model (1) for $k = 0$, ..., $h-1$ where $h \in \mathbb{N}$ is a finite horizon. In the sequel, we will show that these relations allow the determination of the maximal horizon of consistency (possibly infinite) and the calculation of the date of the first token deaths.

A. Expressions of the extremal state trajectories

We consider below a pair of trajectories corresponding to the earliest and greatest trajectories. The dimension of vector x is denoted n. Symbols $\wedge$ and $\backslash$ respectively correspond to the minimization operation and the left $\otimes-$ residual defined in Appendix I.

***Theorem** 1:* Given horizon $h \in \mathbb{N}$, the lowest and greatest state trajectories $(x^-(k), x^+(k)) \in ((\overline{\mathbb{R}}_{max})^n \mathrm{x} (\overline{\mathbb{R}}_{max})^n)$ for $k = 0,...,h$ respectively starting from an initial condition $x^-(0) \geq x_0^- \in (\overline{\mathbb{R}}_{max})^n$ and $x^+(0) \leq x_0^+ \in (\overline{\mathbb{R}}_{max})^n$, are given by the following equalities:

a) Coefficients of matrix w_k by forward iteration

Initialization: $w_0 = A^=$

For $k = 1$ to h, $\quad w_k = A^= \oplus A^- \otimes (w_{k-1})^* \otimes A^+$

b) First estimate (β_k^-, β_k^+) by forward iteration

Initialization: (β_0^-, β_0^+)=(x_0^-, x_0^+)

For $k = 1$ to h, (β_k^-, β_k^+)=$(A^- \otimes (w_{k-1})^* \otimes \beta_{k-1}^-$, $A^+ \backslash (w_{k-1})^* \backslash \beta_{k-1}^+)$

c) Trajectory $(x^-(k), x^+(k))$ by backward iteration

Initialization: $(x^-(h), x^+(h))$=$((w_h)^* \otimes \beta_h^-, (w_h)^* \backslash \beta_h^+)$

For $k = h-1$ to 0, $(x^-(k), x^+(k))$=$((w_k)^* \otimes [A^+ \otimes x^-(k+1) \oplus \beta_k^-], (w_k)^* \backslash [A^- \backslash x^+(k+1) \wedge \beta_k^+])$

Proof. The proof is given in Appendix II. ■

The three steps of the theorem make up two forward/backward algorithms. Identical in the calculation of the two bounds, the first step a) is the forward calculation of parameters w_k which only depends on the model. Starting from the initial condition x_0^- (respectively, x_0^+), the second step b) is also based on a forward iteration. It expresses a first estimate of the lowest (resp., greatest) trajectory denoted β_k^- (resp., β_k^+), which is finally improved by a maximisation (resp., a minimisation) in step c). The final result is the lowest (resp., greatest) trajectory denoted by $x^-(k)$ (resp., $x^+(k)$). Note that each bound can be derived from the other one by duality and each lower (resp., upper) matrix respectively corresponds to an upper (resp., lower) matrix by replacing $\oplus$ by $\wedge$, $\otimes$ by $\backslash$ and conversely.

The following property compares the intermediate trajectories (β_k^-, β_k^+) with $(x^-(k), x^+(k))$.

***Property** 1:* The pair (β_k^-, β_k^+) for $k = 0$ to $+\infty$ is a box (interval vector) containing the extremal trajectories $(x^-(k), x^+(k))$ for any finite horizon h, for a given pair of initial conditions (x_0^-, x_0^+).

Proof. Immediate: $x^-(k)$ (respectively, $x^+(k)$) is the result of a maximization (respectively, a minimization) in step c). ■

Therefore, step b) gives intermediate trajectories (β_k^-, β_k^+) for a given interval $[x_0^-, x_0^+]$, which are formally defined in the infinite horizon. As they are independant of step c), a calculation on a given finite horizon h_1 can be reused in new calculation of the extremal state trajectories for another horizon $h_2 \neq h_1$. If $h_1 < h_2$, only the calculation of (β_k^-, β_k^+) for $k = h_1 + 1, .., h_2$ is necessary.

Remark. Defined on a box $[x_0^-, x_0^+]$, the initial condition is less restrictive than the more usual $x(0) = x_0$ which is a particular case ($x_0^- = x_0^+ = x_0$). Assuming that the system is consistent, the two dual algorithms allow checking the existence of an initial condition $x(0) \in [x_0^-, x_0^+]$ in $\mathbb{R}$ which is the starting point of a finite trajectory: indeed, if A^- has no null row, trajectory x^- is finite and the check is the verification of inequality $x^-(0) \leq x_0^+$ for the lower bound $x^-(k)$. Also, the equality $x^-(0) = x_0^-$ clearly allows checking the acceptability of x_0^- or, in other words, if there is a trajectory starting from x_0^- . The same remarks hold for the dual algorithm under the condition that matrix A^+ has no null column.

B. Maximal horizon of temporal consistency

Assuming the liveness of the Event Graph, we consider the temporal consistency of P-time Event graphs. Clearly, if we can calculate an arbitrary finite trajectory (that is, in $\mathbb{R}$) starting from $x(0) \in (\mathbb{R})^n$, the system is consistent on the given horizon. Therefore, the liveness of tokens is guaranteed and it does not lead to any deadlock situation. In fact, we can prove that the existence of a finite trajectory only depends on matrices w_k and more precisely, that a finite trajectory exists if and only if matrices $(w_k)^*$ converge in $\mathbb{R}_{\max}$ [7].

Let us now consider the problem of the determination of the maximal horizon of temporal consistency. In step c) of the algorithms, the calculations of the state trajectory $x^-(k)$

start from values w_h , β_h^- and β_h^+ and consequently depend on horizon $[0, h]$ where h is a datum. Contrary to step c), the calculations of w_k, β_k^- and β_k^+ start from $A^=$, x_0^- and x_0^+ in steps a) and b): they depend on index k, but not on horizon h as the calculations can continue after h. Therefore, the problem is now to determine the maximal horizon h_{max} where the system can follow a finite trajectory. As the horizon can be finite or infinite, we consider the two following cases.

- Case 1. Matrix $(w_k)^*$ does not belong to $\mathbb{R}_{\max}$. As there is at least an infinite entry $((w_k)^*)_{i,j} = +\infty$, the system is not consistent on horizon $[0, h]$ with $h \geq k$.
- Case 2. Matrix $(w_k)^*$ belongs to $\mathbb{R}_{\max}$. If $w_k = w_{k-1}$, then the P-time Event graph is consistent on an infinite horizon and $h_{max} = +\infty$.

A practical way to determine the horizon of consistency h_{max} is as follows.

Algorithm

Initialization: $k \leftarrow 0$. Calculate and analyze $(w_k)^*$ for $k \geq 0$. Stop if case 1 ($h_{max} = k - 1$ if $k \geq 1$) or case 2 ($h_{max} = +\infty$) defined above is satisfied or repeat with $k \leftarrow k + 1$. ■

As the series w_k is non-decreasing (the proof is given in [7]), each entry converges to a stable finite value or the infinite value $+\infty$.

Example continued.

$$w_0 = A^= = \varepsilon, \; w_1 = \begin{pmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & -1 \end{pmatrix}, \; w_2 = \begin{pmatrix} -8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 \\ -4 & \varepsilon & -1 \end{pmatrix}, \; w_3 = \begin{pmatrix} -8 & \varepsilon & -3 \\ \varepsilon & \varepsilon & 1 \\ -4 & \varepsilon & 1 \end{pmatrix} \text{ and } (w_3)^* = \begin{pmatrix} +\infty & \varepsilon & +\infty \\ +\infty & 0 & +\infty \\ +\infty & \varepsilon & +\infty \end{pmatrix}.$$

As some coefficients of w_3^* are equal to $+\infty$, $h_{max} = 2$ and a trajectory can only be defined on horizon $[0, 2]$. System (1) is only consistent for $k = 0$ and 1.

C. Date of the first token deaths

If the system is only consistent on horizon h_{max}, an admissible trajectory can be calculated but the tokens produced by the firing at date $x(h_{max})$ do not lead to a complete firing of the transitions at the following number of events $h_{max} + 1$. Below, we consider only the case of places with unitary initial marking: by reason of the lack of space, the case of places with a null initial marking is omitted but follows a similar technique. If $m_l = 1$, the time interval of token stay is $[A_{ig}^- \otimes x_g(k),\ A_{gi}^+ \backslash x_g(k]$ for a token generated par the k^{th} firing of transition g in a place $p_l \in P$ such that $x_g \in^\bullet p_l$ and $x_i \in p_l^\bullet$. As there is at least one transition i such that relation $\bigoplus_{j \in^\bullet(^\bullet x_i)} A_{ij}^- \otimes x_j(h_{max}) \leq x_i(h_{max} + 1) \leq \bigwedge_{j \in^\bullet(^\bullet x_i)} A_{ji}^+ \backslash x_j(h_{max})$ is not satisfied, the non-synchronization of transition i leads to some token deaths. Let G be the set of transitions $g \in^\bullet (^\bullet x_i)$ such that $A_{gi}^+ \backslash x_j(h_{max}) = \bigwedge_{j \in^\bullet(^\bullet x_i)} A_{ji}^+ \backslash x_j(h_{max})$. Each input transition $g \in G$ generates a token which dies in the place $p_l \in (x_g)^\bullet \bigcap {}^\bullet(x_i)$ at the date $A_{gi}^+ \backslash x_j(h_{max})$.

However, the firing of transition i is still possible if a new firing of each transition $g \in G$ produces another token. This can be expressed by a shift in the numbering of the events. Therefore, relation $A_{ig}^- \otimes x_g(k) \leq x_i(k+1) \leq A_{gi}^+ \backslash x_g(k)$ for $k < h_{max}$, becomes relation $A_{ig}^- \otimes x_g(k+1) \leq x_i(k+1) \leq A_{gi}^+ \backslash x_g(k+1)$ for $k \geq h_{max}$, in the new algebraic model.

Example continued.

For $x_0^- = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t$, Theorem 1 provides the lowest trajectory x^- which is also trajectory x given in the first table of the example in part II. Using these dates, we can deduce that the date of the first token death is 13. The new model is as follows: for $k \geq 2$, matrices $B^=$, B^- and B^+, replace the previous one in system (1).

$$B^= = \begin{pmatrix} \varepsilon & \varepsilon & -2 \\ \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{pmatrix}, B^- = \begin{pmatrix} \varepsilon & \varepsilon & 3 \\ 3 & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon \end{pmatrix} \text{ and } B^+ = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon \end{pmatrix}.$$ ■

IV. Conclusion

Considering the (max, +) model of P-time Event Graphs, our first objective is the determination of the extremal state trajectories satisfying an initial condition defined on an interval. Based on a specific series of matrices, the proposed resolution is composed of three steps: using the Kleene star, the iterative calculation determines the values of the greatest paths for different horizons; a forward iteration generates a box containing the extremal trajectories; a backward iteration gives the extremal trajectories. The introduction of a nondecreasing series of matrices alleviates the storage as the dimension is the size of the model, which depends on the number of the transitions and the initial marking. Therefore, each calculation processes reduced matrices of dimension $(n \times n)$. The approach can be applied to important processes for large horizons because the algorithms are strongly polynomial: the complexity is $O(h.n^3)$ for a given horizon h if the complexity of the used algorithm of Kleene star is $O(n^3)$ ([7] gives the CPU time for different dimensions and horizons).

The determination of the maximal horizon of temporal consistency is the second objective. The technique is based on the analysis of convergence of matrices w_k^*: each entry can converge to a stable finite value or the infinite value $+\infty$. For a given P-time Event Graph, the case of a convergence to a constant matrix after a transitory period h_{max}, facilitates the storage and the reuse in the calculation of a new trajectory for any horizon. If the system is only consistent on horizon h_{max}, a non-synchronization cannot be avoided at $h_{max} + 1$ and we calculate the date of the first token deaths.

Appendix I

In this section, we shall review a few basic theoretical notions about dioids. For more extensive presentations, the reader is invited to consult the following reference [1].

A monoid is a couple $(S, \oplus)$ where operation $\oplus$ is associative and presents a neutral element. A semi-ring S is a triplet $(S, \oplus, \otimes)$ where $(S, \oplus)$ and $(S, \otimes)$ are monoids, $\oplus$

is commutative, $\otimes$ is distributive in relation to $\oplus$ and the zero element ε of $\oplus$ is the absorbing element of $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$). A dioid D is an idempotent semi-ring (operation $\oplus$ is idempotent, that is $a \oplus a = a$). Let us note that unlike the structures of group and ring, monoids and semi-rings do not have a property of symmetry on S. The set $\mathbb{R} \cup \{-\infty\}$ provided with the maximum operation denoted $\oplus$ and the addition denoted $\otimes$ is an example of dioid which is usually denoted $\mathbb{R}_{max}$ = ($\mathbb{R} \cup \{-\infty\}, \oplus, \otimes$). The neutral elements of $\oplus$ and $\otimes$ are represented by $\varepsilon = -\infty$ and $e = 0$, respectively.

A dioid D is complete if it is closed for infinite sums and the distributivity of the multiplication with respect to addition extends to infinite sums: ($\forall\, c \in D$) ($\forall\, A \subseteq D$) $c \otimes (\bigoplus_{x \in A} x) = \bigoplus_{x \in A} c \otimes x$. For example, ($\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \otimes$) usually denoted $\overline{\mathbb{R}}_{max}$, is complete. The set of nxn matrices with entries in a complete dioid D included with the two operations $\oplus$ and $\otimes$ is also a complete dioid, which is denoted $D^{n\text{x}n}$. Nonsquare matrices can be considered if they are completed with rows or columns with entries equal to ε. The sum and product of matrices are defined conventionally from the sum and product in D.

Let Γ be a subset of vectors over $\overline{\mathbb{R}}_{max}$. The partial natural order denoted $\leq$ is defined as follows: $v \leq w \iff v \oplus w = w$. It is also a componentwise order which allows the comparison of any pair of vectors (v, w) i.e. $v \leq w \iff v_i \leq w_i$, for each component i. In the paper, this concept is applied to control and state trajectories. The element v of subset Γ is called greatest element or maximum element if and only if $w \leq v$ for all $w \in \Gamma$. In other words, it is greater than any other element of the subset: (see part 4.3.1 of [1] for more details). If this greatest element exists, it is unique as the existence of two different maximum elements v and w implies $w \leq v$ and $v \leq w$. Let $v \wedge w$ denote the lower bound of v and w.

A mapping f is monotone or isotone if $x \leq y$ implies $f(x) \leq f(y)$. Let f: $E \rightarrow F$ be an isotone mapping, where (E, $\leq$) and (F,$\leq$) are ordered sets. Mapping f is said to be residuated if for all $y \in F$, the least upper bound of subset $\{x \in E \mid f(x) \leq y\}$ exists and belongs to this subset. The corresponding mapping, denoted $f^d(y)$ is called the residual of f. When f is residuated, f^d is the only isotone mapping, such that $f \circ f^d \leq Id_F$ and $f^d \circ f \geq Id_E$ where Id_F and Id_E are identity mappings. Mapping $x \in (\overline{\mathbb{R}}_{max})^n \mapsto A \otimes x$, defined over $\overline{\mathbb{R}}_{max}$ is residuated [1] and the left $\otimes-$ residual of b by A is denoted by: $A \backslash b = \max\{x \in (\overline{\mathbb{R}}_{max})^n$ such that $A \otimes x \leq b\}$. Moreover, $(A \backslash b)_i = \bigwedge_{j=1}^{m} A_{ji} \backslash b_j$ where A is an $m \times n$ matrix. Using the Kleene star defined by: $A^* = \bigoplus_{i=0}^{+\infty} A^i$, the following theorem will be considered in the dioid of matrices.

***Theorem** 2:* (Theorem 4.75 part 1 in [1]) Consider equation $x = A \otimes x \oplus B$ and inequality $x \geq A \otimes x \oplus B$ with A and B in complete dioid D. Then, $A^* \otimes B$ is the least solution of these two relations. ■

APPENDIX II

Proof. System (1) for $k = 0, \ldots, h-1$ with $x(0) \geq x_0^-$ can be rewritten as follows.

$$\begin{cases} x(0) \geq A^= \otimes x(0) \ \oplus A^+ \otimes x(1) \oplus x_0^- \\ x(k) \geq A^- \otimes x(k-1) \oplus A^= \otimes x(k) \ \oplus A^+ \otimes x(k+1) \\ \text{ for } k = 1 \text{ to } h-1 \\ x(h) \geq A^- \otimes x(h-1) \oplus A^= \otimes x(h) \end{cases} \quad (2)$$

Theorem 2 shows that the smallest solution to this system also satisfies the corresponding equality and we can now consider the above system with equalities. The following proposition $\mathcal{P}(k)$ is now proved by recursion.

$\mathcal{P}(k)$: $x^-(k) = (w_k)^* \otimes [A^+ \otimes x^-(k+1) \oplus \beta_k^-]$

Base case: $\mathcal{P}(0)$

From the first equality of (2), we can write $x(0) = w_0 \otimes x(0) \oplus A^+ \otimes x(1) \oplus \beta_0^-$ where $w_0 = A^=$ and $\beta_0^- = x_0^-$. Therefore, $x(0) = (w_0)^*[A^+ \otimes x(1) \oplus \beta_0^-]$, which proves $\mathcal{P}(0)$.

Case: $\mathcal{P}(1)$

From the second equality of (2), we can write for $k = 1$, $x(1) = A^= \otimes x(1) \oplus A^- \otimes x(0) \oplus A^+ \otimes x(2)$. If $\mathcal{P}(0)$ is used, $x(1)$=$A^= \otimes x(1) \oplus A^- \otimes [(w_0)^*[A^+ \otimes x(1) \oplus \beta_0^-]] \oplus A^+ \otimes x(2)$. The distributivity of $\otimes$ with respect to $\oplus$ leads to $x(1) = [A^= \oplus A^- \otimes (w_0)^* \otimes A^+] \otimes x(1) \oplus A^- \otimes (w_0)^* \otimes \beta_0^- \oplus A^+ \otimes x(2) = w_1 \otimes x(1) \oplus \beta_1^- \oplus A^+ \otimes x(2)$ where $w_1 = A^= \oplus A^-(w_0)^* A^+$ and $\beta_1^- = A^-(w_0)^* \otimes \beta_0^-$. Therefore, $x(1) = (w_1)^* \otimes [A^+ \otimes x(2) \oplus \beta_1^-]$ and $\mathcal{P}(1)$ is proved. Now, this approach is generalized for $k = 1$ to $h-1$.

Case: $\mathcal{P}(k)$ for k from 1 to $h-1$.

Let us assume $\mathcal{P}(k-1)$: $x(k-1) = (w_{k-1})^* \otimes [A^+ \otimes x(k) \oplus \beta_{k-1}^-]$. We will prove that $\mathcal{P}(k-1)$ entails $\mathcal{P}(k)$. From the second equality of (2), we can write $x(k) = A^= \otimes x(k) \oplus A^- \otimes x(k-1) \oplus A^+ \otimes x(k+1)$. As $x(k-1) = (w_{k-1})^* \otimes [A^+ \otimes x(k) \oplus \beta_{k-1}^-]$, the following expression is deduced: $x(k) = A^= \otimes x(k) \oplus A^- \otimes (w_{k-1})^* \otimes [A^+ \otimes x(k) \oplus \beta_{k-1}^-] \oplus A^+ \otimes x(k+1)$. The distributivity of $\otimes$ with respect to $\oplus$ yields $x(k) = [A^= \oplus A^- \otimes (w_{k-1})^* \otimes A^+] \otimes x(k) \oplus A^- \otimes (w_{k-1})^* \otimes \beta_{k-1}^- \oplus A^+ \otimes x(k+1) = w_k \otimes x(k) \oplus \beta_k^- \oplus A^+ \otimes x(k+1)$, where $w_k = A^= \oplus A^- \otimes (w_{k-1})^* \otimes A^+$and $\beta_k^- = A^- \otimes (w_{k-1})^* \otimes \beta_{k-1}^-$. Therefore, $x(k) = (w_k)^*[A^+ \otimes x(k+1) \oplus \beta_k^-]$ and the desired expression is obtained: $\mathcal{P}(k)$ has been deduced from $\mathcal{P}(k-1)$. Moreover, as $\mathcal{P}(0)$ is true, $\mathcal{P}(k)$ has been proved for k from 1 to $h-1$: the recursion is finished. Knowing β_k^-, the calculation of $x(k)$ uses a backward iteration, while the calculation of β_k^- is relevant to a forward iteration. Now, the final case will be proved.

Case: $\mathcal{P}(h)$

The last equality of (2) can be considered like the second equality but without $A^+ \otimes x(k+1)$: the argument of case $\mathcal{P}(k)$ can be taken and we can write $x(h) = (w_h)^* \otimes \beta_h^-$ with $w_h = A^= \oplus A^- \otimes (w_{h-1})^* \otimes A^+$and $\beta_h^- = A^- \otimes (w_{h-1})^* \otimes \beta_{h-1}^-$

The proof of the greatest trajectory is omitted as it can be deduced by duality from the previous proof. Indeed, as mapping $A^= \otimes x(k)$, $A^- \otimes x(k-1)$ and $A^+ \otimes x(k+1)$ are residuated, the application of property f3 in [1] part 4.4.4) gives the following form: it expresses every "upper" constraint on $x(k)$ which can minimize it.

$$\begin{cases} x(0) \leq A^= \backslash x(0) \wedge A^- \backslash x(1) \wedge x_0^+ \\ x(k) \leq A^+ \backslash x(k-1) \wedge A^= \backslash x(k) \wedge A^- \backslash x(k+1) \\ \text{ for } k = 1 \text{ to } h-1 \\ x(h) \leq A^+ \backslash x(h-1) \wedge A^= \backslash x(h) \quad ■ \end{cases}$$

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